International Journal of Pure and Applied Mathematics
Volume 101 No. 3 2015, 433-443
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v101i3.12
ijpam.eu

Hom$_*$ COMMUTING WITH FILTERED PRODUCTS

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Abstract: In a sufficiently rich category, such as a category of R-modules, given an infinite cardinal κ, we examine classes $\mathcal{H}_*^\kappa$ of objects M, such that the following natural monomorphism is an isomorphism:

$$\prod_{i\in I}^{\kappa} \mathrm{Hom}\,(M, A_i) \cong \mathrm{Hom}\,(M, \prod_{i\in I}^{\kappa} A_i),$$

for every family of objects $\{A_i : i \in I\}$ ($\prod^\kappa$ denotes the subproduct of all vectors with support $< \kappa$).

AMS Subject Classification: 16D80, 16D90, 16E30, 16Z99, 18B99, 18E99
Key Words: Hom commuting, filtered products, chains of submodules, regular cardinals

1. Preliminaries

We will assume the axiom of choice or equivalently that every set may be well ordered; one consequence is the existence of arbitrary infinite products in the category of sets. Furthermore we will assume that the categories we work with have arbitrary products and coproducts. We will identify any cardinal κ with the smallest (initial) ordinal of that cardinality, when it is convenient to do so. This is equivalent to the statement that every cardinal is of the form

Received: February 16, 2015

$\aleph_\alpha$, for some ordinal number α; we will use 'fin' to denote a finite cardinal. Thus, an arbitrary non-empty index set I may be assumed to be well ordered by an (initial) ordinal; if needed, we will assume ordinals to be regular, i.e. that $\mathrm{cf}\, I{=}I$. A cardinal κ is regular, if it is not singular, i.e., if it cannot be represented as $\sup\{\alpha_i : i < \theta\}$, where each $\alpha_i < \kappa$ and $\theta < \kappa$ is a limit ordinal. Equivalently, κ is regular iff it cannot be represented as the sum of less than κ smaller cardinals. An ordinal α is a successor ordinal, if $\alpha = \beta + 1$, for some ordinal β; otherwise, α is a limit ordinal. $|I|^+$ will denote the successor cardinal to cardinal $|I|$. An infinite cardinal $\aleph_\alpha$ is a successor cardinal, if it is of the form $\aleph_{\beta+1} = \aleph_\alpha$; if α is a limit ordinal, then $\aleph_\alpha$ is called a limit cardinal. Every successor ordinal is regular, but it is not always the case with limit ordinals, which may be singular. In this study, we are mostly interested in regular ordinals (cardinals).

If $a = (a_i)_{i\in I} \in \prod_{i\in I} A_i$, we will also write a as a formal sum $\sum_{i\in I} a_i$ which is, more precisely, the formal sum $\sum_{i\in I} p_i a_i$, where $p_i : A_i \longrightarrow \prod A_i$ are the natural product injections. In absence of a topology/metric, the sum will "make sense" when there are only totality of finitely many non-zero coordinates, at every coordinate of all I-vectors being summed. This will always be the case if we are doing genuine summation, not just the formal one.

In essence, our note concerns the category of unital (one-sided) R-modules, but we are using the language of general categories to indicate that the results and the proofs carry over to this more general setting, mutatis mutandis.

For an arbitrary family $\{A_i, i \in I\}$ of (non-zero) objects, and an arbitrary infinite cardinal κ, $\prod_{i\in I}^{\kappa} A_i$ will denote the filtered κ-product, namely it consists of all the vectors with support $< \kappa$. The natural κ-product-to-product (κptp) embedding will be denoted by u_κ or $u : \prod_{i\in I}^{\kappa} A_i \longrightarrow \prod_{i\in I} A_i$.

