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# SIMPLER SOLUTION TO GROUP DECISION MAKING PROBLEMS BASED ON INTERVAL-VALUED HESITANT FUZZY SETS AND A NEW APPROACH TO FIND CRITICAL PATH USING HESITANT FUZZY

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**Abstract:** Critical path method is widely used in project scheduling and controlling. The task of finding the crisp critical path has received researcher's attention over the past two decades. It has wide range of applications in planning and scheduling the large projects. The unknowns and vagueness about the time duration for activities in network-planning have led to the development of fuzzy critical path. To express more uncertainty and people's hesitancy in daily life, hesitant fuzzy is a very useful technique. In this paper we introduce a new method to find critical path using hesitant fuzzy in a network where total completion times of a project in more than one path are same. In other words, by considering hesitant fuzzy we can overcome the problem that arises when more than one path are having the same total crisp activity time. We introduce a new operator as well to be used in ranking methods and apply hesitant fuzzy (as hesitant fuzzy is a very useful technique to express people's hesitancy in daily life) for activity times to find out the critical path. To be precise, in this paper a new operator (to be used in ranking methods) and a solution to above mentioned critical path problem (using hesitant fuzzy) have been discussed with the aid of a numerical example.

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## 1. Introduction

The main objective of a project plan is to optimize the resource-utilization so that the overall cost is minimized. This type of management problem can very well be tackled using the network technique called critical path method.Various approaches have been explored for solving fuzzy critical path problems [2], [5], [13], [14], [15] and [16].In reality, it is often difficult to obtain exact estimate of an activity time, due to the uncertainty of information as well as variation in the management scenario. So we can assign membership grades to each of the activity time as per uncertainty and variation of management scenario, and these membership grades (hesitant fuzzy) are used to determine the critical path amongst them.Again,to determine the criticality of the project activities and hence the critical path(s), we faced an issue for certain part of a network where crisp activity durations are same in more than one path. Let us take a network (Fig. 1 rather a part of a network) where we faced the above mentioned problem.

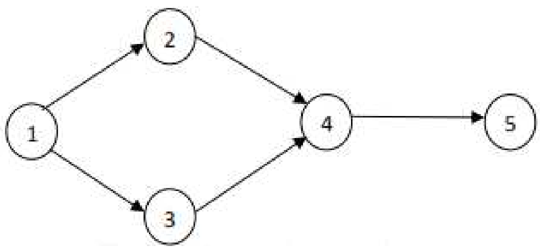


Figure 1: A Fuzzy Project Network.

| Activity | 1 – 2 | 1 – 3 | 2 – 4 | 3 – 4 | 4 – 5 |
|---|---|---|---|---|---|
| Duration(Days) | Approx. 2 | Approx. 3 | Approx 3 | Approx. 2 | Approx. 2 |

Table 1: Fuzzy activity time for each activity in the project.

In this network, the total time durations in both the paths 1-2-4-5 and 1-3-4-5 are same (earliest start time and latest finish time of each activity are also same). To identify the critical path in this situation, we will use hesitant fuzzy (derived from the uncertainty of information as well as variation in the management scenario) for each activity time.Na Chen,Zeshui Xu and Meimei Xia discussed Interval-valued hesitant Fuzzy Preference relations and their applications to group decision making in [1].Consistency measure of hesitant fuzzy preference relation was studied by Bin Zhu [3].Triangular Hesitant Fuzzy Set was successfully applied to teaching quality evaluation by Dejian Yu [4]. In the course of finding the critical path using hesitant fuzzy in this paper, at first we will decide the best hesitant fuzzy set for an activity time from several alternatives. In doing so, we will initially follow the method discussed by Na Chen,Zeshui Xu and Meimei Xia in [1] and finally introduce a new operator to be used in ranking method. This new operator makes the ranking method very simple, very easy to think and calculate and as a result less effort and less time are required to identify the best membership grade for an activity using preference matrix.

This paper is organized as follows: Section 2 gives an introduction of HFSs[11] and [4], Section 3 briefly describes the concept of Interval-valued Hesitant Fuzzy Sets (IVHFS), degree of possibility, score function, distance measure and aggregation operator (IVHFA) for IVHFS which have been earlier introduced in [1] and [12], Section 4 briefly describes fuzzy preference relations (FPRs) and interval valued hesitant fuzzy preference relations (IVHFPRs)[1,3], Section 5 **introduces a new operator to find the score function for interval valued hesitant preference relations to find the best alternative and the steps to be followed to find critical path for a network having same time duration in more than one path**. In Section 6 a numerical example is presented to show how the operator makes the score function derivation for preference relation simpler and easier and validate our approach to find the critical path in above discussed situation. We conclude the paper in Section 7.

