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AN EQUICONVERGENCE THEOREM FOR A PAIR OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract: In this paper, the system $(M + \lambda)\phi = 0$ for a pair of second order differential equations has been considered in $L^2[0, \infty)$. An equiconvergence theorem has been discussed under Fourier conditions.

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1. First Section of the Paper

Many problems of the spectral theory of differential equations concentrated around the problem of eigenvalues and eigenfunctions expansions. From one hand it accrues questions of eigenvalues and eigenfunctions asymptotes; and in the other hand it connects mathematics with the function satisfy any condition which is sufficient for the convergence of an ordinary Fourier series, then the series under consideration converges to $f(x)$. This phenomenon is called *equiconvergence.* (see[2], [3]).

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2. Basic Concept

Consider the differential system

$$(M + \lambda)\phi = 0, \quad 0 \le x < \infty, \tag{2.1}$$

where

$$M = \begin{pmatrix} \frac{d^2}{dx^2} - p(x) & r(x) \\ r(x) & \frac{d^2}{dx^2} - q(x) \end{pmatrix}, \tag{2.2}$$

and

$$\phi = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

is a vector having components $u(x)$ and $v(x)$ and λ is a parameter real or complex. The main object of the paper is to obtain an equiconvergence theorem associated with the system (2.1) under Fourier conditions which is analogous to Titchmarsh [7], [Chap IX]. We have used the notations and results of Bhagat [1] and Kumar [5] wherever necessary. The differential system (2.1) satisfies the following conditions: $p(x)$, $q(x)$ and $r(x)$ are all real-valued and continuous functions in $0 \le x < \infty$. We suppose that any solution of $[\phi = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$ of the system satisfies the two linearly independent boundary conditions at $x = 0$. viz

$$a_{j1}u(0) + a_{j2}u'(0) + a_{j3}v(0) + a_{j4}v'(0), (j = 1, 2) \tag{2.3}$$

where

(a) $a_{jk}, (j = 1, 2, 3, 4)$are real-valued constants.
(b) The set$\{a_{1k}\}$is linearly independent of the set$\{a_{2k}\}$
(c) $a_{14}a_{23} - a_{24}a_{13} + a_{12}a_{21} - a_{11}a_{22} = 0$ (2.4)

Following Pandey and Kumar [8], the bilinear concomitant $[\theta, \phi]$ of two vectors

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

and

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

is defined by

$$[\theta, \phi] = \theta_1\phi_1' - \theta_1'\phi_1 + \theta_2\phi_2' - \theta_2'\phi_2$$

If ϕ and θ are any two solutions of the system (2.1) for the same value of λ then $[\theta, \phi]$ is a function of λ alone. It is an integral function of λ, real for real λ.

Let

$$\phi_j(x,\lambda) \equiv \phi_j(0|x;\lambda) = \begin{pmatrix} u_j(0|x;\lambda) \\ v_j(0|x;\lambda) \end{pmatrix}, (j = 1, 2)$$

be the boundary condition vectors at $x = 0$, given by

$$\left.\begin{array}{ll} u_j(0|0;\lambda) = a_{j2}, & u'_j(0|0;\lambda) = -a_{j1} \\ v_j(0|0;\lambda) = a_{j4}, & v'_j(0|0;\lambda) = -a_{j3} \end{array}\right\} \qquad (j = 1, 2)$$

so that (2.3) and (2.4) can be written as

$$[\phi(x,\lambda) \cdot \phi_j(0|x,\lambda)] = 0 \tag{2.5}$$

and

$$[\phi_1\phi_2] = 0 \tag{2.6}$$

The vectors

$$\theta_k(x,\lambda) \equiv \theta_k(0|x;\lambda) = \begin{pmatrix} x_k(0|x;\lambda) \\ y_k(0|x;\lambda) \end{pmatrix}, (k = 1, 2)$$

which take real constant values (independent of λ) at $x = 0$ are defined by the relations

$$[\phi_j\theta_k] = \delta_{jk}, [\theta_1\theta_2] = 0; (1 \le j, k \le 2). \tag{2.7}$$

Condition (2.7) is equivalent to the condition (2.3) and (2.4).

