International Journal of Pure and Applied Mathematics
Volume 101 No. 3 2015, 425-432
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v101i3.11

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HESSENBERG MATRICES AND THE PELL-LUCAS AND JACOBSTHAL NUMBERS

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Abstract: There are many relationships between the number theory and matrix theory. In this work, we defined two upper Hessenberg Matrices and then we showed that the permanents of these Hessenberg matrices are Pell-Lucas and Jacobsthal numbers.

AMS Subject Classification: 11B37, 11B39
Key Words: contraction, permanent, Pell-Lucas numbers, Jacobsthal numbers, Hessenberg matrix

1. Introduction

In matrix theory, determinant and permanent are two importance consepts. Let $A = [a_{ij}]$ be an $n \times n$ matrix and S_n is a symetric group, denotes the group

Received: February 14, 2015

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of permutations over the set $\{1, 2, ..., n\}$. The determinant of A defined [8] by

$$\det A = \sum_{\sigma \epsilon S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the sum ranges over all the permutations of the integers $1, 2, ..., n$. It can be denoted by $sgn(\sigma) = \pm 1$ the signature of σ, equal to $+1$ if σ is the product an even number of transposition, and -1 otherwise. Similarly the permanent of the matrix is defined by

$$perA = \sum_{\sigma \epsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

Computations of determinants and permanents have a great importance in many branches of mathematics and physics. In literature, there are several methods to compute determinant and permanent. In this article, we will use contraction method defined by Brualdi and et. al.

Let $A = [a_{ij}]$ be an $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We call A *contractible* on column k, if column k contains exactly two nonzero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0, a_{jk} \neq 0$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0, a_{kj} \neq 0$ and $i \neq j$, then the matrix $A_{k:ij} = [A_{ij:k}^T]^T$ is called the contraction of A on row k relative to columns i and j. We know that if A is a nonnegative matrix and B is a contraction of A [7], then

$$perA = perB. \tag{1}$$

The well-known Pell-Lucas and Jacobsthal sequences are recursively defined by

$$\begin{aligned} Q_n &= 2Q_{n-2} + Q_{n-1}, & Q_0 = Q_1 = 2 \\ J_n &= J_{n-1} + 2J_{n-2}, & J_0 = 0, J_1 = 1 \end{aligned}$$

for $n \geq 2$ respectively. The first few values of these sequences are given below:

n	0	1	2	3	4	5	6	7	8	9
Q_n	2	2	6	14	34	82	198	478	1154	2786
J_n	0	1	1	3	5	11	21	43	85	171

It is known that there are a lot of relationships between determinantal representations of matrices and well-known number sequences. For example, the authors [1] derived some relationships between the Fibonacci and Lucas numbers and determinants of matrices. The authors [2] defined two Hessenberg matrices whose determinants are Pell and Perrin numbers. In [3], Lee defined the matrix

$$E_n = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ 0 & 0 & 1 & 1 & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

and showed that

$$per(E_n) = L_{n-1}$$

where L_n is the nth Lucas number. In [4], the authors found $(0, 1, -1)$ tridiagonal matrices whose determinants and permanents are negatively subscripted Fibonacci and Lucas numbers. Let S be a $(1, -1)$ matrix of order n, defined with

$$S = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ -1 & 1 & \ldots & 1 & 1 \\ 1 & -1 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & -1 & 1 \end{bmatrix}. \tag{2}$$

They also give

$$perA = \det(A \circ S) \tag{3}$$

where $A \circ S$ denotes Hadamard product of A and S. In [5], the author investigate general tridiagonal matrix determinants and permanents. Also he showed that the permanent of tridiagonal matrix based on $\{a_i\}, \{b_i\}, \{c_i\}$ is equal to the determinant of matrix based on $\{-a_i\}, \{b_i\}, \{c_i\}$. In [6], the authors gave some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers.

2. Main Theorems

In this section, we define two type upper Hessenberg matrix and show that the permanents of these type matrices are Pell-Lucas and Jacobsthal numbers.

