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MODULAR, PRIMITIVE PROPERTIES AND CHAIN CONDITIONS IN RIGHT TERNARY NEAR-RINGS

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Abstract: A study on Right Ternary Near-Ring (RTNR) paves a way for a deeper understanding of its binary counterpart. In this paper the necessary and sufficient for an RTNR N to have chain conditions is given. The existence of a right unital element in a monogenic N-subgroup of a zero-symmetric RTNR N with Descending Chain Condition (DCC) on left N-subgroups is established. Modular ideals are introduced in this generalised setting and it is proved that the intersection of two modular ideals L_1 and L_2 is a modular ideal if the zero-symmetric distributive RTNR N is the sum of L_1 and L_2. If N is a right and lateral ternary near-ring then for $\nu \in \{0, 1, 2\}$, ν-modular left ideals and ν-primitivity are defined. The interrelationship between the different types of ν-primitive ideals is also discussed.

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1. Introduction

Near-rings are algebraic structures to study about non-linear functions on finite

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groups. With the help of n-ary generalization of algebraic structures their basic properties can deeply be understood.

The authors introduced right ternary near-rings in [7] and in [3] right ternary N-groups were studied. In this paper Descending Chain Condition (DCC) on N-subgroups, ideals are defined and a criterion for N to have DCC is given in terms of its direct summand I and quotient RTNR N/I. If a zero-symmetric RTNR N has DCC on its left N-subgroups then as given in [2], in this generalized setting it is proved that the monogenic N-subgroup of N has a right unital element. The modular ideals are defined and if N is a distributive zero-symmetric RTNR and $N = L_1 + L_2$ where L_1 and L_2 are modular ideals then $L_1 \cap L_2$ is proved to be a modular ideal. If N is a right and lateral ternary near-ring then for $\nu \in \{0, 1, 2\}$, ν-modular left ideals and ν-primitivity are defined.

2. Preliminaries

In this section the basic definitions and results that are needed for the rest of the sections are given.

Definition 1. [7]

(a) Let N be a non-empty set together with a binary operation $+$ and a ternary operation $[\ ] : N \times N \times N \to N$. Then $(N, +, [\ ])$ is a right ternary near-ring(RTNR) if
(RTNR-1) $(N, +)$ is a group.
(RTNR-2) $[[x\ y\ z]\ u\ v] = [x\ [y\ z\ u]\ v] = [x\ y\ [z\ u\ v]] = [x\ y\ z\ u\ v]$ for every $x, y, z, u, v \in N$.
(RTNR-3) $[(x + y)\ z\ w] = [x\ z\ w] + [y\ z\ w]$ for every $x, y, z, w \in N$.

Similarly left ternary near-ring and lateral ternary near ring can be defined.

(b) Let N and N' be RTNRs. Then a mapping $h : N \to N'$ is called an RTNR homomorphism if (a) $h(m+n) = h(m)+h(n)$ and (ii) $h([m\ n\ r]) = [h(m)\ h(n)\ h(r)]$ for every $m, n, r \in N$.

(c) Let N be a right ternary near-ring. Let I be a normal subgroup $(N, +)$. Then I is called (i) a right ideal of N if $[INN] \subseteq I$ (ii) a left ideal if $[t\ t'\ (t''+i)] - [t\ t'\ t''] \in I$ (iii) a lateral ideal if $[t\ (t'+i)\ t''] - [t\ t't''] \in I$ for every $t, t', t'' \in N$, $i \in I$. I is called a two-sided ideal if it is a left and right ideal of N and I is an ideal of N if it is a left, right and lateral ideal of N.

Definition 2. [6] A non-empty subset H of N is called an N-subgroup of N if (i) H is a subgroup of $(R,+)$ (ii) $[NNH] \subseteq H$ (iii) $[NHN] \subseteq H$ (iv) $[HNN] \subseteq H$. If (i) and (ii) hold then H is called a left N-subgroup. If (i) and (iii) hold then H is called a lateral N-subgroup. If (i) and (iv) hold then H is called a right N-subgroup.

Definition 3. [4]

(i) If N is an RTNR then an element $e \in N$ is a left (resp. right, lateral) unital element if $[e\ e\ x] = x$ (resp. $[x\ e\ e] = x$, $[e\ x\ e] = x$) for every $x \in N$. If $[e\ e\ x] = x = [x\ e\ e]$ then e is called a bi-unital element.

(ii) For all $n \in N$, $(0 : n) = \{x \in N | [xsn] = 0\ \forall\ s \in N\}$ is a two-sided ideal of N.

(iii) If I is an ideal of an RTNR N then $(N/I, +, [\ ])$ is called a quotient RTNR.

Definition 4. [3] Let $(N, +, [\ ])$ be an RTNR and $(\Gamma, +)$ be a group with additive identity o. Then Γ is said to be a right ternary N-group if there exists a mapping $[\ ]_\Gamma : N \times N \times \Gamma \to \Gamma$ satisfying the conditions
(i) $[n + m\ x\ \gamma]_\Gamma = [n\ x\ \gamma]_\Gamma + [m\ x\ \gamma]_\Gamma$
(ii) $[[n\ m\ u]\ x\ \gamma]_\Gamma = [n\ [m\ u\ x]\ \gamma]_\Gamma = [n\ m\ [u\ x\ \gamma]_\Gamma]_\Gamma$ for all $\gamma \in \Gamma$ and $n, m, u \in N$.

Definition 5. [5]

(i) Let N be an RTNR. Then $N_0 = \{n \in N | [n\ 0\ 0] = 0\}$ is the zero-symmetric part of N. If $N = N_0$ then N is called a zero-symmetric RTNR.

(ii) An ideal J of an RTNR N is called a completely prime ideal (c-prime ideal) of N if for $x, y, z \in N$, $[x\ y\ z] \in J \Rightarrow$ either $x \in J$ or $y \in J$ or $z \in J$.

Definition 6. [3]

(i) A subgroup Δ of $_N\Gamma$ is said to be an N-subgroup of $_N\Gamma$ if $[N\ N\ \Delta]_\Gamma \subseteq \Delta$.

(ii) If $_N\Gamma$ and $_N\Gamma'$ are any two right ternary N-groups then $h : {_N\Gamma} \to {_N\Gamma'}$ is an N-homomorphism if $h(\gamma+\delta) = h(\gamma)+h(\delta)\ \forall\ \gamma, \delta \in \Gamma$ and $h([n\ x\ \gamma]_\Gamma) = [n\ x\ h(\gamma)]_{\Gamma'}\ \forall\ n, x \in N$.

(iii) A subgroup Δ of Γ is called a normal subgroup of Γ if $\forall\ \gamma \in \Gamma$, $\delta \in \Delta$, $\gamma + \delta - \gamma \in \Delta$.