The two linearly independent solutions of the system (2.1) which are $L^2[0,\infty)$, are given by

$$\psi_r(x,\lambda) = \theta_r(x,\lambda) + \sum_{i=1}^{2} m_{rs}(\lambda)\phi_s(x,\lambda) \quad (r = 1, 2) \tag{2.8}$$

This $m_{rs}(\lambda), (1 \le r, s \le 2)$ are analytic functions of regular in either half planes $im(\lambda) > 0$ and $im(\lambda) < 0$ and

$$\overline{m_{rs}(\lambda)} = m_{rs}\left(\bar{\lambda}\right)$$

İt is also proved that

$$[\phi_j(0|x;\lambda)\psi_r(x,\lambda)] = \delta_{jr}, \quad (1 \le j, r \le 2) \tag{2.9}$$

The Green's matrix

$$\begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{pmatrix}$$

for the system (2.1) is given by

$$\begin{aligned} G_1(x,y;\lambda) &= \begin{pmatrix} \psi_{11}(x,\lambda) & \psi_{21}(x,\lambda) \\ \psi_{12}(x,\lambda) & \psi_{22}(x,\lambda) \end{pmatrix} \cdot \begin{pmatrix} u_1(y,\lambda) & v_1(y,\lambda) \\ u_2(y,\lambda) & v_2(y,\lambda) \end{pmatrix} ; y \in [0,x) \\ &= \begin{pmatrix} u_1(x,\lambda) & u_2(x,\lambda) \\ v_1(x,\lambda) & v_2(x,\lambda) \end{pmatrix} \cdot \begin{pmatrix} \psi_{11}(y,\lambda) & \psi_{12}(y,\lambda) \\ \psi_{21}(y,\lambda) & \psi_{22}(y,\lambda)) \end{pmatrix} ; \text{y}\in(\text{x},\infty). \end{aligned}$$

3. Important Result

Titchmarsh [7], Chap [IX] has proved the following theorems convergence of the expansion under Fourier conditions.

3.1. Theorem

Let $q(x)$ be continuous, increasing and convex downwards. Let $f(y)$ be $L^2[0,\infty)$, $x > 0$ and in the neighbourhood of $y = x$. Let $f(y)$ satisfy any condition which is sufficient for the convergence of an ordinary Fourier series. Then the series

$$\sum_{n=0}^{\infty} C_n \psi_n(x),$$

where

$$c_n = \int_0^{\infty} f(y)\psi_n(y)\,dy, \text{and} \psi_n(x)$$

is the eigenfunction associated with the differential equation

$$\frac{d^2y}{dx^2} + (\lambda - q(x))\,y = 0, (0 \le x < \infty)$$

converges to sum $f(x)$.

4. Special Solution

Let us consider the system of integral equations

$$\left.\begin{aligned} X_j(x) &= \frac{e^{i\mu x}}{2i\mu} + \frac{1}{\mu}\int_x^\infty \{p(y)X_j(y) - r(y)Y_j(y)\}\sin\{\mu(y-x)\}dy,\\ Y_j(x) &= \frac{e^{i\mu x}}{2i\mu} + \frac{1}{\mu}\int_x^\infty \{q(y)Y_j(y) - r(y)X_j(y)\}\sin\{\mu(y-x)\}dy, \end{aligned}\right\} \tag{4.1}$$

where $\lambda = \mu^2 = \sigma + it$ The solution of (4.1) can be obtained by the method of successive approximation as follows: Let

$$X_{j1}(x) = \frac{e^{i\mu x}}{2i\mu}, \quad Y_{j1}(x) = \frac{e^{i\mu x}}{2i\mu} \qquad (j = 1, 2), \tag{4.2}$$

and for $n \geq 1$,

$$\left.\begin{aligned} X_{j,n}(x) &= \frac{e^{i\mu x}}{2i\mu} + \frac{1}{\mu}\int_x^\infty \{p(y)X_{j,(n-1)}(y) - r(y)Y_{j,(n-1)}(y)\}\cdot\sin\{\mu(y-x)\}dy\\ Y_{j,n}(x) &= \frac{e^{i\mu x}}{2i\mu} + \frac{1}{\mu}\int_x^\infty \{q(y)Y_{j,(n-1)}(y) - r(y)X_{j,(n-1)}(y)\}\cdot\sin\{\mu(y-x)\}dy \end{aligned}\right\} \tag{4.3}$$