## 2. Hesitant Fuzzy Sets

### 2.1. Definition

Let X be a reference set. A hesitant fuzzy set A on X is defined in terms of a function $h_A$(x) that returns a subset of [0,1] when it is applied to X, i.e.

$$A = \{\langle x, h_A(x)\rangle \,|x \in X\},$$

where $h_A$(x) is a set of some different values in [0,1],representing the possible membership degrees of the element x ∈ X to A.

#### 2.1.1. Example

Let $X = \{x_1, x_2, x_3\}$ be a reference set. $h_A(x_1) = \{0.2, 0.4, 0.5\}$, $h_A(x_2) = \{0.3, 0.4\}$, and $h_A(x_3) = \{0.3, 0.2, 0.5, 0.6\}$. A is a HFS, namely

$$A = \{\langle x_1, \{0.2, 0.4, 0.5\}\rangle, \langle x_2, \{0.3, 0.4\}\rangle, \langle x_3, \{0.3, 0.2, 0.5, 0.6\}\rangle\}.$$

### 2.2. Heitant Fuzzy Operation

Operational laws on the HFEs $h$, $h_1$ and $h_2$ have been given by Xia and Xu [9] and Huchang Liao and Zeshui Xu [11].

Let $h$, $h_1$ and $h_2$ be three HFEs, then:

(1) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;

(2) $h_1 \ominus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{t\}$; where

$$t = \begin{cases} \dfrac{\lambda_1 - \lambda_2}{1 - \lambda_2} & \text{if } \lambda_1 \geq \lambda_2 \text{ and } \lambda_2 \neq 1; \\ 0 & \text{otherwise.} \end{cases}$$

## 3. Interval-Valued Hesitant Fuzzy Set

### 3.1. Definition

Let X be a reference set,and D[0, 1] be the set of all closed intervals of [0,1], An IVHFS on X is $A = \{\langle x_i, h_A(x_i)\rangle \,|x_i \in X, i = 1, 2, ..., n\}$ where $h_A(x_i) : X \to D[0, 1]$ denotes all possible interval-valued membership degrees of the element $x_i \in X$ to the set A. For convenience, we call $h_A(x_i)$ an interval-valued hesitant fuzzy element (IVHFE), which reads $h_A(x_i) = \{\tilde{\gamma} : \tilde{\gamma} \in h_A(x_i)\}$ Here $\tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$ is an interval number $\tilde{\gamma}^L = inf\tilde{\gamma}$ and $\tilde{\gamma}^U = sup\tilde{\gamma}$ represent the lower and upper limits of $\tilde{\gamma}$, respectively. An IVHFE is the basic unit of an IVHFS, and it can be considered as a special case of the IVHFS. The relationship between IVHFE and IVHFS is similar to that between interval-valued fuzzy number and interval-valued fuzzy set.

#### 3.1.1. Example

Let $X = \{x_1, x_2\}$ be a reference set, and the IVHFEs
$h_A(x_1) = \{[0.1, 0.3], [0.4, 0.5]\}$ and $h_A(x_2) = \{[0.1, 0.2], [0.3, 0.5], [0.7, 0.9]\}$ denote the membership degrees of $x_i (i = 1, 2)$ to a set $\tilde{A}$ respectively. We call $\tilde{A}$ an IVHFS, where

$$\tilde{A} = \{\langle x_1, \{[0.1, 0.3], [0.4, 0.5]\}\rangle, \langle x_2, \{[0.1, 0.2], [0.3, 0.5], [0.7, 0.9]\}\rangle\}.$$

To facilitate to compare the magnitude of different IVHFEs, we give the properties of interval numbers.

