International Journal of Pure and Applied Mathematics
Volume 101 No. 3 2015, 421-424
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v101i3.10
ijpam.eu

SUBSPACE SUPERCYCLICITY OF TUPLES OF OPERATORS

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Abstract: In this paper, we investigate subspace supercyclicity of tuples of operators.

AMS Subject Classification: 47B37, 47B33
Key Words: tuple, subspace supercyclicity, subspace supercylicity criterion

1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbb{C}$ and let M be a nonzero subspace of X. We will let

$$\mathcal{F} = \{T_1{}^{k_1} T_2{}^{k_2} ... T_n{}^{k_n} : k_i \geq 0, i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a M-supercyclic vector for $\mathcal{T}$ if $\mathbb{C}Orb(\mathcal{T}, x) \cap M$ is dense in M and in this case the tuple $\mathcal{T}$ is called M-supercyclic. The set of all M-supercyclic vectors of $\mathcal{T}$ is denoted by $SC(\mathcal{T}, M)$. Also, for all $k \geq 2$, by $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of

Received: January 31, 2015

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an element of $\mathcal{F}$, i.e.

$$\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}\}.$$

We say that $\mathcal{T}_d^{(k)}$ is subspace-supercyclic, with respect to M, provided there exist $x_1, ..., x_k \in X$ such that $\mathbf{C}\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\} \cap M$ is dense in the k copies of M, $M \oplus ... \oplus M$.

Suprisingly, there are something that does not happen for single operators. For example, hypercyclic tuples can arise in finite dimensional, and there are operators that have somewhere dense orbits that are not everywhere dense. Also, we note that there are subspace-hypercyclic operators that are not hypercyclic. For some topics we refer to [1]-[3].

2. Main Results

In this section, we introduce subspace-supercyclicity criterion for tuples of operators and we give some relations between the concept of subspase-supercyclicity and the subspace-superciclicity criterion.

Theorem 2.1. Suppose that $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbf{C}$ and M is a nonzero subspace of X. Then $SC(\mathcal{T}, M)$ is a G_δ set.

Proof. Let $\{B_n : n \in \mathbb{N}\}$ be a countable open basis for the relative topology of M. Note that

$$x \in \bigcap_i \bigcup \{\lambda T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n}(B_i) \cap M : \; K_1, ..., k_n \geq 0, \; \lambda \in \mathbf{C} \backslash \{0\}\}$$

if and only if for any integer $j \geq 1$, there exists a $\lambda \in \mathbf{C}\backslash\{0\}$ and a tuple $(k_1, k_2, ..., k_n)$ of integers such that $\lambda T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x \in B_j$. This occurs if and only if $\mathbf{C}Orb(\mathcal{T}, x) \cap M$ is dense in M or equivalently, if $x \in SC(\mathcal{T}, M)$. Thus $SC(\mathcal{T}, M)$ is indeed a G_δ set. ☐

Corollary 2.2. Suppose that $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbf{C}$ and M is a nonzero subspace of X. Then

$$SC(\mathcal{T}, M) = \bigcap_i \bigcup \{\lambda T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n}(B_i) \cap M : \; K_1, ..., k_n \geq 0, \; \lambda \in \mathbf{C} \backslash \{0\}\}.$$

Proof. By the proof of Theorem 2.1, it is clear. □

Theorem 2.3. Suppose that $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbf{C}$ and M is a nonzero subspace of X. Then the following conditions are equivalent:

i) For any nonempty sets $U \subset M$ and $V \subset M$, both relatively open, there exist a $\lambda \in \mathbf{C}\backslash\{0\}$ and a tuple $(k_1, k_2, ..., k_n)$ of integers such that

$$\lambda^{-1}T_1^{-k_1}T_2^{-k_2}...T_n^{-k_n}(U) \cap V$$

contains a relatively open nonempty subset of M.

ii) For any nonempty sets $U \subset M$ and $V \subset M$, both relatively open, there exist a $\lambda \in \mathbf{C}\backslash\{0\}$ and a tuple $(k_1, k_2, ..., k_n)$ of integers such that

$$\lambda^{-1}T_1^{-k_1}T_2^{-k_2}...T_n^{-k_n}(U) \cap V$$

is nonempty and $T_1^{k_1}T_2^{k_2}...T_n^{k_n}M \subset M$.

