International Journal of Pure and Applied Mathematics
Volume 101 No. 3 2015, 413-420
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v101i3.9

ijpam.eu

FRACTIONAL FOURIER TRANSFORM OF A CLASS OF BOEHMIANS

Abhishek Singh
DST Centre for Interdisciplinary Mathematical Sciences
Faculty of Science
Banaras Hindu University
Varanasi- 221 005, India

Abstract: In this paper we have extended the results of fractional Fourier transform to integrable Boehmians given by Zayed [12] and prove its properties. Further, an inversion theorem of the same is established.

AMS Subject Classification: 44A15, 44A35, 46F12, 46F99
Key Words: fractional Fourier transform, convolution, distribution spaces, Boehmians

1. Introduction

Boehmian, which is a class of generalized functions, have been introduced by Boehme [3]. The construction of Boehmians was given by Mikusiński and Mikusiński [5, 7]. Fourier transforms of integrable Boehmians is studied in [6]. Fractional Fourier transform of distributions of compact support was studied by Zayed [12], Bhosale and Chaudhary [2].

The fractional Fourier transform, which is a generalization of the Fourier transform, was first introduced by Namias [8] in order to solve certain classes of quadratic Hamiltonians. Namias applied his technique to the free and the forced quantum mechanical harmonic oscillator. His results were later modified by McBride and Kerr [4], who among other investigations, also developed an

Received: January 24, 2015

operational calculus for the fractional Fourier transform.

The fractional Fourier transform, with angle α of the function $f(t)$, is defined by

$$\mathcal{F}_\alpha[f](w) = F_\alpha(w) = \int_{-\infty}^{\infty} f(t) K_\alpha(t,w)\,dt, \tag{1}$$

where

$$K_\alpha(t,w) = \begin{cases} (c(\alpha)/\sqrt{2\pi}) \exp\{ia(\alpha)[(t^2+w^2) - 2b(\alpha)wt]\} & \text{if } \alpha \neq 0, \frac{\pi}{2}, \pi \\ \delta(t-w) & \text{if } \alpha = 0 \\ \delta(t+w) & \text{if } \alpha = 0 \\ (1/\sqrt{2\pi}) e^{-iwt} & \text{if } \alpha = \frac{\pi}{2} \end{cases}$$

with

$$a(\alpha) = (\cot\alpha)/2, \quad b(\alpha) = \sec\alpha, \quad \text{and} \quad c(\alpha) = \sqrt{1 - i\cot\alpha}\,.$$

As special cases, where $\alpha = 0, \frac{\pi}{2}$, and π, convert the fractional Fourier transform (1) into the classical Fourier transform of f.

Let W be the subspace of the space of all integrable function $L_1(\mathbb{R})$, with the property that $f \in W$ if and only if the Fourier transform $\hat{f}$ of f is also in W. Let f and g be in W and denote their convolution h by $f * g$, i.e.,

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\,dt. \tag{2}$$

Definition 1. For any function $f(x)$, let us define the function $\tilde{f}(x)$ and $\tilde{\tilde{f}}$ by $\tilde{f}(x) = f(x)e^{ia(\alpha)x^2}$ and $\tilde{\tilde{f}} = e^{-ia(\alpha)x^2} f(x)$, respectively. Then for any two functions f and g, the operation $\star$ is given by

$$h(x) = (f \star g)(x) = \frac{c(\alpha)}{\sqrt{2\pi}}\, e^{-iax^2} (\tilde{f} * \tilde{g})(x),$$

where $*$ is defined in (2). And likewise, we define the operation $\otimes$ by

$$(f \otimes g)(x) = \frac{e^{ia(\alpha)x^2}}{\sqrt{2\pi}} (\tilde{\tilde{f}} * \tilde{\tilde{g}})(x).$$

Theorem 1. [12] *Let $h(x) = (f \star g)(x)$ and F_α, G_α, H_α denote the fractional Fourier transforms of f, g and h, respectively. Then*

$$H_\alpha(u) = F_\alpha(u) G_\alpha(u) e^{-ia(\alpha)u^2}. \tag{3}$$

Moreover,

$$\mathcal{F}_\alpha[f(x)g(x)e^{ia(\alpha)x^2}](u) = c(-\alpha)(F_\alpha \otimes G_\alpha)(u). \tag{4}$$

The fractional Fourier transform on generalized function by the embedding method, constructed by Zayed [12].