For $\kappa \geq |I|^+$ ($\kappa = \aleph_0$) we have respectively

$$\prod_{i\in I}^{\kappa} A_i = \prod_{i\in I} A_i \qquad \left(\prod_{i\in I}^{\kappa} A_i = \coprod_{i\in I} A_i\right).$$

Fact. *For every object M, we have a natural monomorphism*

$$\phi : \prod_{i\in I}^{\kappa} \mathrm{Hom}\,(M, A_i) \longrightarrow \mathrm{Hom}\,(M, \prod_{i\in I}^{\kappa} A_i) \qquad (*)$$

given by $(f_i : M \longrightarrow A_i)_{i\in I} \mapsto f = \sum_{i\in I} p_i f_i : M \longrightarrow \prod_{i\in I}^{\kappa} A_i$ (with coordinates $\pi_i f = f_i$), $i \in I$. This monomorphism is an isomorphism in case $\kappa \geq |I|^+$, as well as when I is a finite index set, regardless of κ, thus, we can assume in the sequel, when needed, without loss of generality, that $\aleph_0 \leq \kappa \leq |I|$. When

$|I|$ is an infinite cardinal and $\kappa \leq |I|$, then we do not necessarily have an isomorphism. For instance, if $\kappa = \aleph_0$, (*) is not an isomorphism, for every M and every family of non-zero modules A_i, even when all $A_i = R$. Thus we have the following

Task 1. Investigate, for various infinite cardinals κ, and if possible characterize, objects M such that for every infinite index set I, every family of (non-zero) objects $\{A_i, i \in I\}$ monomorphism (*) is an isomorphism. Call every such object M a *Hom$_*$-κ-commuting object*. Given an infinite cardinal κ, denote by $\mathcal{H}_*^\kappa$ the class of all Hom$_*$-κ-commuting objects. For $\kappa = \aleph_0$ the Hom$_*$-κ-commuting object was introduced by Mitchell (no later than 1965, [2]), under the name *small object* (see e.g [3], where small objects go under the name of Σ-*type* object (or Σ-*generated*) object).

2. Equivalent Definitions

While this is an ambitious task, we show in the sequel how to arrive at a number of illuminating results. We begin with the following:

Lemma 1. *Given an infinite cardinal κ, an additive category $\mathcal{C}$ with arbitrary coproducts and products, an arbitrary non-empty index set I and a morphism $f : M \longrightarrow \prod_{i\in I}^\kappa A_i$, the following statements are equivalent:*

1. *The morphism f factors through a κ'- subproduct, $\kappa' < \kappa$, namely there is a subset $J \subseteq I$, $|J| = \kappa' < \kappa$ and a factorization $M \xrightarrow{f'} \prod_{i\in J}^\kappa A_i \xrightarrow{p_J} \prod_{i\in I}^\kappa A_i$, $f = p_J f'$ (here p_J is the natural embedding of smaller κ-product into the larger; note that $\prod_{i\in J}^\kappa A_i = \prod_{i\in J} A_i$, since $|J| < \kappa$).*
2. *$f = \sum_{i\in J} p_i f_i$, for some $J \subseteq I$, $|J| < \kappa$ and for some morphisms $f_i : M \longrightarrow A_i$, $i \in J$.*
3. *If, for some $J \subseteq I, |J| < \kappa$, $q_{I\setminus J} : \prod_{i\in I}^\kappa A_i \longrightarrow \prod_{i\in I\setminus J}^\kappa A_i$ is the canonical quotient map $= \pi_{I\setminus J}| \prod_{i\in I}^\kappa A_i$, then $q_{I\setminus J} f = 0$.*
4. *There is a subset $J \subseteq I$, $|J| < \kappa$ such that $\pi_{I\setminus J} u f = 0$.*

Proof. As before, denote by π'_i and p'_i the natural product projections and injections associated with the product indexed by J, and likewise by u' the corresponding κptp-morphism. Note that
(a) $\forall i \in J\; p_J p'_i = p_i$ and (b) $\forall i \in J\;\; \pi'_i u' = \pi_i u p_J$.

The proof is as follows:

(1)$\Rightarrow$ (2): $f = p_J f' = p_J(\sum_{i\in J} p'_i \pi'_i u')f' =$ (by (a))$=$
$= \sum_{i\in J} p_i \pi'_i u' f' =$ (by (b)) $=$
$= \sum_{i\in J} p_i \pi_i u p_J f' = \sum_{i\in J} p_i \pi_i u f$; denote $f_i = \pi_i u f$, for all $i \in J$.