Since $p(x), q(x)$ and $r(x)$ all are $L^2[0,\infty)$, therefore suppose that

$$J = \max\left[\int_x^\infty |p(x)|dx, \int_x^\infty |q(x)|dx, \int_x^\infty |r(x)|dx\right]. \tag{4.4}$$

Then

$$\begin{aligned} X_{j,2}(x) - X_{j,1}(x) &= \frac{1}{\mu}\int_x^\infty [p(y)X_{j,1}(y) - r(y)Y_{j,1}(y)]\cdot\sin\mu(y-x)dy\\ &= \frac{1}{\mu}\int_x^\infty \frac{e^{i\mu y}}{2i\mu}\cdot\frac{e^{i\mu(y-x)} - e^{i\mu(y-x)}}{2i}[p(y) - r(y)]\,dy\\ &= \frac{e^{i\mu x}}{4\mu^2}\int_x^\infty \left[1 - e^{i\mu(y-x)}\right][p(y) - r(y)]\,dy. \end{aligned}$$

Therefore

$$|X_{j2}(x) - X_{j1}(x)| \leq \frac{e^{-ix}}{|\mu|^2} J \qquad (j = 1, 2) \tag{4.5}$$

Similarly

$$|Y_{j2}(x) - Y_{j1}(x)| \leq \frac{e^{-ix}}{|\mu|^2} J \qquad (j = 1, 2) \tag{4.6}$$

Hence by using (4.5) and (4.6), we have

$$|X_{j3}(x) - X_{j2}(x)| \leq \frac{e^{-ix}}{|\mu|^4} J^2 \qquad (j = 1, 2) \tag{4.7}$$

and,

$$|Y_{j3}(x) - Y_{j2}(x)| \leq \frac{e^{-ix}}{|\mu|^4} J^2 \qquad (j = 1, 2) \tag{4.8}$$

From above it follows that if $|\mu|^2 > J$, then the series

$$\sum_{n=1}^{\infty} (X_{j,n+1}(x) - X_{j,n}(x))$$

and

$$\sum_{n=1}^{\infty} (Y_{j,n+1}(x) - Y_{j,n}(x))$$

are convergent.

Let $X_j(x) = \lim_{n\to\infty} X_{j,n}(x)$ and $Y_j(x) = \lim_{n\to\infty} Y_{j,n}(x)$.

Now for every n

$$\begin{aligned} |X_{j,n}(x)| &\leq |X_{j1}(x)| + |X_{j2}(x) - X_{j1}(x)| + \cdots + |X_{j,n}(x) - X_{j,n-1}(x)| \\ &\leq e^{-tx}[\frac{1}{|\mu|} + \frac{J}{|\mu|^2} + \frac{J^2}{|\mu|^4} + \cdots + \frac{J^{n-1}}{|\mu|^{2(n-1)}} \\ &\leq \frac{e^{-tx}}{\left(1 - \frac{J}{|\mu|^2}\right)}. \end{aligned}$$

Hence for $n \to \infty$

$$|X_j(x)| = \lim_{n\to\infty} |X_{j,n}(x)| \leq \frac{e^{-tx}}{\left(1 - \frac{J}{|\mu|^2}\right)} \qquad (j = 1, 2). \tag{4.9}$$

Similarly

$$|Y_j(x)| = \lim_{n\to\infty} |Y_{j,n}(x)| \leq \frac{e^{-tx}}{\left(1 - \frac{J}{|\mu|^2}\right)} \qquad (j = 1, 2). \tag{4.10}$$

Therefore by dominated convergence, it follows that the limit operation can be taken under the integral sign and that $X_j(x), Y_j(x)$ for $(j = 1, 2)$ satisfy the basic equation. Now

$$X_j'(x) = \frac{e^{i\mu x}}{2} - \int_x^\infty cos(\mu(y-x)) \left\{p(y)X_j(y) - r(y)Y_j(y)\right\} dy. \tag{4.11}$$