Let $U_n = [u_{ij}]$ be an n-square upper Hessenberg matrix with $u_{11} = u_{21} = 2$ and $u_{(i,i)} = 1$ for $i = 2, 3, \dots, n$ and $u_{(i+1,i)} = 1$ for $i = 3, 4, \dots, n-1$ and $u_{(i,j)} = 2$ for $j - i \geq 1$ and $u_{(i,j)} = 0$ for $i - j \geq 2$. Clearly:

$$U_n := \begin{bmatrix} 2 & 2 & 2 & \dots & \dots & \dots & \dots & 2 & 2 \\ 2 & 1 & 2 & 2 & \dots & \dots & \dots & \dots & 2 \\ 0 & 1 & 1 & 2 & \ddots & & & & \vdots \\ \vdots & \ddots & 1 & 1 & 2 & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & 1 & 1 & 2 & \ddots & \vdots \\ & & & & \ddots & 1 & 1 & 2 & 2 \\ \vdots & & & & & \ddots & 1 & 1 & 2 \\ 0 & \dots & & & & \dots & 0 & 1 & 1 \end{bmatrix}. \tag{4}$$

Theorem 1. *Let U_n be an $n-$ square matrix as in (4), then*

$$perU_n = perU_n^{(n-2)} = Q_n$$

where Q_n is the nth Pell-Lucas number.

Proof. By definition of the matrix U_n it can be contracted on first column. Let $U_n^{(r)}$ be the rth contraction of U_n. If $r = 1$, then

$$U_n^{(1)} := \begin{bmatrix} 6 & 8 & 8 & \dots & \dots & \dots & \dots & 8 & 8 \\ 1 & 1 & 2 & 2 & \dots & \dots & \dots & \dots & 2 \\ 0 & 1 & 1 & 2 & \ddots & & & & \vdots \\ \vdots & \ddots & 1 & 1 & 2 & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & 1 & 1 & 2 & \ddots & \vdots \\ & & & & \ddots & 1 & 1 & 2 & 2 \\ \vdots & & & & & \ddots & 1 & 1 & 2 \\ 0 & \dots & & & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

$U_n^{(1)}$ also can be contracted according to the first column,that is

$$U_n^{(2)} := \begin{bmatrix} 14 & 20 & 20 & \dots & \dots & \dots & \dots & 20 & 20 \\ 1 & 1 & 2 & 2 & \dots & \dots & \dots & \dots & 2 \\ 0 & 1 & 1 & 2 & \ddots & & & & \vdots \\ \vdots & \ddots & 1 & 1 & 2 & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & 1 & 1 & 2 & \ddots & \vdots \\ & & & & \ddots & 1 & 1 & 2 & 2 \\ \vdots & & & & & \ddots & 1 & 1 & 2 \\ 0 & \dots & & & & \dots & 0 & 1 & 1 \end{bmatrix}$$

Going with this process, in rth step we have

$$U_n^{(r)} := \begin{bmatrix} Q_{r+1} & Q_r + Q_{r+1} & Q_r + Q_{r+1} & \dots & \dots & \dots & \dots & Q_r + Q_{r+1} & Q_r + Q_{r+1} \\ 1 & 1 & 2 & 2 & \dots & \dots & \dots & \dots & 2 \\ 0 & 1 & 1 & 2 & \ddots & & & & \vdots \\ \vdots & \ddots & 1 & 1 & 2 & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & 1 & 1 & 2 & \ddots & \vdots \\ & & & & \ddots & 1 & 1 & 2 & 2 \\ \vdots & & & & & \ddots & 1 & 1 & 2 \\ 0 & \dots & & & & \dots & 0 & 1 & 1 \end{bmatrix}$$

where $1 \leq r \leq n-4$.Hence

$$U_n^{(n-3)} = \begin{bmatrix} Q_{n-2} & Q_{n-2} + Q_{n-3} & Q_{n-2} + Q_{n-3} \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

which by contraction of $V_n^{(n-3)}$ on first column,

$$U_n^{(n-2)} := \begin{bmatrix} Q_{n-1} & Q_{n-1} + Q_{n-2} \\ 1 & 1 \end{bmatrix}$$