(2)$\Rightarrow$(3): $q_{I\setminus J} f = q_{I\setminus J}(\sum_{i\in J} p_i f_i) = \sum_{i\in J} q_{I\setminus J} p_i f_i = 0$.

(3)$\Rightarrow$(4): If u'' denotes the κptp map associated with the κproduct on the index set $I \setminus J$, then the proof is established by noting that $u'' q_{I\setminus J} = \pi_{I\setminus J} u$ and thus $\pi_{I\setminus J} u f = u'' q_{I\setminus J} f = 0$. In fact the same observation proves the reverse implication.

(4)$\Rightarrow$(1): Equality $\pi_{I\setminus J} u f = 0$ ensures $\operatorname{Im} f \subseteq \operatorname{Ker}(\pi_{I\setminus J} u) \subseteq \prod^{\kappa}_{i\in J} A_i$ and this in turn ensures validity of (1).

We extend this result as follows, by not assuming a priori that index sets are the same or that the components A_i are the same, in each of the equivalent statements:

Proposition 2. *Given an additive category $\mathcal{C}$ with arbitrary (κ-products and) coproducts and an object M, the following are equivalent:*

1. *For every non-empty index set I and an arbitrary family of objects $\{A_i : i \in I\}$ in $\mathcal{C}$, every morphism $f : M \longrightarrow \prod^{\kappa}_{i\in I} A_i$ in $\mathcal{C}$ factors through a κ'-subproduct, $\kappa' < \kappa$, namely there is a subset $J \subseteq I$, $|J| = \kappa' < \kappa$ and a factorization $M \xrightarrow{f'} \prod^{\kappa}_{i\in J} A_i \xrightarrow{p_J} \prod^{\kappa}_{i\in I} A_i$, with $f = p_J f'$.*

2. *For every non-empty index set I and an arbitrary family of objects $\{A_i : i \in I\}$ in $\mathcal{C}$ and every morphism $f : M \longrightarrow \prod^{\kappa}_{i\in I} A_i$ in $\mathcal{C}$, there is a $J \subseteq I$, $|J| < \kappa$, such that $f = \sum_{i\in J} p_i f_i$, for some morphisms $f_i : M \longrightarrow A_i$, $i \in J$.*

3. *For every non-empty index set I and an arbitrary family of objects $\{A_i : i \in I\}$ in $\mathcal{C}$ and every morphism $f : M \longrightarrow \prod^{\kappa}_{i\in I} A_i$ in $\mathcal{C}$ there is a $J \subseteq I$, $|J| < \kappa$, such that $q_{I\setminus J} f = 0$, i.e. $\operatorname{Im} f \subseteq \prod^{\kappa}_{i\in J} A_i$.*

4. *The functor $\operatorname{Hom}_{\mathcal{C}}(M, -)$ commutes with κ-products, i.e., for every non-empty index set I and an arbitrary family of objects $\{A_i : i \in I\}$*

$$\phi : \prod_{i\in I}^{\kappa} \operatorname{Hom}(M, A_i) \longrightarrow \operatorname{Hom}(M, \prod_{i\in I}^{\kappa} A_i) \qquad (*)$$

via the natural isomorphism of Abelian groups
$\phi : (f_i)_{i\in I} \mapsto \sum_{i\in I} p_i f_i$.

Proof. Lemma 1 establishes equivalence of the first three statements, since, a posteriori, it turns out that the index set I and the product components A_i may be the same in each of the equivalent statements.
(2)⇒(4): Given a morphism $f : M \longrightarrow \prod_{i\in I}^{\kappa} A_i$ in $\mathcal{C}$, there is a $J \subseteq I$, $|J| < \kappa$ such that $f = \sum_{i\in J} p_i f_i$, for some morphisms $f_i : M \longrightarrow A_i$. Note now that $h = (f_{i0})_{i\in I} \in \prod_{i\in I}^{\kappa} \mathrm{Hom}_{\mathcal{C}}(M, A_i)$ with $f_{i0} = f_i$, for $i \in J$ and $f_{i0} = 0$ for $i \in I \setminus J$ is such that $\phi(h) = f$.
(4)⇒(2): Given a morphism $f : M \longrightarrow \prod_{i\in I}^{\kappa} A_i$ in $\mathcal{C}$, there is an $h = (f_i)_{i\in I} \in \prod_{i\in I}^{\kappa} \mathrm{Hom}_{\mathcal{C}}(M, A_i)$ where, for some $J \subseteq I$, $|J| < \kappa$, $f_i = 0$ whenever $i \in I \setminus J$ and $\phi(h) = f$, i.e. $\sum_{i\in J} p_i f_i = f$. □