Definition 2. The fractional Fourier transform of a generalized function f with compact support is defined as

$$\mathcal{F}_\alpha[f](x) = \langle f(t), K_\alpha(t,x)\rangle,$$

where, kernel $k_\alpha(t,x)$, is an infinitely differentiable function in both t and x, it belong to the space ε consisting of all infinitely differentiable functions. The dual space ε^* of ε is the space of generalized function with compact supports. $\mathcal{F}_\alpha(x)$, of the generalized function with compact support f is an infinitely differentiable function that grows no faster than a polynomial on the real axis as $|x| \to \infty$.

Theorem 2. $(Inversion)$[12] *Let $F_\alpha(x)$ is the fractional Fourier transform of generalized function with compact support f. Then*

$$f(t) = \lim_{r\to\infty} \int_{-r}^{r} F_\alpha(x) K_{-\alpha}(t,x)dx$$

in the Schwartz distribution space $\mathcal{D}'$.

Definition 3. [12] The fractional Fourier transform with angle α of an integrable Boehmian $[f_n/\varphi_n]$ is defined by

$$\mathcal{F}_\alpha[f_n/\varphi_n] = [\mathcal{F}_\alpha[f_n]/\mathcal{F}_\alpha[\varphi_n]].$$

If $[f_n/\varphi_n]$ and $[g_n/\psi_n]$ are two different representations of the same integrable Boehmians, i.e. $f_n \star \psi_n = g_n \star \varphi_n$ for all n, then

$$\mathcal{F}_\alpha[f_n \star \psi_n](u) = e^{-ia(\alpha)u^2}\mathcal{F}_\alpha[f_n]\mathcal{F}_\alpha[\psi_n],$$

and

$$\mathcal{F}_\alpha[g_n \star \varphi_n](u) = e^{-ia(\alpha)u^2}\mathcal{F}_\alpha[g_n]\mathcal{F}_\alpha[\varphi_n],$$

which implies that

$$\mathcal{F}_\alpha[f_n \otimes \psi_n][u] = \mathcal{F}_\alpha[g_n \otimes \varphi_n](u),$$

or equivalently

$$\mathcal{F}_\alpha[f_n/\varphi_n] = \mathcal{F}_\alpha[g_n/\psi_n].$$

If $f(x)$ has the fractional Fourier transform $\mathcal{F}_\alpha(f)$ and $\varphi(x)$ has the fractional Fourier transform $\mathcal{F}_\alpha(\varphi)$, then the Parseval identity for fractional Fourier trasform is given by

$$\int_{\infty}^{\infty} f(x)\varphi(x)dx = \int_{\infty}^{\infty} \mathcal{F}_\alpha(f)(\omega)\mathcal{F}_\alpha(\varphi)(\omega)d\omega. \tag{5}$$

In the next Section, we introduce integrable Boehmians and investigate fractional Fourier transform on integrable Boehmians by using Definition 3. Inversion and properties are also discussed in this Section.

2. Fractional Fourier Transformation of Integrable Boehmians

Let L_1 is the space of complex valued Lebesgue integrable functions on the real line $\mathbb{R}$, norm of which in L_1 is $||f|| = \int_{\mathbb{R}} |\ f(x)|\, dx$. If $f, g \in L_1$, then the convolution product $f * g$, i.e., $(f*g)(x) = \int_{\mathbb{R}} f(u)g(x-u)du$ is an element of L_1 and that $||f * g|| \leq ||f|| \cdot ||g||$.