(iv) A normal subgroup Δ of $_N\Gamma$ is called an ideal of Γ if $\forall\ \gamma \in \Gamma$, $\forall\ \delta \in \Delta$ and $\forall\ n, x \in N$, $[n\ x\ (\gamma + \delta)]_\Gamma - [n\ x\ \gamma]_\Gamma \in \Delta$.

(v) If I is an ideal of a right ternary N-group N then N/I is called a quotient right ternary N-group.

(vi) A right ternary N-group Γ is said to be simple if $\{o\}$ and Γ are the only ideals of ${}_N\Gamma$.

(vii) A right ternary N-group Γ is said to be N-simple if $\Omega = [N\ O\ o]_\Gamma$ and Γ are the only N-subgroups of ${}_N\Gamma$.

(viii) If N is an RTNR and Δ_1, Δ_2 are any two non-empty subsets of a right ternary N-group Γ, then
$(\Delta_1 : \Delta_2) = \{x \in N | [x\ n\ \delta_2]_\Gamma \in \Delta_1,\ \forall\ n \in N,\ \delta_2 \in \Delta_2\}$, $(o : \Delta) = \{x \in N | [x\ n\ \delta]_\Gamma = o\ \forall\ n \in N,\ \delta \in \Delta\}$ and $(o : \delta) = \{x \in N | [x\ n\ \delta]_\Gamma = o\ \forall\ n \in N\}$ (annihilator of $\delta \in \Delta$) and is a two-sided ideal.

(ix) A right ternary N-group Γ is said to be faithful if $(o : \Gamma) = \{0\}$.

(x) Let Γ be a right ternary N-group of an RTNR N. Then N is monogenic by $\gamma \in \Gamma$ if for every $x \in N$, $[N\ x\ \gamma]_\Gamma = \Gamma$ and γ is called a generator of Γ. A right ternary N-group Γ is strongly monogenic if Γ is monogenic and $[N\ x\ \gamma]_\Gamma = \{o\}$ or Γ for every $x \in N$ and $\gamma \in \Gamma$.

(xi) A monogenic N-group Γ with $\Gamma \neq \{o\}$ is said to be of type 0 if Γ is simple, type 1 if Γ is simple and strongly monogenic and type 2 if Γ is N_0-simple.

Proposition 7. *[3]*

(i) If e is a left unital element of N and ${}_N\Gamma$ is faithful then e is a bi-unital element of N.

(ii) If ${}_N\Gamma$ is monogenic by γ_0 then ${}_N\Gamma \cong_N N/(o : \gamma_0)$.

(iii) Let Γ be an N-group. Then ${}_N\Gamma$ is of type 2 $\Rightarrow$ ${}_N\Gamma$ is of type 1 $\Rightarrow$ ${}_N\Gamma$ is of type 0.

3. Chain Conditions

In this section the chain conditions on ideals and N-subgroups are given and it is proved that in a zero-symmetric RTNR with DCC on left N-subgroups, an N-subgroup M which is monogenic by m_0 has a right unital element.

Definition 8. If the set of ideals of an RTNR N satisfies the Descending Chain Condition (DCC) on ideals of N. i.e., for every descending chain of ideals : $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ of N, there exists a positive integer k such that $I_k = I_n$ for all $n \geq k$ then N is said to have DCCI.

Equivalently, if for every descending chain of ideals : $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ of N, there exists a positive integer k such that $I_k = I_{k+1} = \dots$ then N is said to have DCCI.

Similarly DCCR (on right ideals), DCCL (on left ideals), DCCM (on lateral ideals) and DCCRN (on right N-subgroups), DCCLN (on left N-subgroups), DCCMN (on lateral N-subgroups) and DCCN (on N-subgroups) are defined.

Notation 9. The notations ACCI, ACCR, ACCL, ACCM, ACCRN, ACCLN, ACCMN, ACCN are used for ascending chain conditions on ideals, right ideals, left ideals, lateral ideals, right N-subgroups, left N-subgroups, lateral N-subgroups and N-subgroups respectively.

Proposition 10. *If I is an ideal of an RTNR N and if J_1 and J_2 are any two ideals of N containing I such that $J_2/I \subseteq J_1/I$ then $J_2 \subseteq J_1$.*

Proof. Consider $x \in J_2$. Then $x+I \in J_2/I \Rightarrow x+I \in J_1/I \Rightarrow x+I = x_1+I$ for some $x_1 \in J_1 \Rightarrow x - x_1 \in I \Rightarrow x - x_1 \in J_1 \Rightarrow x \in J_1$. Thus $J_2 \subseteq J_1$. □

Definition 11. Let $\{I_k\}_{k\in K}$ (where K is an index set) be a collection of ideals of N. Then their sum $\sum\limits_{k\in K} I_k$ is called a direct sum (internal) if each element of $\sum\limits_{k\in K} I_k$ has a unique representation as a finite sum of elements of different I_k's. The direct sum is denoted by $I_1 \dot{+} I_2 \dot{+} I_3 \dot{+} \dots$ or $\sum\limits_{k\in K}{}^{\bullet} I_k$.

Lemma 12. *For each family $\{I_k\}_{k\in K}$ of ideals of an RTNR N, the following conditions are equivalent:*

(i) *The sum of the I_k's is direct.*

(ii) *The sum of the normal subgroups $(I_k, +)$ is direct*

(iii) $I_k \cap \sum\limits_{\substack{\ell\in k\\ \ell\neq k}} I_\ell = \{0\}\ \forall\ k \in K$.

Proposition 13. *For each family* $\{I_k\}_{k\in K}$ *of ideals of an RTNR* N*, let* $\sum_{k\in K} I_k$ *direct and* $a, a' \in I_i$*,* $b, b' \in I_j$ *where* $i \neq j$*.*
Then

(i) $a + b = b + a$.

(ii) $[a\ a + b\ a' + b'] = [a\ a\ a']$;
$[b\ b + a\ b' + a'] = [b\ b\ b']$.

(iii) $[a\ b\ b] = [a\ b\ 0] = [a\ 0\ b]$; $[b\ a\ a] = [b\ 0\ a] = [b\ a\ 0]$;
$[a\ b\ a] = [a\ 0\ a]; [a\ a\ b] = [a\ a\ 0]$; $[b\ a\ b] = [b\ 0\ b]; [b\ b\ a] = [b\ b\ 0]$.

(iv) If $N = N_0$ *then* $[a\ b\ b] = 0 = [a\ b\ a] = [a\ a\ b] = [b\ a\ a] = [b\ a\ b] = [b\ b\ a]$.