Finally, we have the following series of equivalent properties that could be used to define Hom$_*$-κ-commuting objects:

Theorem 3. *In an additive category with infinite products and coproducts, given an infinite (regular) cardinal κ, the following are equivalent, for an object M:*

1. *for every infinite set I, and every family of objects $\{A_i : i \in I\}$, the natural monomorphism defined in (*) is an isomorphism:*
$$\prod_{i\in I}^{\kappa} \mathrm{Hom}\,(M, A_i) \cong \mathrm{Hom}\,(M, \prod_{i\in I}^{\kappa} A_i) \qquad (*I)$$
(arbitrary Hom definition);

2. *M is a Hom$_*$-κ-commuting with families of cardinality κ, i.e., for every index set J with $|J| = \kappa$, and, for an arbitrary family of objects $\{A_i : i \in J\}$,*
$$\prod_{i\in J}^{\kappa} \mathrm{Hom}\,(M, A_i) \cong \mathrm{Hom}\,(M, \prod_{i\in J}^{\kappa} A_i) \qquad (*\kappa)$$
via the natural isomorphism of Abelian groups
(κ-Hom definition);

3. *for an arbitrary non-empty index set I, for every family of objects $\{A_i : i \in I\}$, and every morphism $f : M \longrightarrow \prod_{i\in I}^{\kappa} A_i$, there is a $J' \subseteq I$ with $|J'| < \kappa$, such that for all $i \in I \setminus J'$, $\pi_i u f = 0$*
(arbitrary coordinatewise definition);

4. *for $|J| = \kappa$, every family $\{A_i, i \in J\}$ of objects, and every morphism $f : M \longrightarrow \prod_{i\in J}^{\kappa} A_i$, $\pi_i u_J f = 0$, for all $i \in J \setminus J'$, for some $J' \subseteq J$, $|J'| < \kappa$.*
(κ-coordinatewise definition);

5. *for every infinite index set I, for every family $\{A_i : i \in I\}$ of objects, and every morphism $f : M \longrightarrow \prod_{i\in I}^{\kappa} A_i$, there is a well-ordering on I such that, there is an $i_0 \in I, i_0 < \kappa$ with $\pi_{i>i_0} f = 0$ (arbitrary tailwise definition);*

6. *for every family $\{A_i\}_{i\in J}$ of objects, with $|J| = \kappa$ (J well-ordered), and every morphism $f : M \longrightarrow \prod_{i\in J}^{\kappa} A_i$, there is an $i_0 \in J$ with $i_0 < \kappa$ and $\pi_{i>i_0} f = 0$*
(κ-tailwise definition).

(5) and (6) require regularity of κ.

Proof. Note that the index sets in (1), (3) and (5) are arbitrary, unlike (2), (4), (6) where index sets are of cardinality κ. Thus, each of (1),(3),(5) implies respectively (2),(4),(6). Equivalence of (1), (3) and (5) (arbitrary index sets) and equivalence of (2), (4) and (6) follow from Lemma 1 and Proposition 2. We only need to prove that one of the even numbered statements implies any of the odd numbered ones, to complete the proof of equivalence of all the statements. First, we show that regularity of κ is needed when working with tailwise definitions.

(3)⇒ (5): Assume that $f : M \longrightarrow \prod_{i\in I}^{\kappa} A_i$ is an arbitrary morphism. By (3), there is a $J' \subseteq I$ with $|J'| < \kappa$, such that for all $i \in I \setminus J'$, $\pi_i u f = 0$. We can well order I in such a way that $|J'|$ is its initial segment (of cardinality $< \kappa$). Because, κ is regular, there is an $i_0 < \kappa$ in I with $\pi_i u f = 0$, for all $i > i_0$, which is the same as $\pi_{i>i_0} f = 0$. This completes the proof of all the equivalences.