Proof. Let (i) holds and let U and V be nonempty relatively open subsets of M. Hence there exist a $\lambda \in \mathbf{C}\backslash\{0\}$ and a tuple $(k_1, k_2, ..., k_n)$ of integers such that $\lambda^{-1}T_1^{-k_1}T_2^{-k_2}...T_n^{-k_n}(U) \cap V$ contains a relatively open nonempty set W in M. To show that $T_1^{k_1}T_2^{k_2}...T_n^{k_n}M \subset M$, let $x \in M$ and note that

$$\lambda T_1^{k_1}T_2^{k_2}...T_n^{k_n}W \subset U \cap \lambda T_1^{k_1}T_2^{k_2}...T_n^{k_n}V \subset U \subset M.$$

Hence

$$T_1^{k_1}T_2^{k_2}...T_n^{k_n}W \subset M.$$

If $x_0 \in W$, then there exists $r > 0$ small enough such that $x_0 + rx \in W$, since W is relatively open. Thus

$$T_1^{k_1}T_2^{k_2}...T_n^{k_n}(x_0 + rx) = T_1^{k_1}T_2^{k_2}...T_n^{k_n}x_0 + rT_1^{k_1}T_2^{k_2}...T_n^{k_n}x \in M,$$

which implies that $T_1^{k_1}T_2^{k_2}...T_n^{k_n}x \in M$. Thus $T_1^{k_1}T_2^{k_2}...T_n^{k_n}M \subset M$.

Now, let (ii) holds. Note that for all $\lambda \in \mathbf{C}\backslash\{0\}$ and all tuples $(k_1, k_2, ..., k_n)$ of integers, $\lambda T_1^{k_1}T_2^{k_2}...T_n^{k_n} : M \to M$ is continuous and so

$$\lambda^{-1}T_1^{-k_1}T_2^{-k_2}...T_n^{-k_n}(U) \cap V$$

is relatively open and nonempty subset of M. This completes the proof. □

Lemma 2.4. Suppose that $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of operators acting on a separable infinite dimensional Banach space X over $\mathbf{C}$ and M is a nonzero subspace of X. If any of the conditions in Theorem 2.3 is satisfied, then $SC(\mathcal{T}, M)$ is a dense subset of M.

Proof. Let $\{B_n : n \in \mathbb{N}\}$ be a countable open basis for the relative topology of M. In Theorem 2.2 (i), put $U = B_i$ and $V = B_j$, then there exist $\lambda_{i,j} \in \mathbf{C}\backslash\{0\}$ and $K_{i,j}^m \in \mathbb{N}\cup\{0\}$ for $m = 1, ..., n$ satisfying that $\lambda_{i,j}^{-1} T_1^{-k_{i,j}^1} T_2^{-k_{i,j}^2} ... T_n^{-k_{i,j}^n}(B_i) \cap B_j$ is relatively open. Hence, the sets

$$G_i = \bigcup_j \lambda_{i,j}^{-1} T_1^{-k_{i,j}^1} T_2^{-k_{i,j}^2} ... T_n^{-k_{i,j}^n}(B_i) \cap B_j$$

are relatively open. Also, each G_i is dense since it intersects each relatively open set in M. Hence, $\bigcap_i G_i$ is also dense. Since

$$\bigcap_i \bigcup_j \lambda_{i,j}^{-1} T_1^{-k_{i,j}^1} T_2^{-k_{i,j}^2} ... T_n^{-k_{i,j}^n}(B_i) \cap B_j$$

is a subset of

$$\bigcap_i \bigcup \{\lambda^{-1} T_1^{-k_1} T_2^{-k_2} ... T_n^{-k_n}(B_i) \cap M : \; K_1, ..., k_n \geq 0, \lambda \in \mathbf{C}\backslash\{0\}\},$$

by Corollary 2.2, it is clear. □

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