Proof. For each family $\{I_k\}_{k\in K}$, let $\sum_{k\in K} I_k$ be direct and $a, a' \in I_i$, $b, b' \in I_j$ where $i \neq j$.

(i) By the assumption and grouping the terms of $a+b-(b+a)$ in two different ways, (i) follows.

(ii) Consider $[a\ a + b\ a' + b'] = [a\ a + b\ a' + b'] - [a\ a\ a' + b'] + [a\ a\ a' + b'] - [a\ a\ a'] + [a\ a\ a']$. Since I_j is a lateral ideal and I_i is a right ideal of N and $I_i \cap I_j = \{0\}$ it follows that $[a\ a + b\ a' + b'] = [a\ a\ a']$. In a similar manner the other equations can be proved.

(iii) Since $[a\ b\ b+0] - [a\ b\ 0] \in I_i \cap I_j$ as I_i is a right ideal and I_j is a left ideal of N, $[a\ b\ b] = [a\ b\ 0]$ and $[a\ b+0\ b] - [a\ 0\ b] \in I_i \cap I_j$ as I_i is a right ideal and I_j is a lateral ideal of N. Hence $[a\ b\ b] = [a\ 0\ b]$. In a similar manner the other equations can be proved.

(iv) The proof follows from the definition of a zero-symmetric RTNR.

□

Definition 14. An ideal I of an RTNR N is called a direct summand of N if there exists an ideal J of N such that $N = I \dot{+} J$. The ideal J is called a direct complement of I in N.

Definition 15.

(i) Let $\{\Delta_k\}_{k\in K}$ (where K is an index set) be a collection of ideals of ${}_N\Gamma$. Then their sum $\sum_{k\in K} \Delta_k$ is called a direct sum (internal) if each element of $\sum_{k\in K} \Delta_k$ has a unique representation as a finite sum of elements of different Δ_k's. The direct sum is denoted by $\Delta_1 \dot{+} \Delta_2 \dot{+} \Delta_3 \dot{+} \cdots$ or $\sum_{k\in K}{}^{\bullet}\Delta_k$.

(ii) An ideal Δ_1 of an N-group ${}_N\Gamma$ is called a direct summand of ${}_N\Gamma$ if there exists an ideal Δ_2 of ${}_N\Gamma$ such that ${}_N\Gamma = \Delta_1 \dot{+} \Delta_2$.

Theorem 16. *Let I be an ideal of an RTNR N and I be a direct summand of N. Then every ideal of I is an ideal of N.*

Proof. Let $N = I \dot{+} J$ and K be an ideal of I. Let $x \in K$, $n \in N$. Then $n + x - n = (i + j) + x - (i + j) = i + (j + x - j) - i = i + x - i = x \in K$, proving that K is normal in N. Now let $x \in K$ and $n, n', n'' \in N$. Then using Proposition 13 $[x\ n\ n'] = [x\ i + j\ i' + j'] = [x\ i\ i'] \in K$ which proves that K is a right ideal of N. Also since
$[n\ n'\ n'' + x] - [n\ n'\ n''] = [i + j\ i' + j'\ i'' + j'' + x] - [i + j\ i' + j'\ i'' + j''] = [i\ i'\ i'' + x] - [i\ i'\ i''] \in K$. This implies that K is a left ideal of N. Similarly it can be proved that K is a lateral ideal of N. Hence K is an ideal of N. □

Remark 17. A similar statement of the above thorem is true in right ternary N-groups also. i.e., if an ideal Δ of ${}_N\Gamma$ is a direct summand of ${}_N\Gamma$ then every ideal of Δ is an ideal of ${}_N\Gamma$.

Theorem 18. *In an RTNR N the following statements hold good.*

(i) If J is an ideal of N and N has the DCCI then so does N/J.

(ii) If J is an ideal of N and J is a direct summand, then N has the DCCI iff J and N/J have DCCI.

Proof.

(i) Let J be an ideal of N and N have DCCI. Consider the descending chain of ideals of N/J namely $K_1/J \supseteq K_2/J \supseteq \ldots$ where $K_1, K_2, \ldots$ are ideals of N containing J. Then by the above proposition $K_1 \supseteq K_2 \supseteq \ldots$ is the descending chain of ideals of N. Since N has DCCI, there exists positive integers n and ℓ such that $K_\ell = K_n$ for $n \geq \ell \Rightarrow K_\ell/J = K_n/J$ for $n \geq \ell$. Thus N/J has DCCI.

(ii) Let J be an ideal of N and a direct summand of N. Then by Theorem 16 any ideal of J is an ideal of N. Since N has DCCI it follows that J has DCCI. Now by (i), N/J has DCCI.

Conversely let J and N/J have DCCI. Let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ be a descending chain of ideals of N. Then $K_1 + J \supseteq K_2 + J \supseteq \dots$ is a descending chain of ideals of N containing J. This implies that $K_1 + J/J \supseteq K_2 + J/J \supseteq \dots$ is a descending chain of ideals of N/I. Moreover, $K_1 \cap J \supseteq K_2 \cap J \supseteq \dots$ is a descending chain of ideals of J. Since both J and N/J have DCCI there exists a positive integer ℓ such that $K_\ell + J/J = K_n + J/J$ and $K_\ell \cap J = K_n \cap J \; \forall \; n \geq \ell$.

Now to prove that $K_\ell \subseteq K_n$ for $n \geq \ell$, let $x \in K_\ell$. Consider $x = x + 0 \in K_\ell + J = K_n + J$, by Proposition 10. This implies that there exists $y \in K_n$ and $j \in J$ such that $x = y + j \Rightarrow x - y \in J$. Also $y \in K_n \Rightarrow y \in K_\ell$, by the assumption. Since K_ℓ is an ideal, $x - y \in K_\ell$. Thus $x - y \in J \cap K_\ell = J \cap K_n \subseteq K_n$. Since $y \in K_n$, $x \in K_n$. Thus $K_\ell \subseteq K_n$. Hence $K_\ell = K_n$ for $n \geq \ell$ showing that N has DCCI. □

Note 19.

(i) The above theorem holds for DCCR, DCCL and DCCM also.

(ii) A similar statement of the above theorem is true in N-groups also.

In the following theorem it is proved that if Γ is a right ternary N-group with $[N \; x \; \gamma]_\Gamma \neq \{0\} \; \forall \; \gamma \in \Gamma$ and every monogenic N-subgroup of Γ has ACCI then the annihilator of γ is a completely prime two sided ideal of N.