(4)⇒ (6): The proof is the same, mutatis mutandis, as for (3)⇒ (5).

(2)⇒(4): Let $f : M \longrightarrow \prod_{i\in J}^{\kappa} A_i$ be an arbitrary morphism in $\mathcal{C}$. By the assumption, isomorphism ($*\kappa$) holds, hence we can find morphisms $f_i : M \longrightarrow A_i$, with $\text{supp}(f_i)_{i\in J} < \kappa$ i.e. there is a $J' \subseteq J$ with $|J'| < \kappa$ with $f_i = 0$, for all $i \in J \setminus J'$ and such that $f = \sum_{i\in J} p_i f_i$ (summation of $< \kappa$ non-zero summands, indexed by J'). Then $\forall\, i \in J \setminus J'$, $\pi_i u_J f = \pi_i \sum_{i\in J} u_J p_i f_i = \pi_i(\sum_{i\in J\setminus J'} u p_i f_i + \sum_{i\in J'} u p_i f_i) = 0$.

(4)⇒ (3): It is only non-trivial to consider cases when $|I| > \kappa$. If, on the contrary, there is a $J \subseteq I$ with $|J| = \kappa$, such that $\forall i \in J$, $\pi_i u f \neq 0$, then we consider the cut of f: $g = \pi_J f : M \longrightarrow \prod_{i\in J}^{\kappa} A_i$. By (4), there exists a $J' \subseteq J$ with $|J'| < \kappa$ and $\forall i \in J \setminus J'$, $\pi_i u g \neq 0$, which would be a contradiction. □

3. Examples, a Characterization and Constructions

gen M will denote the cardinality of a minimal set of generators of M (and sometimes such a set of generators itself).[1]

Proposition 4. *Let $\kappa \leq |I|$ be an infinite cardinal and let M be an R-module with gen $M < \kappa$. Then the natural monomorphism*

$$\phi : \prod_{i\in I}^{\kappa} \mathrm{Hom}\,(M, A_i) \longrightarrow \mathrm{Hom}\,(M, \prod_{i\in I}^{\kappa} A_i) \qquad (*)$$

is an isomorphism, for every family of objects $A_i, i \in I$, namely M is a Hom$_*$-κ-commuting object. Consequently, if gen $M < \kappa$, then $M \in \mathcal{H}_*^{\kappa'}$, for every $\kappa' \geq \kappa$. In particular, every finitely generated R-module M is Hom$_* - \kappa$-commuting, for every infinite κ.

Proof. The reason for surjectivity is that $\forall f : M \longrightarrow \prod_{i\in I}^{\kappa} A_i$, the image *gen* $\mathrm{Im}\, f \leq$ *gen* $M < \kappa$, and we may assume *gen* $f(M) = \{a^j = (a_i^j)_{i\in I}, j \in J\}$ with $|J| < \kappa$ and support of every $a^j < \kappa$. For every $i \in I$, we define $f_i : M \longrightarrow A_i$ as follows: For $m \in M$, let $f(m) = \sum_{j\in J_m} r_j a^j =$ (each $|J_m| \leq |J|$ is finite) $\sum_{j\in J_m} r_j (a_i^j)_{i\in I} = \left(\sum_{j\in J_m} r_j a_i^j\right)_{i\in I}$ (finite sums); define $\forall i \in I$ $f_i(m) = \sum_{j\in J_m} r_j a_i^j$. Supp $(f_i)_{i\in I} < \kappa$ since $\forall m \in M$ supp $f(m) < \kappa$ and $|J| < \kappa$. Clearly, by definition, $\phi(f_i)_{i\in I} = f$, which proves surjectivity. ☐

When $\kappa = \aleph_0$, we get the fact that, if M is a finitely generated object, then $\mathrm{Hom}\,(M, -)$ commutes with countable coproducts.