A sequence of continuous real functions $\delta_n \in L_1$ will be called a delta sequence if:

(i) $\int_{\mathbb{R}} \delta_n(x)dx = 1 \ \ , \forall n \in \mathbb{N}$;

(ii) $||\delta_n|| < M$, for some $M \in \mathbb{R}$ and all $n \in \mathbb{N}$, and

(iii) $\lim\limits_{n\to\infty} \int_{|x|>\epsilon} |\delta_n(x)|\, dx = 0$, for each $\epsilon > 0$.

If (φ_n) and (ψ_n) are delta sequences, then $(\varphi_n * \psi_n) \in L_1$. If $f \in L_1$ and (δ_n) is a delta sequence, then $||f * \delta_n - f|| \to 0$ as $n \to \infty$. Delta sequences which appear, are called approximate identities or summability kernels. A pair of sequence (f_n, φ_n) is called a quotient of the sequence, denoted by f_n/φ_n, where $f_n \in L_1(n = 1, 2, \ldots), (\varphi_n)$ is a delta sequence and $f_m * \varphi_n = f_n * \varphi_m, \forall m, n \in \mathbb{N}$. Two quotients of sequences f_n/φ_n and $g_n/\psi_n$are equivalent if $f_n * \psi_n = g_n * \varphi_n, \forall$ $n \in \mathbb{N}$. The equivalence class of quotient of sequence is called an integrable Boehmian, the space of all of which is denoted by B_{L_1}.

A function $f \in L_1$ can be identified with the Boehmian $[f * \delta_n/\delta_n]$, where (δ_n) is a delta sequence. If $F = [f_n/\varphi_n]$, then $f * \delta_n = f_n$. Therefore, $f * \delta_n \in L_1, \forall\ n \in \mathbb{N}$. Two types of convergences for the Boehmian are stuided by Mikusinski [5]. A sequence of Boehmian F_n is called Δ - convergent to Boehmian F ($\Delta - \lim F_n = F$) if there exists a delta sequence (δ_n) such that $(F_n - F) * \delta_n \in L_1$, for every $n \in \mathbb{N}$ and that $||(F_n - F) * \delta_n|| \to 0$ as $n \to \infty$.A sequence of Boehmian F_n is said to be δ - convergent to F ($\delta - \lim F_n = F$) if there exists a delta sequence (δ_n) such that $F_n * \delta_k \in L_1$ and $F * \delta_k \in L_1$, for every $n, k \in \mathbb{N}$ and that $||(F_n - F) * \delta_k|| \to 0$ for each $k \in \mathbb{N}$.

From the canon of above two definitions, we write that if $\Delta - \lim_{n\to\infty} F_n = F$ and $\Delta - \lim_{n\to\infty} G_n = G$, then $\Delta - \lim_{n\to\infty} F_n * G_n = F * G$. If (δ_n) is a delta sequence, then δ_n/δ_n represents an integrable Boehmian. since the Boehmian $[\delta_n/\delta_n]$ corresponds to the dirac delta distribution δ, all the derivatives of δ are also integrable Boehmian. Further, if (δ_n) is infinitely differentiable and bounded, then the k-th derivative of δ is defined as $\delta^{(k)} = [\delta_n^{(k)}/\delta_n]$, $\delta^{(k)} \in B_{L_1}, k \in \mathbb{N}$. The k-th derivative of Boehmian $F \in B_{L_1}$ is defined by $F^{(k)} = F * \delta^{(k)}$. From the continuity of convolution in B_{L_1} , if $\Delta - \lim F_n = F$ then $\Delta - \lim F_n^{(k)} = F^{(k)}$, for $k \in \mathbb{N}$.

Theorem 3. *If $[f_n/\delta_n] \in B_{L_1}$, then the sequence*

$$\mathcal{F}_\alpha[f](u) = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ja[t^2+u^2-2but]} dt \tag{6}$$

converges uniformly on each compact set in $\mathbb{R}$.