Theorem 20. *Let Γ be a right ternary N-group such that $[N \; x \; \Gamma]_\Gamma \neq \{0\}$ and every monogenic N-subgroup of Γ has an ascending chain condition on ideals. Then there exists $\gamma \in \Gamma$ such that $(o : \gamma)$ is a completely prime two sided ideal of N.*

Proof. To prove the existence of $\gamma \in \Gamma$ such that $(o : \gamma)$ is a completely prime left ideal, let $A = \{(o : \gamma) | o \neq \gamma \in \Gamma, [N \; x \; \gamma]_\Gamma \neq \{o\}\}$. Suppose A does not contain a maximal element. Then there exists an infinite strict increasing sequence $(o : \gamma_1) \subset (o : \gamma_2) \subset (o : \gamma_3) \subset \dots$ of elements of A and in the factor right ternary N-group $N/(o : \gamma)$ the following is a strict increasing sequence $(o : \gamma_2)/(o : \gamma_1) \subset (o : \gamma_3)/(o : \gamma_1) \subset \dots$. This implies that

$$N/(o : \gamma_1) \;\text{ has no } ACCI. \tag{1}$$

Let Δ be a monogenic N-subgroup of Γ. Then $\Delta = [N \; x \; \gamma_1]_\Gamma$ for $x \in N$.
Now $N/(o : \gamma_1) \simeq [N \; x \; \gamma_1]_\Gamma$, by Proposition 7(ii).

Since Δ has ACCI so does $N/(o:\gamma_1)$ which contradicts the statement (1).

Hence A contains a maximal element say $(o:\gamma)$. Now let $(o:\gamma) = P$. Suppose P is not a completely prime two-sided ideal of N. Then $a \notin P$, $b \notin P$ and $c \notin C$ but $[a\ b\ c] \in P$. This implies that $[a\ x\ \gamma]_\Gamma \neq o$, $[b\ x\ \gamma]_\Gamma \neq o$, $[c\ x\ \gamma]_\Gamma \neq o\ \forall\ x \in N$ but $[[a\ b\ c]\ x\ \gamma]_\Gamma = o \Rightarrow a \in (o:\gamma)$ i.e., $a \in P$ which contradicts the assumption that $a \notin P$. Thus $[a\ b\ c] \in P \Rightarrow a \in P$ or $b \in P$ or $c \in P$. Hence $(o:\gamma)$ is a completely prime two-sided ideal of N. □

Lemma 21. *Let a zero-symmetric RTNR N have the DCCLN and let an N-subgroup M of N be monogenic by m_0. Let there exist an $m_1 \in M$ such that $(0:m_1) = \{0\}$. Then*

(i) M has a right unital element

(ii) $(0:m_0) = \{0\}$.

Proof.

(i) Let N have the DCCLN and M be an N-subgroup of N monogenic by $m_0 \in M$. Throughout the proof all the annihilators are taken in M. Since M is monogenic by m_0 for all $x \in M$,

$$[M\ x\ m_0] = M \tag{2}$$

Let $m_1 \in M$ be such that

$$(0:m_1) = \{0\} \tag{3}$$

Define $h: M \to [M\ x\ m_1]$ by $h(t) = [t\ x\ m_1]\ \ \forall\ \ t \in M$. Then h is an N-monomorphism. Obviously $[M\ x\ m_1] \subseteq [N\ N\ M] \subseteq M$ and more precisely $[M\ x\ m_1] = M$. Suppose not. i.e., $[M\ x\ m_1] \subset M$. Then $[M\ x\ m_1\ x\ m_1] \subset [M\ x\ m_1]$ and it gives an infinite sequence of left N-subgroups in N which contradicts the hypothesis that N has DCCLN. Therefore $[M\ x\ m_1] = M$ for all $x \in M$. This implies that there exists $e \in M$ such that $[e\ e\ m_1] = m_1$. Now for every $m \in M$, $[m\ e\ e] - m \in (0:m_1) = \{0\}$ and hence $[m\ e\ e] = m$. This proves that M has a right unital element.

(ii) Since M is monogenic by m_0, from (2), as $e \in M$ there exists $m_2 \in M$ such that

$$[m_2\ n\ m_0] = e\ \forall\ n \in M. \tag{4}$$

Since e is a right unital element $(0:e) = \{0\}$. Also since M is monogenic by m_0 and $(0:e) = \{0\}$, $(0:m_2) = \{0\}$.

Finally it is claimed that $(0 : m_0) = \{0\}$.
Since $(0 : m_2) = \{0\}$, by the argument used in (i)

$$[M\ n\ m_2] = M \tag{5}$$

Let m_3 be an arbitrary element in $(0 : m_0)$. Then from (5), there exists $m_4 \in M$ such that

$$m_3 = [m_4\ n\ m_2]\ \forall\ n \in M \tag{6}$$

Now, $m_4 = [m_4\ e\ e] = [m_4\ e\ [m_2\ n\ m_0]] = [m_3\ n\ m_0] = 0$
Since $m_4 = 0$, by (6) $m_3 = 0$. Thus $(0 : m_0) = \{0\}$. Hence the proof. □

Theorem 22. *Let a zero-symmetric RTNR have the DCCLN and let an N-subgroup M of N be monogenic by m_0. Then*

(i) *M has a right unital element.*

(ii) $(0 : m_0) = \{0\}$.

Proof. In view of the above lemma it is enough to show that there exists an $m_1 \in M$ such that $(0 : m_1) = \{0\}$. The annhilators are taken in N-subgroups as in the previous lemma. Let an N-subgroup M' of N be monogenic by $m'_0 \in M'$

$$\text{but}\quad (0 : m'_0) \neq \{0\} \tag{7}$$

Let M' be minimal for containing such an m'. i.e., M' is minimal such that

$$(0 : m'_0) \cap M' \neq \{0\} \tag{8}$$

Let $m'_1 \in M'$ be such that $(0 : m'_1)$ is minimal in $\{(0 : m') | [M'\ x\ m'] = M'\ \forall\ x \in M'\}$. This implies that $[M'\ x\ m'_1] = M' \Rightarrow$ there exists $m'_2 \in M'$ such that

$$[m'_2\ x\ m'_1] = m'_1\ \forall\ x \in M' \tag{9}$$

Let

$$u \in (0 : m'_2) \Rightarrow [u\ v\ m'_2] = 0\ \forall\ v \in M' \tag{10}$$

Now $[u\ y\ m'_1] = [u\ y\ [m'_2\ x\ m'_1]] = [[u\ y\ m'_2]\ x\ m'_1] = 0$ by (10).
Thus $u \in (0 : m'_1)$. Hence

$$(0 : m'_2) \subseteq (0 : m'_1) \tag{11}$$

Now there are two cases where M' is monogenic by m'_2 and M' is not monogenic by m'_2.