Also

$$X_j''(x) = \frac{e^{i\mu x}}{2}[(i\mu) - \mu \int_x^\infty \sin\left[\mu(y-x)\right] \left\{p(y)X_j(y) - r(y)Y_j(y)\right\} dy$$
$$+ \left\{p(y)X_j(y) - r(y)Y_j(y)\right\},$$

i.e.

$$X_j'' + (\lambda - p(x))\, X_j(x) = -r(x)Y_j(x).$$

Simlarly

$$Y_j'' + (\lambda - q(x))\, Y_j(x) = -r(x)X_j(x).$$

Thus $X_j(x)$ and $Y_j(x)$ satisfy the basic equations. Therefore $X_j(x)$ and $Y_j(x)$ are the solutions of basic equation. Let us mark:

$$X_j(x) = \frac{e^{i\mu x}}{2i\mu} + \frac{1}{\mu}\int_x^\infty \left\{p(y)X_j(y) - r(y)Y_j(y)\right\} \sin\left\{\mu(y-x)\right\} dy.$$

Here the term under the integral sign vanishes when $x \to \infty$. Therefore

$$X_j(x) = \frac{e^{i\mu x}}{2i\mu} \cdot [1 + c_{j1}], \tag{4.12}$$

where

$$c_{j1} = o\left[\int_x^\infty \left\{p(y)X_j(y) - r(y)Y_j(y)\right\} \cdot \left\{e^{-i\mu(y-2x)} - e^{-i\mu y}\right\} dy\right] = o(1),$$

for large x.

Similarly

$$Y_j(x) = \frac{e^{i\mu x}}{2i\mu} \cdot [1 + c_{j2}], \tag{4.13}$$

where

$$c_{j2} = o\left[\int_x^\infty \{q(y)Y_j(y) - r(y)X_j(y)\} \cdot \left\{e^{-i\mu(y-2x)} - e^{-i\mu y}\right\} dy\right] = o(1),$$

for large x.

4.1. Theorem

It can be verified that $\phi_j(x, \lambda)$ for $(j = 1, 2)$ satisfy the system of integral equations

$$\left.\begin{aligned} u_j(x,\lambda) &= u_j(0)\cos\mu x + \frac{u'_j(0)}{\mu}\sin\mu x \\ &\quad + \frac{1}{\mu}\int_0^x \{p(y)u_j(y,\lambda) - r(y)v_j(y,\lambda)\}\sin\mu(x-y)dy, \\ v_j(x,\lambda) &= v_j(0)\cos\mu x + \frac{v'_j(0)}{\mu}\sin\mu x \\ &\quad + \frac{1}{\mu}\int_0^x \{q(y)v_j(y,\lambda) - r(y)u_j(y,\lambda)\}\sin\mu(x-y)dy, \end{aligned}\right\} \tag{4.14}$$

where $\lambda = \mu^2$, for real μ such that $|\mu| > 0$ as $x \to \infty$.

Let μ be complex. Then we have from (4.1) for fixed $t > 0$,

$$\begin{aligned} u_j(x,\lambda) &= \frac{1}{2}u_j(0)e^{-i\mu x} - \frac{u'_j(0)}{2i\mu}e^{-i\mu x} + o(e^{-tx}) \\ &\quad - \frac{1}{2i\mu}\int_0^x \{p(y)u_j(y,\lambda) - r(y)v_j(y,\lambda)\}dy \\ &\quad + o\left[\int_0^x \{|p(y)u_j(y,\lambda)| - |r(y)v_j(y,\lambda)|\}e^{-t(x-y)}dy\right], \text{ as } x \to \infty. \end{aligned}$$

The last term becomes

$$o\left\{\int_0^x (|p(y)| - |r(y)|)\, e^{t(2y-x)}dy\right\} = o\left\{\int_0^{\frac{x}{2}} + \int_{\frac{x}{2}}^x \{|p(y)| - |r(y)|\}\, e^{t(2y-x)}dy\right\}$$

$$
\begin{aligned}
&= o\left[\int_0^{\frac{x}{2}} \{|p(y)| - |r(y)|\}\, dy\right] \\
&+ o\left[\int_{\frac{x}{2}}^{x} e^{tx}\left(|p(y)| - |r(y)|\right) dy\right] \\
&= o(e^{tx}) \qquad (j = 1, 2)
\end{aligned}
$$