By (1), we have

$$perU_n = perU_n^{(n-2)} = Q_n.$$

□

Let $V_n = [v_{ij}]_{n\times n}$ be an $n-$square upper Hessenberg matrix in which $v_{11} = 3$,$v_{13} = v_{21} = v_{32} = 1, v_{33} = -1$ and $v_{(i,i)} = 1$ for $i = 4, 5, \ldots, n$ and $v_{(i+1,i)} = 2$ for $i = 3, 4, \ldots, n-1$ and $v_{(i,i+1)} = 1$ for $i = 1, 2, 3, 4, \ldots, n-1$ and otherwise 0.That is:

$$V_n := \begin{bmatrix} 3 & 1 & 1 & 0 & \cdots & & & \cdots & 0 \\ 1 & 0 & 1 & 0 & & & & & \vdots \\ 0 & 1 & -1 & 1 & 0 & & & & \\ \vdots & & 2 & 1 & 1 & 0 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 2 & 1 & 1 & 0 & \vdots \\ & & & & & 2 & 1 & 1 & 0 \\ \vdots & & & & & & 2 & 1 & 1 \\ 0 & \cdots & & & & \cdots & 0 & 2 & 1 \end{bmatrix}$$

Theorem 2. *Let V_n be an $n-$ square matrix as in (5), then*

$$perV_n = perV_n^{(n-2)} = J_n$$

where J_n is the nth Jacobsthal number.

Proof. By definition of the matrix V_n, it can be contracted on first column. Let $V_n^{(r)}$ be the rth contraction of V_n. If $r = 1$, then

$$V_n^{(1)} := \begin{bmatrix} 1 & 4 & 0 & \cdots & & & & \cdots & 0 \\ 1 & -1 & 1 & & & & & & \vdots \\ 0 & 2 & 1 & 1 & & & & & \\ \vdots & & 2 & 1 & 1 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 2 & 1 & 1 & & \vdots \\ & & & & & 2 & 1 & 1 & 0 \\ \vdots & & & & & & 2 & 1 & 1 \\ 0 & \cdots & & & & \cdots & 0 & 2 & 1 \end{bmatrix}$$

Since $V_n^{(1)}$ also can be contracted according to the first column,

$$V_n^{(2)} := \begin{bmatrix} 3 & 1 & 0 & \dots & & & & \dots & 0 \\ 2 & 1 & 1 & & & & & & \vdots \\ 0 & 2 & 1 & 1 & & & & & \\ \vdots & & 2 & 1 & 1 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 2 & 1 & 1 & & \vdots \\ & & & & & 2 & 1 & 1 & 0 \\ \vdots & & & & & & 2 & 1 & 1 \\ 0 & \dots & & & & \dots & 0 & 2 & 1 \end{bmatrix}$$

Continuing this method, we obtain the rth contraction

$$V_n^{(r)} := \begin{bmatrix} J_{r+1} & J_r & 0 & \dots & & & & \dots & 0 \\ 2 & 1 & 1 & & & & & & \vdots \\ 0 & 2 & 1 & 1 & & & & & \\ \vdots & & 2 & 1 & 1 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 2 & 1 & 1 & & \vdots \\ & & & & & 2 & 1 & 1 & 0 \\ \vdots & & & & & & 2 & 1 & 1 \\ 0 & \dots & & & & \dots & 0 & 2 & 1 \end{bmatrix}$$

where $2 \leq r \leq n-4$. Hence

$$V_n^{(n-3)} := \begin{bmatrix} J_{n-2} & J_{n-3} & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

which by contraction of $V_n^{(n-3)}$ on first column,

$$V_n^{(n-2)} := \begin{bmatrix} J_{n-1} & J_{n-2} \\ 2 & 1 \end{bmatrix}$$

By (1), we have

$$perV_n = perV_n^{(n-2)} = J_n.$$

□

Corollary 3. *Let us consider (3) and define* $\overline{U}_n = U_n \circ S$ *and* $\overline{V}_n = V \circ S$*.Then*

$$det\overline{U}_n = perU_n = Q_n$$
$$\det \overline{V}_n = perV_n = J_n..$$

3. Acknowledgments

The author is very grateful to an anonymous referee for helpful comments and suggestions to improve the presentation of this paper.

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