Case (i): Suppose M' is monogenic by m'_2. Then by the minimality of $(0 : m'_1)$,

$$(0 : m'_1) \subseteq (0 : m'_2). \tag{12}$$

From (11) and (12) $(0 : m'_1) = (0 : m'_2)$. Using (9) for every $m', \ell \in M'$
$[m'\ \ell\ m'_1] = [m'\ \ell\ m'_2\ \ell\ m'_1]$ taking $x = \ell$ in (9) $\Rightarrow [m'\ \ell\ m'_2] - m' \in (0 : m'_1) = (0 : m'_2)$
$\Rightarrow [[m'\ \ell\ m'_2]\ t\ m'_2] = [m'\ t\ m'_2]\ \forall\ \ell, t \in M'$
$\Rightarrow [[m'\ m'_2\ m'_2]\ m'_2\ m'_2] = [m'\ m'_2\ m'_2]$
$\Rightarrow m'_2$ is a right unital element in $[M'\ m'_2\ m'_2] \Rightarrow (0 : m'_2) = \{0\}$. Since M' is monogenic by m'_2, by the same argument given in (ii) of the above lemma $(0 : m'_0) = \{0\}$ which contradicts (7).
Case (ii): Suppose M' is not monogenic by m'_2. Then

$$[M'\ z\ m'_2] \subset M'\ \forall\ z \in M' \tag{13}$$

Consider the sequence $M' \supset [M'\ m'_2\ m'_2] \supset [[M'\ m'_2\ m'_2]\ m'_2\ m'_2] \supset \ldots$
Since N has DCCLN there exists a positive integer k such that
$[M'\ m_2'^{2k}] = [M'\ m_2'^{2(k+1)}] = \ldots$
Thus

$$[[M'\ (m'_2)^{2k}](m'_2)^{2(k+1)}] = [M'(m'_2)^{2k}] \tag{14}$$

Let

$$m'_3 = [(m'_2)^{2k+1}] = [m'_2(m'_2)^{2k}] \tag{15}$$

From (14) and (15) it follows that, m'_3 generates $[M'\ (m'_2)^{2k}]$. By the minimality of M', $(0 : m'_3) \cap [M'\ (m'_2)^{2k}] = \{0\}$. Again using the minimality of M' each generator m'_4 of $[M'\ m_2'^{2k}] = [M'\ m_2^{2(k+1)}] = [M'\ m'_2\ m'_3]$ should have $(0 : m'_4) \cap [M'\ m'_2\ m'_3] = \{0\}$. But by letting $m'_4 = [m'_1\ m'_2 m'_3]$ in the following argument it is proved that this stament is violated. It is noted that using (15) and (9), $[m'_3\ x\ m'_1] = m'_1$. Also $[M'\ m'_2 m'_3]$ is monogenic by m'_4 since for $x \in M'$, $[[M'\ m'_2\ m'_3]\ x\ m'_4] = [M'\ m'_2\ m'_3]$. Now by (8)

$$(0 : m'_4) \cap [M'\ m'_2\ m'_3] = \{0\} \tag{16}$$

Furthermore, $(0 : m'_3) \neq \{0\}$. Otherwise $M' \cong_N [M'\ m'_2\ m'_3]$ which contradicts (13).
Let $0 \neq m'_5 \in (0 : m'_3) \Rightarrow [m'_5\ r\ m'_3] = 0\ \forall\ r \in M'$.
Since M' is monogenic by m'_1 there exists $m'_6 \in M'$ such that $[m'_6\ x\ m'_1] = m'_5$ as $m'_5 \in M'$. Consider
$0 = [m'_5\ m'_2\ m'_3] = [[m'_6\ x\ m'_1]\ m'_2\ m'_3] = [m'_6\ x\ [m'_3\ t\ m'_1]\ m'_2\ m'_3]$
$\quad = [[m'_6\ x\ m'_3]\ t\ [m'_1\ m'_2\ m'_3]] \Rightarrow [m'_6\ x\ m'_3] \in (0 : [m'_1\ m'_2\ m'_3])$

Now $[[m_6' \; x \; m_3'] \; x \; m_1'] = [m_6' \; x \; m_1'] = m_5' \neq 0$. Thus $(0 : [m_1' \; m_2' \; m_3']) \cap [M' \; m_2' m_3'] \neq \{0\}$ i.e., $(0 : m_4') \cap [M' \; m_2' \; m_3'] \neq \{0\}$ which contradicts (16). Hence the proof. □

4. Modularity

In this section modular left ideals and modular ideals are defined. It is proved that the modular left ideals of N are annihilators of generators of monogenic N-groups.

Definition 23. A left ideal L of an RTNR N is called left modular if for every $n \in N$ there exists $e \in N$ such that $n - [n \; e \; e] \in L$. In this case it is said that L is left modular by e.

Similarly a right ideal R of an RTNR N is called right modular if $n - [e \; e \; n] \in R$ and a lateral ideal M of N is lateral modular if $n - [e \; n \; e] \in M$. An ideal I of an RTNR is modular if I is left modular, lateral modular and right modular i.e., if there is an $e \in N$ such that $n - [n \; e \; e]$, $n - [e \; n \; e]$ and $n - [e \; e \; n]$ are all in I.

Example 24. Let $N = \{0, x, y, z\}$ be as given in [2]. Define $+$ as in Table 1 and the ternary operation $[\;]$ on N by $[x \; y \; z] = (x.y).z$ for every $x, y, z \in N$ where $\cdot$ defined as in Table 2.

Table 1

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	z	0

Table 2

$\cdot$	0	x	y	z
0	0	0	0	0
x	x	x	x	x
y	y	y	y	y
z	z	z	z	z

Let $L = \{0, x\}$. Then it is a left ideal of N and is left modular by $0, x, y, z$.

Lemma 25. *Let N be an RTNR. Then*

(i) If L_1 and L_2 are left ideals of N with $L_1 \subseteq L_2$ and L_1 is left modular by e then L_2 is left modular by e.

(ii) $\{0\}$ is left modular iff N contains a right unital element.

(iii) In a zero-symmetric RTNR if L is left modular by e then $e \in L$ iff $L = N$.

Proof.

(i) Let L_1 be left modular by e. Then for every $n \in N$, $n - [n\ e\ e] \in L_1$. This implies that L_2 is left modular by e as $L_1 \subseteq L_2$.

(ii) Let e be a right unital element of N. Then for every $n \in N$, $[n\ e\ e] = n \Rightarrow n - [n\ e\ e] \in \{0\} \Rightarrow \{0\}$ is left modular. The converse follows by retracing the steps.

(iii) If $e \in L$, as L is a left N-subgroup of N_0 and $n - [n\ e\ e] \in L$, $[n\ e\ e] \in L$. Hence $N = L$. Conversely if $N = L$ and $e \notin L$ then $[n\ e\ e] \notin L$, a contradiction as L is modular by e.