Proof. If (δ_n) is a delta sequence, then $\mathcal{F}_\alpha(\delta_n)$ converges uniformly on each compact set to a constant function 1. Hence, for each compact set Q, $\mathcal{F}_\alpha(\delta_k) > 0$ on Q, for almost all $q \in Q$ and moreover,

$$\mathcal{F}_\alpha(f_n) = \mathcal{F}_\alpha(f_n)\frac{\mathcal{F}_\alpha(\delta_q)}{\mathcal{F}_\alpha(\delta_q)} = e^{ia(\alpha)u^2}\frac{\mathcal{F}_\alpha(f_n \star \delta_q)}{\mathcal{F}_\alpha(\delta_q)},$$

$$= e^{ia(\alpha)u^2}\frac{\mathcal{F}_\alpha(f_q \star \delta_n)}{\mathcal{F}_\alpha(\delta_k)} = \frac{\mathcal{F}_\alpha(f_q)}{\mathcal{F}_\alpha(\delta_q)}\mathcal{F}_\alpha(\delta_n), \text{ on } Q. \tag{7}$$

This justifies that the FrFT of an integrable Boehmian $F = [f_n/\delta_n]$ can be defined as the limit of $\mathcal{F}_\alpha(f_n)$ in the space of continuous functions on $\mathbb{R}$. Thus, the FrFt of an integrable Boehmian is a continuous function.

Theorem 4. *Let $F, G \in B_{L_1}$. Then the following results hold:*

(i) $\hat{\mathcal{F}}_\alpha(\lambda F) = \lambda \hat{\mathcal{F}}_\alpha F$;

(ii) $\hat{\mathcal{F}}_\alpha(F + G) = \hat{\mathcal{F}}_\alpha F + \hat{\mathcal{F}}_\alpha G$;

(iii) If $\hat{\mathcal{F}}_\alpha(f) = 0$, *then* $F = 0$;

(iv) If $\Delta - \lim_{n\to\infty} F_n = F$,

then$\mathcal{F}_\alpha(F_n) \to \mathcal{F}_\alpha(F)$ *uniformly on each compact set.*

Proof. Properties (i)-(ii) can easily be proved for integrable Boehmians by invoking the properties of the fractional Fourier transform. By uniqueness theorem of the fractional Fourier transform Property (iii) can easily be proved [9].

To prove (iv), let (δ_n) be a delta sequence such that $F_n * \delta_q, F * \delta_q \in L_1$, $\forall n, q \in \mathbb{N}$ and $||(F_n - F) * \delta_q|| \to 0$, as $n \to \infty$, $\forall q \in \mathbb{N}$ and Q be a compact set in $\mathbb{R}$. Then $\mathcal{F}_\alpha(\delta_q) > 0$ on Q for some $q \in \mathbb{N}$. Since $\mathcal{F}_\alpha(\delta_q)$ is a continuous function, thus, $\mathcal{F}_\alpha(F_n)\cdot\mathcal{F}_\alpha(\delta_q) \to \mathcal{F}_\alpha(F)\,\mathcal{F}_\alpha(\delta_q)$ uniformly on Q. Since $\mathcal{F}_\alpha(F_n)\cdot \mathcal{F}_\alpha(\delta_q) - \mathcal{F}_\alpha(F)\cdot\ \mathcal{F}_\alpha(\delta_q) = \mathcal{F}_\alpha((F_n - F) * \delta_q)$ and $||(F_n - F) * \delta_q|| \to 0$, as $n \to \infty$, hence (iv) is proved. □

Lemma 1. *If* $f \in L_1$*and*

$$f_n(x) = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ja[t^2+u^2-2but]} \mathcal{F}_\alpha(u)dt \tag{8}$$