### 3.2. Degree of Possibility

Let $\widetilde{a} = [\widetilde{a}^L, \widetilde{a}^U]$ and $\widetilde{b} = [\widetilde{b}^L, \widetilde{b}^U]$ and let $l_{\widetilde{a}} = \widetilde{a}^U$ - $\widetilde{a}^L$ and $l_{\widetilde{b}} = \widetilde{b}^U$ - $\widetilde{b}^L$; then the degree of possibility of $\widetilde{a} \geq \widetilde{b}$ is formulated by

$$p(\widetilde{a} \geq \widetilde{b}) = \max\left[1 - \max\left\{\left(\frac{\tilde{b}^U - \tilde{a}^L}{l_{\tilde{a}} + l_{\tilde{b}}}\right), 0\right\}, 0\right]. \quad (1)$$

Equation (1) is proposed in order to compare two interval numbers, and to rank all the input arguments. We define the score function for IVHFEs below.

### 3.3. Score Function

For an IVHFE $\tilde{h}$, $s(\tilde{h}) = \frac{1}{l_h}\sum_{\tilde{\gamma}\in\tilde{h}}\tilde{\gamma}$ is called the score function of $\tilde{h}$ with $l_h$ being the number of the interval values in $\tilde{h}$, and $s(\tilde{h})$ is an interval value belonging to [0,1]. For two IVHFEs $\tilde{h_1}$ and $\tilde{h_2}$, if $s(\tilde{h_1}) \geq s(\tilde{h_2})$, then $\tilde{h_1} \geq \tilde{h_2}$.

Note that we can compare two score functions using Equation (1). Moreover, using this Definition we can judge the magnitude of two IVHFEs.

### 3.4. Distance Measures for IVHFEs

Since the number of interval values for different IVHFEs could be different, we arrange them in any order using Eq. (1). To calculate the distance between two IVHFEs, we let $l = \max\{l_\alpha, l_\beta\}$ with $l_\alpha$ and $l_\beta$ being the number of intervals in IVHFEs $\alpha$ and $\beta$. To operate correctly, we give the following regulation: when $l_\alpha \neq l_\beta$, we can make them equivalent through adding elements to the IVHFE that has a less number of elements. In terms of pessimistic principles, the smallest element can be added while the opposite case will be adopted following optimistic principles. In this study we adopt pessimistic or optimistic for our convenience. We arrange the elements in $\alpha$ and $\beta$ in an increasing order. Let $\alpha_{\sigma(i)}$ and $\beta_{\sigma(i)}$ $(i = 1, 2, ..., l)$ be the $i$th values which are smallest (largest) values in case of pessimistic (optimistic) principle in $\alpha$ and $\beta$. Then the distance measures for IVHFEs are given by

$$d_1(\alpha, \beta) = \frac{1}{2l}\sum_{i=1}^{l}\left(\left|\alpha^L_{\sigma(i)} - \beta^L_{\sigma(i)}\right| + \left|\alpha^U_{\sigma(i)} - \beta^U_{\sigma(i)}\right|\right) \quad (2)$$

Equation (2) can be considered as the extension of the well known Hamming distance under the interval-valued hesitant fuzzy environment. They satisfy the following properties:

(1) $0 \leq d(\alpha, \beta) \leq 1$;

(2) $d(\alpha, \beta) = 0$, if and only if $\alpha = \beta$;

(3) $d(\alpha, \beta) = d(\beta, \alpha)$.

### 3.5. Aggregation Operators for Interval-Valued Hesitant Fuzzy Information

#### 3.5.1. Definition of IVHFA Operator

Let $\tilde{h_j}(j = 1, 2...n)$ be a collection of IVHFEs, and $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of $\tilde{h_j}(j = 1, 2...n)$ with $w_j \in$ [0,1],$\sum_{j=1}^{n} w_j = 1$ and $\lambda > 0$, then an interval-valued hesitant fuzzy weighted averaging (IVHWFA) operator is a mapping IVHFWA:$\tilde{H}^n \rightarrow \tilde{H}$,where

$$IVHFWA(\tilde{h_1}, \tilde{h_2}, ..., \tilde{h_n}) = \oplus(w_j\tilde{h_j})$$
$$= \{[1 - \prod_{j=1}^{n}(1 - \tilde{\gamma_j}^L)^{w_j}, 1 - \prod_{j=1}^{n}(1 - \tilde{\gamma_j}^U)^{w_j}]; \tilde{\gamma_1} \in \tilde{h_1}, \tilde{\gamma_2} \in \tilde{h_2}...\tilde{\gamma_n} \in \tilde{h_n}\},$$

and an interval-valued hesitant fuzzy averaging (IVHFA) operator is given by

$$IVHFA(\tilde{h_1}, \tilde{h_2}, ..., \tilde{h_n}) = \oplus(\frac{1}{n}\tilde{h_j})$$
$$= \{[1 - \prod_{j=1}^{n}(1 - \tilde{\gamma_j}^L)^{\frac{1}{n}}, 1 - \prod_{j=1}^{n}(1 - \tilde{\gamma_j}^U)^{\frac{1}{n}}]; \tilde{\gamma_1} \in \tilde{h_1}, \tilde{\gamma_2} \in \tilde{h_2}...\tilde{\gamma_n} \in \tilde{h_n}\}.$$