□

Theorem 26. *A left ideal L of an RTNR N is left modular iff there exists a right ternary N-group ${}_N\Gamma$ and $\gamma \in \Gamma$ such that Γ is monogenic by γ and $L = (o : \gamma)$.*

Proof. Let a left ideal L of N be left modular by e. Since L is a left ideal of N, L is an ideal of right ternary N-group N and hence N/L is a factor right ternary N-group. Now for every $x \in N$

$$[N\ x\ e + L] = \{[n\ x\ e] + L | n \in N\} = \{n' + L | n' \in N\} = N/L$$

Taking $N/L = {}_N\Gamma$ it is seen that N/L is monogenic by $\gamma = e + L$. Moreover $n \in (o : \gamma) \Rightarrow n \in (L : e + L) \Rightarrow [n\ e\ e] \in L \Rightarrow n \in L$, as L is left modular. Similarly if $u \in L$ then $u \in (L : e + L) \Rightarrow u \in (o : \gamma)$. Thus $L = (o : \gamma)$.

Conversely let ${}_N\Gamma$ be monogenic by γ. Then $[e\ e\ \gamma] = 0$. Since $[([n\ e\ e] - n)\ e\ \gamma] = 0$. Thus $[n\ e\ e] - n \in (o : \gamma)$. Hence L is left modular by e. □

Corollary 27. *If L is a left ideal of N and L is left modular then $(L : N) \subseteq L$.*

Proof. Since L is left modular by the above theorem there exists an N-group ${}_N\Gamma$ monogenic by $\gamma \in \Gamma$ such that $L = (o : \gamma)$. Since $(o : \Gamma) = \bigcap_{\gamma \in \Gamma} (o : \gamma)$, $(o : \Gamma) \subseteq (o : \gamma) = L$. Taking $\Gamma = N/L$ it follows that $(L : N) \subseteq L$. □

Remark 28. If L is a lateral and left ideal of N then $(L : N)$ will be an ideal of N.

In the following theorem it is shown that the intersection of two modular ideals is again modular if the zero-symmetric distributive RTNR N is the sum of those two ideals.

Theorem 29. *Let N be a zero-symmetric distributive RTNR and L_1 and L_2 be any two modular ideals of N such that $N = L_1 + L_2$. Then $L_1 \cap L_2$ is modular.*

Proof. Let N be a distributive RTNR. Let $N = L_1 + L_2$ where L_1 and L_2 are modular by e_1 and e_2 respectively. Decompose e_1, e_2 as $e_1 = h + f$, $e_2 = h' + f'$, where $h, h' \in L_1$ and $f, f' \in L_2$. Let $e = h' + f$. It is now claimed that $L_1 \cap L_2$ is modular by e. Consider
$n - [n\ e\ e] = n - [n\ e\ (h' + f)] = n - [n\ e\ f] - [n\ e\ h']$
$= n - [n\ e\ (-h + e_1)] + \ell_1$ where $\ell_1 = -[n\ e\ h'] \in L_1$
$= n - [n\ e\ e_1] + \ell_1' + \ell_1$, where $\ell_1' = -[n\ e\ h] \in L_1$, as L_1 is a left ideal
$= n - [n\ e_1\ e_1] + [n\ e_1\ e_1] - [n\ e\ e_1] + \ell_1''$, where $\ell_1'' = \ell_1' + \ell_1$
$= \ell_1''' + [n\ (e_1 - e)\ e_1] + \ell_1''$, where $\ell_1''' = n - [n\ e_1\ e_1]$
$= [n\ h - h'\ e_1] + \ell$, where $\ell = \ell_1''' + \ell_1''$, as L_1 is a normal subgroup of $N \in L_1$, as L_1 is a left N-subgroup of N.

Similarly $n - [n\ e\ e] \in L_2\ \forall\ n \in N$. Also $n - [e\ n\ e] \in L_1 \cap L_2$ and $n - [e\ e\ n] \in L_1 \cap L_2$. Thus $L_1 \cap L_2$ is modular by e. □

Definition 30. Let N be a right and lateral ternary near-ring. Let I be an ideal of N. Then a group $(\Gamma, +)$ is called a right-lateral ternary N-group if $[\ ]_\Gamma : N \times N \times \Gamma \to \Gamma$ such that

(i) $[n + m\ x\ \gamma]_\Gamma = [n\ x\ \gamma]_\Gamma + [m\ x\ \gamma]_\Gamma$

(ii) $[n\ m + x\ \gamma]_\Gamma = [n\ m\ \gamma]_\Gamma + [n\ x\ \gamma]_\Gamma$

(iii) $[[n\ m\ u]\ x\ \gamma]_\Gamma = [n\ [m\ u\ x]\ \gamma]_\Gamma = [n\ m\ [u\ x\ \gamma]_\Gamma]_\Gamma$
for every $n, m, x, u \in N$ and $\gamma \in \Gamma$.

Remark 31. Let N be a right and lateral ternary near-ring. If Δ_1 and Δ_2 are any two non-empty subsets of a right-lateral ternary N-group then
$(\Delta_1 : \Delta_2) = \{n \in N | [n\ x\ \delta_2]_\Gamma \in \Delta_1 \text{ and } [x\ n\ \delta_2]_\Gamma \in \Delta_1\ \forall\ x \in N, \delta_2 \in \Delta_2\}$
and $(o : \Delta) = \{n \in N | [n\ x\ \delta]_\Gamma = o \text{ and } [x\ n\ \delta]_\Gamma = o\ \forall\ x \in N, \delta \in \Delta\}$

It is noted that if Δ_1 and Δ are ideals of ${}_N\Gamma$ then $(\Delta_1 : \Delta_2)$ and $(o : \Delta)$ are left ideals of N and if Δ_2 is an N-subgroup of Γ, $(\Delta_1 : \Delta_2)$ is a two sided ideal of N. If $N = N_0$ or in N for every $x, y, z \in N$, $[x\ y\ z] = [y\ x\ z]$ and Δ is an N-subgroup then $(o : \Delta)$ is an ideal of N.

The following lemma gives the process of transforming ${}_N\Gamma$ group into ${}_{N/I}\Gamma$ group where $I \subseteq (o : \Gamma)$ is an ideal of a right and lateral ternary near-ring N.

Lemma 32. *Let $(\Gamma, +)$ be a group and N be an RTNR such that $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N$. Then*

(i) If ${}_N\Gamma$ is a right-lateral ternary N-group and I is an ideal of N such that $I \subseteq (o : \Gamma)$ by defining $[\ ]^*_\Gamma : N/I \times N/I \times \Gamma \to \Gamma$ as $[n + I\ x + I\ \gamma]^*_\Gamma = [n\ x\ \gamma]_\Gamma$, Γ transforms into a right-lateral ternary N/I-group ${}_{N/I}\Gamma$.