Also

$$
\begin{aligned}
\int_0^\infty \{p(y)u_j(y,\lambda) - r(y)v_j(y,\lambda)\}\, e^{-\mu(x-y)} dy &= o\left[e^{tx}\int_x^\infty (|p(y)| - |r(y)|) dy\right] \\
&= o(e^{tx}) \qquad (j = 1, 2).
\end{aligned}
$$

Thus

$$
u_j(x,\lambda)v_j(x,\lambda) = o(e^{tx}) \qquad (j = 1, 2) \tag{4.15}
$$

Therefore

$$
\left.\begin{aligned}
u_j(x,\lambda) &= e^{-i\mu x}\{M_{j1}(\lambda) + o(1)\} \\
v_j(x,\lambda) &= e^{-i\mu x}\{M_{j2}(\lambda) + o(1)\}
\end{aligned}\right\} \qquad (j = 1, 2) \tag{4.16}
$$

where

$$
\left.\begin{aligned}
M_{j1}(\lambda) &= \frac{1}{2}u_j(0) - \frac{1}{2i\mu}u'(0) \\
&\quad - \frac{1}{2i\mu}\int_0^\infty e^{i\mu y}\{p(y)u_j(y,\lambda) - r(y)v_j(y,\lambda)\}\, dy \\
M_{j2}(\lambda) &= \frac{1}{2}v_j(0) - \frac{1}{2i\mu}v'(0) \\
&\quad - \frac{1}{2i\mu}\int_0^\infty e^{i\mu y}\{(q(y)v_j(y,\lambda) - r(y)v_j(u,\lambda)\}\, dy
\end{aligned}\right\} \qquad (j = 1, 2) \tag{4.17}
$$

Also for $|\mu| \geq |\mu_0|$

$$
\left.\begin{aligned}
u_j(x,\lambda) &= u_j(0)\cos\mu x + o\left\{\frac{e^|tx|}{|\mu|}\right\} \\
v_j(x,\lambda) &= v_j(0)\cos\mu x + o\left\{\frac{e^|tx|}{|\mu|}\right\},
\end{aligned}\right\} \qquad (j = 1, 2) \tag{4.18}
$$

4.2. Important Result

The two linearly independent solutions of the system (2.1) which are $L^2[0,\infty)$ given by (2.8). Now by (4.16) $u_j(x,\lambda), v_j(x,\lambda)(j=1,,2)$ are large when the imaginary part of λ is large and positive $\phi_j(x,\lambda)(j=1,2)$ are not $L^2[0,\lambda)$. But from (4,12) and (4.13), we see that $X_j(x,\lambda), Y_j(x,\lambda)(j=1,2)$ are small when imaginary part of λ is large and positive. Therefore we conclude that $\phi_j(x,\lambda)$ and $\beta_j(x,\lambda) = \begin{pmatrix} X_j(x,\lambda) \\ Y_j(x,\lambda) \end{pmatrix} (j=1,2)$ are linearly independent. Then

$$\psi_r(x,\lambda) = \sum_{s=1}^{2} K_{rs}(\lambda)\cdot\beta_s(x,\lambda) + \sum_{s=1}^{2} l_{rs}(\lambda)\phi_s(x,\lambda) \qquad (r=1,2). \tag{4.19}$$

Since $\psi_r(x,\lambda)$ $(r=1,2)$ are $L^2[0,\infty)$, but $\phi_j(x,\lambda)(j=1,2)$ are not $L^2[0,\infty)$, therefore, $l_{rs}(\lambda)(1\le r,s\le 2)$. Hence

$$\psi_r(x,\lambda) = \sum_{s=1}^{2} K_{rs}(\lambda)\cdot\beta_s(x,\lambda) \qquad (r=1,2) \tag{4.20}$$

From asymptotic formulae (4.16), (4.12) and (4.13), we obtain as $x\to\infty$.