## 4. Interval-Valued Hesitant Preference Relations

In the process of group decision making (GDM), preference relations are very popular tools to express the decision makers' (DMs) preferences when they compare a set of alternatives. Various types of preference relations have been given for different environments. Orlovsky [6] proposed the concept of fuzzy preference relations.

### 4.1. Definition of Fuzzy Preference Relations

Let $X = \{x_1, x_2, ..., x_n\}$ be a discrete set of alternatives. A fuzzy preference relation R on the set X is represented by a complementary matrix $R = (r_{ij})_{n\times n} \subset X \times X$ with the condition that $r_{ij} \geq 0$; $r_{ij} + r_{ji} = 1$ ; $r_{ii} = 0.5$; for all $i, j = 1, 2, ..., n$, where $r_{ij}$ denotes the preference degree of the alternative $x_i$ over $x_j$. In particular, $r_{ij} = 0.5$ shows no difference between $x_i$ and $x_j$, while $r_{ij} > 0.5$ shows that $x_i$ is preferred to $x_j$, and $r_{ij} < 0.5$ shows that $x_j$ is preferred to $x_i$.

The larger the $r_{ij}$, the greater the preference degree of the alternative $x_i$ over $x_j$. Xu [10] introduced the concept of interval fuzzy preference relations to express the uncertainty and vagueness.

Let $X = \{x_1, x_2, ..., x_n\}$ be a discrete set of alternative. An interval fuzzy preference relation $\tilde{R} = (\tilde{r_{ij}}) \subset X \times X$ which satisfies $(\tilde{r_{ij}}) = [\tilde{r_{ij}}^L, \tilde{r_{ij}}^U]$, $\tilde{r_{ij}}^U \geq \tilde{r_{ij}}^L \geq 0$, $\tilde{r_{ij}}^L + \tilde{r_{ji}}^U = \tilde{r_{ij}}^U + \tilde{r_{ji}}^L = 1$, $\tilde{r_{ii}}^L = \tilde{r_{ii}}^U = 0.5$ for all $i, j = 1, 2 \ldots, n$. where $(\tilde{r_{ij}})$ shows the interval-valued preference degree of the alternative $x_i$ over $x_j$,and $\tilde{r_{ij}}^L$ and $\tilde{r_{ij}}^U$ are the lower and upper limits of $(\tilde{r_{ij}})$respectively. Previous approaches, based on interval-valued fuzzy preference relations, treat a GDM problem through aggregating the individual fuzzy preference relation into the groups one, causing the loss of information. Here we introduce interval-valued hesitant preference relations, allowing the DMs to provide several possible interval fuzzy preference values when they compare two alternatives;that is, it avoids performing information aggregation and directly reflects the differences of preference information of different DMs. If a decision organization, which has multiple experts, provides some interval fuzzy preference values to describe the degrees that $x_i$ is superior to $x_j$, which are denoted by $\tilde{r_{ii}}^1, \tilde{r_{ii}}^2, \tilde{r_{ii}}^3 ..., \tilde{r_{ii}}^L$ then preference information $(\tilde{r_{ij}})$ i.e. $x_i$ is preferred to $x_j$, can be considered as an IVHFE $\tilde{r_{ij}} = \{\tilde{r_{ii}}^s, s = 1, 2, ...l\}$. All $(\tilde{r_{ij}})$ $(i, j = 1, 2, ..., n)$ constitute an interval-valued hesitant preference relation.