(ii) If ${}_{N/I}\Gamma$ is a right-lateral ternary N/I-group then $[\ ]_\Gamma : N \times N \times \Gamma \to \Gamma$ defined by $[n\ x\ \gamma]_\Gamma = [n + I\ x + I\ \gamma]^*_\Gamma$ turns ${}_{N/I}\Gamma$ into a right-lateral ternary N-group where $I \subseteq (o : \Gamma)$.

Proof.

(i) Let $I \subseteq (o : \Gamma)$. Now for $n, x \in N$, $[n + I\ x + I\ \gamma]^*_\Gamma = [n\ x\ \gamma]_\Gamma$. It can be easily seen that Γ is a right-lateral ternary N/I-group.

(ii) Retracing the steps given in (i), it can be seen that ${}_{N/I}\Gamma$ becomes right-lateral ternary N-group ${}_N\Gamma$.

□

Proposition 33. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N$. Then*

(i) ${}_N\Gamma$ is monogenic iff ${}_{N/I}\Gamma$ is monogenic.

(ii) Δ is an ideal of ${}_N\Gamma$ iff Δ is an ideal of ${}_{N/I}\Gamma$.

Proof.

(i) If ${}_N\Gamma$ is monogenic then $[N\ x\ \gamma]_\Gamma = \Gamma$ and $[x\ N\ \gamma]_\Gamma = \Gamma\ \forall\ x \in N$. Since $[n + I\ x + I\ \gamma]^*_\Gamma = [n\ x\ \gamma]_\Gamma$ it follows that $[N/I\ x + I\ \gamma]^*_\Gamma = \Gamma$ and $[x + I\ N/I\ \gamma]^*_\Gamma = \Gamma\ \forall\ x + I \in N/I$ thus ${}_{N/I}\Gamma$ is monogenic by γ. The converse follows by retracing the steps.

(ii) Let Δ be an ideal of ${}_N\Gamma$. Consider for $n, x \in N$ and $\gamma \in \Gamma$, $\delta \in \Delta$,
$[n + I\ x + I\ \gamma + \delta]^*_\Gamma - [n + I\ x + I\ \gamma]^*_\Gamma = [n\ x\ \gamma + \delta]_\Gamma - [n\ x\ \gamma]_\Gamma \in \Delta$.

Thus Δ is an ideal of ${}_{N/I}\Gamma$. The converse follows by retracing the steps. □

Corollary 34. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N$. Then ${}_{N/I}\Gamma$ is of type ν iff ${}_N\Gamma$ is of type ν where $\nu \in \{0, 1, 2\}$.*

Proof. The proof follows from Proposition 33 and Definition 6(xii). □

Remark 35. If ${}_{N/I}\Gamma$ is faithful then $I = (o : \Gamma)$.

Definition 36. Let N be an RTNR and $\nu \in \{0,1,2\}$. Then a left ideal L of N is called left ν-modular if L is left modular and N/L is an N-group of type ν.

Theorem 37. *Let $\{N_i\}_{i\in I}$ be a collection of all RTNRs such that $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N_i$ and*

$$N = \{(n_1, n_2, \dots)|n_i \in N_i\} = N_1 \times N_2 \times \dots N_k \dots .$$

Let L_i be a left ideal of N_i for some $i \in I$. Let $\overline{L_i} = \prod_{j\in I} M_j$ where $M_j = \begin{cases} N_j, & \text{if } i \neq j \\ L_i, & \text{if } i = j \end{cases}$ which is a left ideal of N. Then for $\nu \in \{0,1,2\}$. L_i is left ν-modular in N_i if and only if $\overline{L_i}$ is left ν-modular in N.

Proof. Let L_i be ν-modular in N_i. Then L_i is left modular in N_i and therefore $\overline{L_i}$ is left modular in N. Also the right-lateral ternary N_i-group N_i/L_i is of type ν. Now $N/\overline{L_i}$ becomes a right-lateral N_i-group by defining

$[n_i\ n_i'\ (\dots n_i'' \dots) + \overline{L_i}]_{N/\overline{L_i}} = (0\dots 0[n_i\ n_i'\ n_i'']0\dots) + \overline{L_i}$. Now define $h : N_i \to N/\overline{L_i}$ by $h(n_i) = n + \overline{L_i}$ where $n = (\dots n_i \dots)$.
Then h is an N_i-homomorphism and $ker\ h = \{n_i|n+\overline{L_i} = \overline{L_i}\} = \{n_i|n \in \overline{L_i}\} = L_i$.

Since h is an epimorphism, $N_i/L_i \cong_{N_i} N/\overline{L_i}$ and also as N_i/L_i is of type ν, $N/\overline{L_i}$ is of type ν. Denote $\overline{O_i} = J_i$. Then $J_i \subseteq (o : N/\overline{L_i})$. Obviously $N/J_i \cong N_i$. Thus $N/\overline{L_i}$ is a N/J_i-group and is transformed into a right-lateral ternary N-group by defining $[n\ x\ n + \overline{L_i}] = [n + J_i\ x + J_i\ n + \overline{L_i}]^*_{N/\overline{L_i}}$ and is of type ν.

Conversely if $\overline{L_i}$ is left ν-modular in N then L_i is obviously left ν-modular in N_i and $N/\overline{L_i}$ is a right-lateral ternary N-group of type ν. Now N_i/L_i becomes a right-lateral ternary N-group by defining $[(n_1, n_2, \dots)(n_1', n_2', \dots)(n_i'' + L_i)] = [n_i\ n_i'\ n_i''] + L_i$.

Define a map $g : N \to N_i/L_i$ by $g((n_1, n_2, \dots)) = n_i + L_i$. Then g is an N-epimorphism and $ker\ g = \{(n_1, n_2, \dots)|g((n_1, n_2, \dots)) = L_i\} = \{(n_1, n_2, \dots)|n_i + L_i = L_i\} = \{(n_1, n_2, \dots)|n_i \in L_i\} = \overline{L_i}$.

Thus $N/\overline{L_i} \cong_N N_i/L_i$. Since $N/\overline{L_i}$ is of type ν, N_i/L_i is of type ν. Moreover $\overline{O_i} = J_i \subseteq (L_i : N_i)$. Now $N/\overline{L_i}$ is right-lateral ternary N/J_i group and since

$N_i \cong N/J_i$ it follows that N_i/L_i is a right-lateral ternary N_i-group of type ν. Thus L_i is left ν-modular. Hence the proof. $\square$

5. Primitive RTNR

In this section ν-primitivity ($\nu \in \{0,1,2\}$) of an RTNR is defined. The interrelationship between the different types of ν-primitive ideals is discussed.