Then (f_n) *converges to* f *in the* L_1*norm.*

Theorem 5. *Let* $F \in B_{L_1}$ *and*

$$f_n(x) = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ja[t^2+u^2-2but]} \mathcal{F}_\alpha(u)dt$$

Then $\delta - \lim\limits_{n\to\infty} f_n = F$ *(hence also* $\Delta - \lim\limits_{n\to\infty} f_n = F$*).*

Proof. Let $F = [g_n/\varphi_n]$ and $q \in \mathbb{N}$. Then, using parseval relation (5) we obtain,

$$\int_{-\infty}^{+\infty} \mathcal{F}_\alpha(f_n)\mathcal{F}_\alpha(\delta_k)df = \int_{-\infty}^{+\infty} f_n(x)\delta_k(x)dx$$

$$\int_{-\infty}^{+\infty} \mathcal{F}_\alpha(f_n)\mathcal{F}_\alpha(\delta_k)df = \int_{-\infty}^{+\infty} \mathcal{F}_\alpha(f_n) \int_{-\infty}^{+\infty} \delta_k e^{-ja[t^2+u^2-2but]}du\ dt$$

$$= \int_{-\infty}^{+\infty} \delta_k(t)dt \int_{-\infty}^{+\infty} \mathcal{F}_\alpha(f_n)e^{-ja[t^2+u^2-2but]}du$$

$$= \int_{-\infty}^{+\infty} f_n(t)\delta_k(t)dt.$$

Therefore, by Lemma 1, $||f_n * \delta_q - F * \delta_n|| \to 0$, as $n \to \infty$. Since q is an arbirtary positive integer, we have $\delta - \lim\ f_n = F$. Similarly, this can easily be proved for $\Delta - \lim\ f_n = F$. □

Acknowledgments

This work is supported by University Grants Commission, under the Dr. D.S. Kothari Post Doctoral Fellowship, Sanction No. F. 4-2/2006(BSR)/13-663/2012.

References

[1] L.B. Almeida, Product and convolution theorems for the fractional Fourier transform, *IEEE Trans. Signal Processing Letters*, **4** (1997), 15-17.

[2] B.N. Bhosale and M.S. Chaudhary, Fractional Fourier transform of distributions of compact support, *Bull. Cal. Math. Soc.*, **94**, No. 5 (2002), 349-358.

[3] T.K. Boehme, The support of Mikusński Operators, *Trans. Amer. Math. Soc.*, **176** (1973), 319-334.

[4] A. McBride and F. Kerr, On Namias's fractional Fourier transforms, *IMA J. Appl. Math.*, **39** (1987), 159-175

[5] P. Mikusiński, Convergence of Boehmians, *Japan. J. Math.*, **9**, No. 1 (1983), 159-179.

[6] P. Mikusiński, Fourier Transform for Integrable Boehmians, *Rocky Mountain J. Math.*, **17** (1987), 577-582.

[7] P. Mikusiński, Boehmians and generalized functions, *Acta Math. Hungarica*, **51** (1988), 159-179.

[8] V. Namias, The fractional order Fourier transform and its application to quantum mechanics, *J. Inst. math. Appl.*, **25** (1980), 241-265.

[9] H.M. Ozaktas, Z. Zalevsky, M.A. Kutay, *The Fractional Fourier Transform*, Wiley, Chichester (2001).

[10] Abhishek Singh, D. Loonker, and P.K. Banerji, Fourier-Bessel transform for tempered Boehmians, *Int. Journal of Math. Analysis*, **4**, No. 45 (2010), 2199-2210.

[11] Abhishek Singh, P. K. Banerji and S. L. Kalla, A uniqueness theorem for Mellin transform for quotient space, *Sci., Ser. A, Math. Sci. (N.S.)*, **23** (2012), 25-30.

[12] A.I. Zayed, Fractional Fourier transform of generalized functions, *Integ. Trans. Spl. Funct.*, **7** (1998), 299-312.