### 4.2. Definition of Interval-Valued Hesitant Preference Relation

Let $X = \{x_1, x_2, ..., x_n\}$ be a fixed set. An interval-valued hesitant preference relation on X is denoted by a matrix $\tilde{R} = (\tilde{r_{ij}})$, where

$$(\tilde{r_{ij}}) = \left\{ \{\tilde{r_{ij}}^s\}, s = 1, 2, ...l_{(\tilde{r_{ij}})} \right\}$$

is an IVHFE, indicating all possible degrees to which $x_i$ is preferred to $x_j$ with representing the number of intervals in an IVHFE. Moreover, $(\tilde{r_{ij}})$ should satisfy $\inf \tilde{r_{ij}}^{(\sigma(s))} + \sup \tilde{r_{ji}}^{\sigma(l_{\tilde{r_{ij}}} - s + 1)} = \sup \tilde{r_{ij}}^{(\sigma(s))} + \inf \tilde{r_{ji}}^{\sigma(l_{\tilde{r_{ij}}} - s + 1)} = 1$, $\tilde{r_{ii}} = [0.5, 0.5]$, $i, j = 1, 2, ..., n$. where we arrange the elements in $\tilde{r_{ii}}$ in an increasing order, and let $\tilde{r_{ij}}^{\sigma(s)}$ be the sth smallest value in $\tilde{r_{ij}}$. $\inf \tilde{r_{ij}}^{\sigma(s)}$ and $\sup \tilde{r_{ij}}^{\sigma(s)}$ denote the lower and upper limits of $\tilde{r_{ij}}^{\sigma(s)}$ respectively.

## 5. Algorithm and New Operator to Calculate Score Function for Interval Valued Hesitant Fuzzy Preference Relations

**Step 1.** The decision organizations are to decide the best among a set of alternative durations of a particular activity in a critical path.The decision organizations provide interval preference relations in the form

$$A^{(k)} = \left( a_{ij}{}^{(k)} \right)_{n \times n}.$$

Calculate in this step the IVHFA operator (using eqn. mentioned in Subsection 3.5.1) to aggregate all $a_{ij}{}^{(k)}$ $(j = 1, 2, .., n)$ that correspond to the alternative $x_i$. K denotes the Kth organization.

**Step 2.** Calculate the difference between any two organizations $O_l$ and $O_k$ by equation (2). The distance matrix thus obtained:

$$D_{lk} = \left( d_{ij}{}^{(lk)} \right)_{n \times n} = \left( d\left( a_{ij}{}^{(l)}, a_{ij}{}^{(k)} \right) \right)_{n \times n}.$$

$l, k = 1, 2, ..., m$ ($m$ =number of organizations) where:

(1) $d_{ij}{}^{(lk)} \geq 0$, specially if l=k then $d_{ij}{}^{(lk)} = 0$. i,j = 1,2,...,n;

(2) $d_{ij}{}^{(lk)} = 0$. i=j, meaning diagonal elements are zero;

(3) $d_{ij}{}^{(lk)} = d_{ji}{}^{(lk)}$.

$D_{lk}$, as is seen, is a symmetric matrix. Now compute the average value of the matrix $D_{lk}$ by $d_{lk} = \frac{1}{n^2}$[sum of the elements of the matrix $D_{lk}$]. Deviation of the organization O from the rest organizations: $d_l = \sum_{k=1,k\neq l}^{m} d_{lk}$.The smaller the deviation, the closer the preference information given by $O_l$ and the rest organizations, and hence the larger the weight $w_l$ (given by $O_l$): $w_l = \frac{(d_l)^{-1}}{\sum_{l=1}^{m}(d_l)^{-1}}$, $l = 1, 2.., m$.

**Step 3.** Here we introduce a new operator to find the score function for interval valued hesitant preference relations to find the best alternative. Let $\tilde{a}_i^{(k)} = IVHFA(\tilde{a}_{i1}^{(k)}, \tilde{a}_{i2}^{(k)}, ..., \tilde{a}_{in}^{(k)})$.

If $\tilde{a}_i = IVHFWA(\tilde{a}_i^{(1)}, \tilde{a}_i^{(2)}, ..., \tilde{a}_i^{(k)})$, score function of $\tilde{a}_i$ is determined by the equation (3) given below. This formula makes the calculation easier for the below reasons:

(1) No need to determine $\tilde{a}_i$, i.e.,IVHFWA operator.

(2) Easier to think and calculate, because determining of IVHFWA operator itself is very tedious job and after that the score function has to be calculated.

(3) Using the below new proposed operator, we will get the score function by only one equation. Very less time and minimal effort is required.