Task 2. Find and characterize categories (rings R) such that the only Hom$_*$-κ-commuting modules are those that are $< \kappa$-generated, as well as those where there are Hom$_*$-κ-commuting modules generated by at least κ elements.

Proposition 5. *Let κ be any infinite limit cardinal and $M \in R$***Mod** *be such that* **no** *ascending (smooth) κ-chain of proper submodules of M like $M_0 < M_1 < \cdots < M_\alpha < \cdots < M$, $\alpha < \rho \leq \kappa$, fills the whole of M, i.e. $\cup_{\alpha<\rho} M_\alpha = \sum M_\alpha \neq M$. Then M is Hom$_*$-κ-commuting .*

Proof. By Theorem 3, it is sufficient to prove this for index sets I with $\kappa = |I|$; we well order I so that I represents the smallest ordinal of cardinality κ. For every $\alpha < \kappa$ we will denote by $\Pi_\alpha = \prod_{i\in I}^{\alpha\kappa} A_i$ the truncated κ-product that consists of elements $(a_i)_{i\in I} \in \Pi^\kappa = \prod_{i\in I}^{\kappa} A_i$ with $a_i = 0$, for all $i > \alpha$. Note

[1] This practical notation was first introduced in [1] and it has since been adopted in a number of cases, by other authors, without reference.

that $\{\Pi_\alpha\}_{\alpha<\kappa}$ is a smooth κ-chain with union Π^κ. Let $f \in \mathrm{Hom}\,(M, \prod_{i<\kappa}^{\kappa} A_i)$; denote by $M_\alpha = f^{-1}(\Pi_\alpha)$ (the inverse image of the α-truncated k-product of A_i's). Since $\{\Pi_\alpha\}_{\alpha<\kappa}$ is a smooth κ-chain uniting in Π^κ, $\{M_\alpha\}_{\alpha<\kappa}$ is likewise a smooth κ-chain uniting in $M = f^{-1}(\Pi^\kappa)$. By the assumption, this may happen only if not all the links are proper subobjects of M, i.e., if there exists an $\alpha < \kappa$ with $M_\alpha = f^{-1}(\Pi_\alpha) = M$. This means that $f : M \longrightarrow \Pi_\alpha = \prod_{i\in I}^{\alpha\kappa} A_i$. But $\prod_{i\in I}^{\alpha\kappa} A_i \cong \prod_{i\leq\alpha} A_i$ and, in this case $f = \Sigma_{i<\alpha} p_i f_i$, for some $f_i : M \longrightarrow A_i$, which proves the statement. Note that the same proof goes if it is done by transfinite induction on κ. □

Proposition 6. *For an arbitrary infinite cardinal κ, let M be Hom$_*$-κ-commuting. Then: No strictly ascending (smooth) ρ-chain $\rho \leq \kappa$ of proper submodules of M like $M_0 < M_1 < \cdots < M_\alpha < \cdots < M$, $\alpha < \rho \leq \kappa$ fills the whole of M, i.e. $\cup_{\alpha<\rho} M_\alpha = \sum M_\alpha \neq M$.*

Proof. (here $\rho = \kappa$) Assume that M is $\mathrm{Hom}_* - \kappa$-commuting and suppose, that on the contrary, for some such chain, we have $\cup_{i<\kappa} M_i = \sum M_i = M$. Define $f(x) = (x + M_i)_{i<\kappa}$ and hope that f is a morphism $f : M \longrightarrow \prod_{i\in\kappa}^{\kappa} M/M_i$. For every $x \in M$, there is the smallest $i_x < \kappa$ with $x \in M_{i_x}$ and then, $\forall i' \geq i_x,\ x \in M_{i'}$ which implies that $|\mathrm{supp} f(x)| \leq |i_x| < \kappa$, thus, indeed $f \in \mathrm{Hom}\,(M, \prod_{i\in\kappa}^{\kappa} M/M_i)$. Then, by the assumption, $f = (f_i)_{i<\kappa}$, for some $(f_i)_{i<\kappa} \in \prod_{i<\kappa}^{\kappa} \mathrm{Hom}\,(M, M/M_i)$. This means that there exists a $\kappa_1 < \kappa$, such that $\forall i > \kappa_1 \ \ f_i = 0$. This means that $\forall i > \kappa_1 \ \ \forall x \in M$, $x \in M_i$. This is a contradiction, since it is assumed that all M_i are proper submodules of M. Hence, our claim holds. □