$$\left.\begin{aligned} u_j'(x,\lambda) &\sim -\mu e^{i\mu x} M_{j1}(\lambda) \\ v_j'(x,\lambda) &\sim -\mu e^{i\mu x} M_{j2}(\lambda) \\ X_j'(x,\lambda) &\sim -\frac{e^{i\mu x}}{2}(1+c_{j1}(\lambda)) \\ u_j'(x,\lambda) &\sim -\frac{e^{i\mu x}}{2}(1+c_{j2}(\lambda)), \end{aligned}\right\} \qquad (j=1,2) \tag{4.21}$$

where dashes denote differentiation with respect to x. Using (4.12),(4.13), (4.16), (4.20) and (4.21) we obtain from (2.9)

$$\left.\begin{aligned} K_{11}\left[M_{11}(1+c_{11}) + M_{12}(1+c_{12})\right] + K_{12}\left[M_{11}(1+c_{21}) + M_{12}(1+c_{22})\right] + 1 = 0 \\ K_{11}\left[M_{21}(1+c_{11}) + M_{22}(1+c_{12})\right] + K_{12}\left[M_{21}(1+c_{21}) + M_{22}(1+c_{22})\right] + 1 = 0 \\ K_{21}\left[M_{11}(1+c_{11}) + M_{12}(1+c_{12})\right] + K_{22}\left[M_{11}(1+c_{21}) + M_{12}(1+c_{22})\right] + 1 = 0 \\ K_{21}\left[M_{21}(1+c_{11}) + M_{22}(1+c_{12})\right] + K_{22}\left[M_{21}(1+c_{21}) + M_{22}(1+c_{22})\right] + 1 = 0 \end{aligned}\right\} \tag{4.22}$$

From (4.22), we get

$$\left.\begin{aligned} K_{11} &= \frac{(M_{21}c_{21}+M_{22}c_{22})+(M_{21}+M_{22})}{(M_{11}M_{22}-M_{12}M_{21})[(c_{12}c_{21}-c_{11}c_{22})+(c_{12}+c_{21}-c_{11}-c_{22})]} \\ K_{12} &= -\frac{(M_{21}c_{11}+M_{22}c_{12})+(M_{21}+M_{22})}{(M_{11}M_{22}-M_{12}M_{21})[(c_{12}c_{21}-c_{11}c_{22})+(c_{12}+c_{21}-c_{11}-c_{22})]} \\ K_{21} &= -\frac{(M_{11}c_{21}+M_{12}c_{22})+(M_{11}+M_{12})}{(M_{11}M_{22}-M_{12}M_{21})[(c_{12}c_{21}-c_{11}c_{22})+(c_{12}+c_{21}-c_{11}-c_{22})]} \\ K_{22} &= \frac{(M_{11}c_{11}+M_{12}c_{12})+(M_{11}+M_{12})}{(M_{11}M_{22}-M_{12}M_{21})[(c_{12}c_{21}-c_{11}c_{22})+(c_{12}+c_{21}-c_{11}-c_{22})]} \end{aligned}\right\} \tag{4.23}$$

Thus from (4.20) and (4.23), we get

$$\psi_{11}(y,\lambda) = -\frac{e^{i\mu y}}{2i\mu}\cdot\frac{M_{22}}{(M_{11}M_{22}-M_{12}M_{21})} \tag{4.24}$$

$$|\psi_{11}(y,\lambda)| = -\frac{|e^{i\mu y}||M_{22}|}{2|\mu||M_{11}M_{22}-M_{12}M_{21}|}$$

i.e.

$$|\psi_{11}(y,\lambda)| < \frac{\alpha\cdot\ e^{i\mu y}}{|\mu|} \tag{4.25}$$

where

$$\alpha = \frac{|M_{22}|}{2|M_{11}M_{22}-M_{12}M_{21}|} = constant.$$

4.3. Convergence Theorem

Let $p(x), q(x)$ and $r(x)$ be continuous, increasing and convex downwards. Suppose

$$\begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix}$$

be $L^2[0,\infty)$. Let $x>0$ and in the neighbourhood of $y=x$. Let $f(y)$ satisfy any condition which is sufficient for the convergence of an ordinary Fourier series, then the series

$$\sum_{n=0}^{\infty} c_n\psi_{1n}(x)\text{and}\sum_{n=0}^{\infty} c_n\psi_{2n}(x) \tag{4.26}$$

converge to the sum $f(x)$.