Definition 38. Let N be an RTNR and ${}_N\Gamma$ be a right ternary N-group. Let $\nu \in \{0,1,2\}$. Then

(i) N is called ν-primitive on ${}_N\Gamma$ if Γ is faithful and Γ is of type ν.

(ii) N is ν-primitive if there exists an N-group ${}_N\Gamma$ such that N is ν-primitive on ${}_N\Gamma$.

Definition 39. If N is an RTNR such that $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N$ then an ideal I of N is called a ν-primitive ideal of N if N/I is ν-primitive where $\nu \in \{0,1,2\}$.

Theorem 40. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N$ and I be an ideal of N and $\nu \in \{0,1,2\}$. Then the following conditions are equivalent:*

(i) I is ν-primitive.

(ii) There exists an N-group ${}_N\Gamma$ such that $I = (o : \Gamma)$ and ${}_N\Gamma$ is of type ν.

(iii) There exists a left ideal L of N such that $I = (L : N)$ and L is left ν-modular.

Proof.
(i) $\Longrightarrow$ (ii) If I is ν-primitive by Definition 39, N/I is ν-primitive on ${}_{N/I}\Gamma$.

This implies that there exists an N/I group Γ such that N/I is ν-primitive on ${}_{N/I}\Gamma$.

$\Rightarrow {}_N\Gamma$ is of type ν and $I = (o : \Gamma)$, by Corollary 34.

(ii) $\Rightarrow$ (iii) Let $\Gamma = [N\ x\ \Gamma]_\Gamma \neq \{o\}\ \forall\ x \in N$ and $(o : \gamma) = L$. Then L is left modular by Theorem 26. By Proposition 7(ii), Γ is N-isomorphic to N/L and since ${}_N\Gamma$ is of type ν so is N/L. Thus by Definition 38, L is left ν-modular. Moreover $I = (o : \Gamma) = (o : N/L) = (L : N)$.

(iii) $\Rightarrow$ (i) Let $\Gamma = N/L$. Since L is left ν-modular, N/L is an N group of type ν i.e., ${}_N\Gamma$ is of type ν and hence by Remark 35, ${}_{N/I}\Gamma$ is of type ν. Also

$I = (L : N) = (o : N/L) = (o : \Gamma)$. This shows that ${}_{N/I}\Gamma$ is faithful. Thus N/I is ν-primitive on ${}_{N/I}\Gamma$. Hence I is ν-primitive. □

Corollary 41. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N$. Then the following conditions are equivalent.*

(i) N is ν-primitive where $\nu \in \{0, 1, 2\}$.

(ii) $\{0\}$ is a ν-primitive ideal.

(iii) There exists a left ideal L of N such that $(L : N) = \{0\}$ and L is left ν-modular.

Proof.
(i) ⇒ (ii) follows directly from Definition 38 and Definition 39.
(ii) ⇒ (iii) Let $\Gamma = [N\ x\ \Gamma]_\Gamma \neq \{o\}\ \forall\ x \in N$ and $(o : \gamma) = L$. Then L is left modular by Theorem 26. Since $\{0\}$ is a ν-primitive ideal, N is ν-primitive and hence $(o : \Gamma) = \{0\}$ i.e., Γ is faithful and Γ is of type ν. By Proposition 7(ii), Γ is N-isomorphic to N/L. Thus $(L : N) = \{0\}$ and N/L is a right-lateral ternary N-group of type ν and hence L is left ν-modular.
(iii) ⇒ (i) Let $\Gamma = N/L$. Then $(o : \Gamma) = \{0\}$ and since L is left ν-modular, N/L is an N-group of type ν i.e., ${}_N\Gamma$ is of type ν. This implies that N is ν-primitive. □

Proposition 42. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N$ and $\nu \in \{0, 1, 2\}$. Let Γ be an N-group. Then*

(i) If N is ν-primitive on ${}_N\Gamma$ then $N \neq \{0\}$, $\Gamma \neq \{o\}$ and if an ideal I of N is ν-primitive then $I \neq N$.

(ii) 2-primitivity ⇒ 1-primitivity ⇒ 0-primitivity.

Proof. (i) If N is ν-primitive on ${}_N\Gamma$ then Γ is of type $\nu \Rightarrow \Gamma \neq \{o\}$. Moreover if $N = \{0\}$ then $\Gamma = \{o\}$, as N is monogenic which is not true and hence $N \neq \{0\}$. If an ideal I of N is ν-primitive then N/I is of type ν and hence by the above argument $I \neq N$.
(ii) Let N be 2-primitive. Then there is an N-group Γ such that Γ is faithful and is of type 2. Thus (ii) follows by Proposition 7(iii). □

Lemma 43. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N$ and N contain a left unital element. Then every ν-primitive ideal I of N is modular where $\nu \in \{0, 1, 2\}$.*

Proof. Let N be a right ternary near-ring such that $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N$. Let I be a ν-primitive ideal of N. Then $_{N/I}\Gamma$ is faithful and is of type ν. Let e be a left unital element in N. Then I is left modular. Since $_{N/I}\Gamma$ is faithful and $e + I$ is a left unital element, by Proposition 7(i) $e + I$ is a bi-unital element of N/I. Hence I is right modular and also I is lateral modular as $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N$. Thus I is modular. □

Theorem 44. *Let N be an RTNR such that $[x\ y\ z] = [y\ x\ z]$ for every $x, y, z \in N$ and N be simple. Let a right-lateral ternary N-group $_N\Gamma$ be of type ν. Then N is ν-primitive on $_N\Gamma$ where $\nu \in \{0, 1, 2\}$.*

Proof. Since $(o : \Gamma)$ is an ideal of N and N is simple, $(o : \Gamma) = \{0\}$ or $(o : \Gamma) = N$. If $(o : \Gamma) = N$ then $[N\ x\ \gamma]_\Gamma = \{o\}\ \forall\ \gamma \in \Gamma$ and $x \in N$ which is not true as Γ is of type ν. Thus $(o : \Gamma) = \{0\}$ and hence N is ν-primitive on $_N\Gamma$. □

6. Conclusion

In this paper the chain conditions on the substructures of an RTNR N were given and the existence of a right unital element in a monogenic N-subgroup of a zero-symmetric RTNR N with DCC on left N-subgroups was established. If N is a right and lateral ternary near-ring then a right-lateral ternary N-group was transformed into a right-lateral ternary N/I-group and vice-versa which led to define left ν-modularity and ν-primitivity of ideals where $\nu \in \{0, 1, 2\}$. A further exploration on ν-primitive ideals and prime ideals will lead to the study of various types of radicals.

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