Corollary 7. *Let κ be an infinite cardinal and $M \in R$***Mod** *such that gen $M = \kappa$. Then,*

1. *If κ is a limit cardinal, M is not Hom$_*$-κ-commuting .*
2. *Furthermore, if κ is a limit cardinal, gen $M \leq \kappa$ and M is Hom$_* - \kappa$-commuting, then gen $M < \kappa$.*

Proof. (1) The reason is as follows: As usual, κ denotes the initial ordinal representing cardinal κ. If gen $M = \{m_\alpha : \alpha < \kappa\}$, denote $M_\alpha = \langle\{m_i : i \leq \alpha\}\rangle$. Then $M_\alpha, \alpha < \kappa$ is an ascending κ-chain of proper subobjects of M, with $\cup_{\alpha<\kappa} M_\alpha = M$, because κ is a limit cardinal. By Proposition 6, this would be impossible, if M were to be Hom$_*$-κ-commuting .

(2) This is a reformulation of (1). □

In any Abelian category with products and coproducts, we have the following:

Proposition 8. *Let κ be an infinite (regular) cardinal. Then:*

1. *The 0 object is Hom$_*$-κ-commuting , thus the class of Hom$_*$-κ-commuting objects in any category with the zero object is non-empty. Moreover all the $<\kappa$-generated objects are in $\mathcal{H}_*^\kappa$.*

2. *An object is in $\mathcal{H}_*^\kappa$ iff all its quotient objects are in $\mathcal{H}_*^\kappa$.*

3. *If $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is an exact sequence and A, C are in $\mathcal{H}_*^\kappa$ objects, then B is likewise in $\mathcal{H}_*^\kappa$.*

4. *For an (infinite) index set I, and an arbitrary family of objects $\{B_i : i \in I\}$, the κ-product $M = \prod_{i\in I}^\kappa B_i$ is in $\mathcal{H}_*^\kappa$, iff only finitely many $B_i \neq 0$.*

5. *Given a finite ascending sequence $0 = A_0 < A_1 < \cdots < A_n = C$ of subobjects of an object C, such that all the factors A_{i+1}/A_i, $i = 0, 1, \ldots, n-1$ are in $\mathcal{H}_*^\kappa$, then C is likewise in $\mathcal{H}_*^\kappa$. The claim is no longer true, if the length of chain is infinite (such as of countable cofinality).*

6. *Let $F : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ be an equivalence of two categories with κ-products and $M \in \mathcal{A}_1$. Then M is Hom$_*$-κ-commuting in $\mathcal{A}_1$ if and only if $F(M)$ is Hom$_*$-κ-commuting in $\mathcal{A}_2$.*

Proof. (1) is trivial, in view of Proposition 4.

(2) holds, because a morphism $f : M/D \longrightarrow \prod_{i\in I}^\kappa A_i$ gives rise to a morphism $fq : M \longrightarrow \prod_{i\in I}^\kappa A_i$ (q is the canonical quotient map). If M is Hom$_*$-κ-commuting then, by Proposition 2, fq is expressible as a sum $\sum_{i\in J} p_i f_i$, for some $|J| < \kappa$ and some morphisms $f_i : M \longrightarrow A_i$. Given $\overline{x} \in M/D$, define, $\forall i \in I$, $\overline{f}_i : M/D \longrightarrow A_i$ by $\overline{f}_i(\overline{x}) = f_i(x)$. $\overline{f}_i$ are morphisms since f_i and q are; moreover every $\overline{f}_i$ is well-defined, for if $\overline{x} = \overline{x}'$, then $f(\overline{x}) = fq(x) = fq(x') = f(\overline{x}')$ hence $\sum p_i f_i(x) = \sum p_i f_i(x')$ and thus $\sum p_i f_i(x - x') = 0$, $i \in J$. The latter is in $\prod_{i\in I}^\kappa A_i$, since $|J| < \kappa$ thus every $p_i f_i(x - x') = 0$ and this is possible only if for every $i \in I$, $f_i(x) = f_i(x')$. We can now represent $f = \sum p_i \overline{f}_i$ as a J-sum which, by Proposition 2, means that the quotient M/D is Hom$_*$-κ-commuting . The other implication is trivial (once we take $D = 0$).