Let us consider

$$\phi(x,\lambda)=\begin{pmatrix}\phi_1(x,\lambda)\\ \phi_2(x,\lambda)\end{pmatrix}=\int_0^\infty G(x,y;\lambda)f(y)dy \tag{4.27}$$

Therefore

$$\begin{aligned}
\phi_1(x,\lambda) &= \psi_{11}(x,\lambda)\left[\int_0^{x-\delta}+\int_{x-\delta}^{x}\right]u_1(y,\lambda)f_1(y)dy\\
&+u_1(x,\lambda)\left[\int_x^{x+\delta}+\int_{x+\delta}^{x+\Delta}+\int_{x+\Delta}^{\infty}\right]\psi_{21}(y,\lambda)f_1(y)dy\\
&+\psi_{21}(x,\lambda)\left[\int_0^{x-\delta}+\int_{x-\delta}^{x}\right]u_2(y,\lambda)f_1(y)dy\\
&+u_2(x,\lambda)\left[\int_x^{x+\delta}+\int_{x+\delta}^{x+\Delta}+\int_{x+\Delta}^{\infty}\right]\psi_{21}(y,\lambda)f_1(y)dy\\
&+\psi_{11}(x,\lambda)\left[\int_0^{x-\delta}+\int_{x-\delta}^{x}\right]v_1(y,\lambda)f_2(y)dy\\
&+\psi_{21}(x,\lambda)\left[\int_0^{x-\delta}+\int_{x-\delta}^{x}\right]v_2(y,\lambda)f_2(y)dy\\
&+u_1(x,\lambda)\left[\int_x^{x+\delta}+\int_{x+\delta}^{x+\Delta}+\int_{x+\Delta}^{\infty}\right]\psi_{12}(y,\lambda)f_2(y)dy\\
&+u_2(x,\lambda)\left[\int_x^{x+\delta}+\int_{x+\delta}^{x+\Delta}+\int_{x+\Delta}^{\infty}\right]\psi_{22}(y,\lambda f_2(y)dy.
\end{aligned}$$

$$\begin{aligned}
\phi_1(x,\lambda) &= A_1+A_2+A_3+A_4+A_5+A_6+A_7+A_8+A_9+A_{10}+A_{11}+A_{12}\\
&+A_{13}+A_{14}+A_{15}+A_{16}+A_{17}+A_{18}+A_{19}+A_{20}.
\end{aligned} \tag{4.28}$$

where δ and Δ are positive number $\delta<\Delta, \delta<1, \Delta>1$.

Similarly we can write the expression for ϕ_2. We shall integrate $\phi_1(x, \lambda)$ round a contour C in the $\lambda-$ plane which is symmetrical with respect to the real axis and such that its upper half corresponds to the straight lines joining $T, T+iT, iT$ in the plane of $\mu = \sqrt{\lambda}$.
If there is a pole at $\lambda = T^2$, a small loop can be made to the right of it. If there is no eigenvalue to the left of $-T^2$, we have

$$\frac{1}{2\pi i}\int_{c} \phi_1(x,\lambda)d\lambda = \sum_{\lambda_n \leq T^2} c_n \phi_{1n}(x)$$

and

$$\frac{1}{2\pi i}\int_{c} \phi_2(x,\lambda)d\lambda = \sum_{\lambda_n \leq T^2} c_n \phi_{2n}(x).$$

Let

$$I_1 = \left[\int_{T}^{T+i} + \int_{T+i}^{T+iT} + \int_{T+iT}^{T^{-1}+iT} + \int_{T^{-1}+iT}^{iT}\right] A_1(x,\mu^2)2\mu d\mu$$
$$= I_{11} + I_{12} + I_{13} + I_{14}.$$

and similarly for $A_2, A_3, \cdots, A_{20}$. The proof follows exactly following Titchmarch [7, ChapIX].
so that the boundary conditions to be satisfy by any solution

$$[\phi(x,\lambda)\phi_j(x,\lambda)] = 0 \qquad (j = 1, 2).$$

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