(3) (Draw the commutative diagram to follow the argument easier). Consider an arbitrary morphism $f : B \longrightarrow \prod_{i\in I}^\kappa A_i \in \mathcal{C}$. Then $f\alpha : A \longrightarrow \prod_{i\in I}^\kappa A_i$ and A is assumed to be in $\mathcal{H}_*^\kappa$, hence there is a $J \subseteq I$, $|J| < |I|$ such that

$q_{I\setminus J} f\alpha = 0$, by Proposition 2(3). Denote $\chi = q_{I\setminus J} f$. By the universal property of the quotient (cokernel) construction, there is a unique $\gamma : C \longrightarrow \prod^{\kappa}_{i\in I\setminus J} A_i$ such that $\gamma\beta = \chi$. However C is assumed to be in $\mathcal{H}^{\kappa}_{*}$, hence, there is a $J' \subseteq I\setminus J$, $|J'| < \kappa$ with $q'_{I\setminus J\setminus J'}\gamma = 0$, where $q'_{I\setminus J\setminus J'} : \prod^{\kappa}_{i\in I\setminus J} A_i \longrightarrow \prod^{\kappa}_{i\in I\setminus J\setminus J'} A_i$ is the map corresponding to γ, via Proposition 2(3). This implies $0 = q'_{I\setminus J\setminus J'}\gamma\beta = q'_{I\setminus J\setminus J'} q_{I\setminus J} f = q_{I\setminus (J\cup J')} f$. Since both $|J|, |J'| < \kappa$, so is their union and $q_{I\setminus (J\cup J')} f = 0$, which, by Proposition 2(3) establishes the fact that B is in $\mathcal{H}^{\kappa}_{*}$.

(4) If only finitely many components are $\neq 0$, M is clearly Hom$_*$-κ-commuting , by way of canonical isomorphisms for finite products/coproducts. Assume for a moment that there are infinitely many $B_i \neq 0$ (say countably many). Then $M_{\mathbb{N}} = \prod^{\kappa}_{i\in\mathbb{N}} B_i$ is the ascending union of its proper subobjects $M_n = \prod^{\kappa}_{i\in n} B_i$, $n \in \mathbb{N}$. By Proposition 6 this means that $\prod^{\kappa}_{i\in\mathbb{N}} B_i$ is not Hom$_*$-κ-commuting . But then $\prod^{\kappa}_{i\in I} B_i$ cannot be Hom$_*$-κ-commuting , for otherwise, its quotient $M_{\mathbb{N}}$ would have to be such as well, but this is impossible by (2) of this proposition.

(5) The first claim follows from statements (0-3) of this proposition; the second claim follows from the fact that (say for a countable index set) a countable direct sum of Hom$_*$-κ-commuting objects is the union of an ascending chain of its proper subobjects (and by Proposition 6, not Hom$_*$-κ-commuting),

(6) This statement is fully categorical and is straightforward.

Our present effort concentrates on

Task 3. Give fairly detailed account of how the classes $\mathcal{H}^{\kappa}_{*}$ and $\mathcal{H}^{\kappa'}_{*}$ relate, for different cardinals κ and κ'.

Remarks

The first version of this paper, dated February 11, 2007, was communicated to C. U. Jensen. Its content has been presented at several conferences since then.

References

[1] Radoslav Dimitric, On pure submodules of free modules and κ-free modules, *CISM Courses and Lectures*, No. **287**, *Abelian Groups and Modules*, 373-381, Springer-Verlag, New York (1984).

[2] Barry Mitchell, *Theory of Categories* Academic Press, New York (1965).

[3] Rudolf M. Rentschler, Sur les modules M tels que Hom $(M, -)$ commute avec les sommes directes, *C.R. Acad. Sc. Paris, Série A*, **268** (1969), 930-933.