The score function for the ith alternative is given by

$$s(\tilde{a}_i) = \left[1 - \frac{1}{l_i^{(1)} l_i^{(2)} ... l_i^{(k)} \left( \sum_{j=1}^{l_i^{(1)}} \left(1 - \tilde{\gamma_{ij}}^{L(1)}\right)^{W_1} \sum_{j=1}^{l_i^{(2)}} \left(1 - \tilde{\gamma_{ij}}^{L(2)}\right)^{W_2} ... \sum_{j=1}^{l_i^{(k)}} \left(1 - \tilde{\gamma_{ij}}^{L(k)}\right)^{W_k} \right)}, 1 - \frac{1}{l_i^{(1)} l_i^{(2)} ... l_i^{(k)} \left( \sum_{j=1}^{l_i^{(1)}} \left(1 - \tilde{\gamma_{ij}}^{U(1)}\right)^{W_1} \sum_{j=1}^{l_i^{(2)}} \left(1 - \tilde{\gamma_{ij}}^{U(2)}\right)^{W_2} ... \sum_{j=1}^{l_i^{(k)}} \left(1 - \tilde{\gamma_{ij}}^{U(k)}\right)^{W_k} \right)} \right]. \quad (3)$$

$l_i^{(k)}$ = number of intervals in IVHFA operator of kth organization for ith alternative,

$\tilde{\gamma_{ij}}^{L(k)}$ = lower value of the jth interval obtained in the IVHFA operator of kth organization for ith alternative,

$\tilde{\gamma_{ij}}^{U(k)}$ = upper value of the jth interval obtained in the IVHFA operator of kth organization for ith alternative,

Computing the score functions of $\tilde{a}_i$ $(i = 1, 2, ..., n)$ by above new formula, rank all the alternatives $x_i$ (i = 1,2,..,n).

**Step 4.** Now use the best alternative (activity time) HFS to use forward pass and backward pass method to find out the critical path for a network.

## 6. Numerical Example

Three estimators are to estimate the degrees of activity timings (crisp activity times are given in table 1) in fig 2. The first estimator thinks the minimum possible degree of time for activity 1-2 is 0.3, the maximum possible is 0.6 and most possible is 0.4 and second estimator thinks (0.4,0.5,0.6) and third estimator thinks (0.4,0.6,0.7). Hence we get three alternatives for activity 1-2: $X_1$, $X_2$, $X_3$. In the similar manner we will get three alternatives for activity 1-3, 2-4, 3-4 and 4-5.

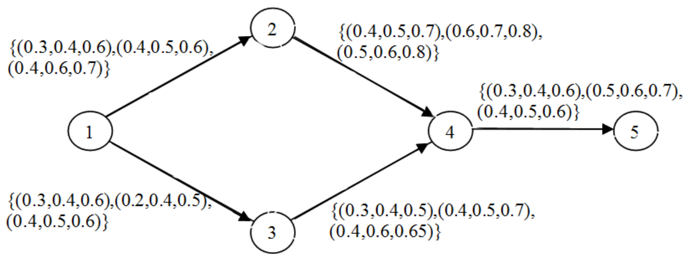


Figure 2: A Fuzzy Project Network.

To select the best alternative from the above three estimates in each activity (given three estimators) two decision organizations $O_k(K = 1, 2)$ are assigned.

Due to uncertainties, the decision makers in each organization give their preference information over alternatives in the form of interval values. Take $O_1$ as an example, the DMs evaluate the degrees to which $X_1$ is preferred to $X_2$. Some give [0.4,0.5] and the other give [0.7,0.9]. Consider that these DMs in the organization $O_1$ can not be persuaded each other, the preference information that $X_1$ is preferred to $X_2$ provided by the decision organization $O_1$ can be considered as an IVHFE, i.e., $[0.4, 0.5], [0.7, 0.9]$. The preference information of these two organizations for the activities 1-2, 2-4, 1-3, 3-4 and 4-5 are given below:

### 6.1. Method of Solution

To get the optimal alternative the following steps are adopted: For activities 1-2, 1-3 and 4-5:

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | [0.5,0.5] | [0.4,0.5], [0.7,0.9] | [0.5,0.6], [0.8,0.9] |
| $X_2$ | [0.1,0.3], [0.5,0.6] | [0.5,0.5] | [0.4,0.5] |
| $X_3$ | [0.1,0.2], [0.4,0.5] | [0.5,0.6] | [0.5,0.5] |

Table 2: The preference relation of the decision organization $O_1$

Table 3: The preference relation for the activities $1-2$, $1-3$, $4-5$ (see Table 2, Table 3)

Table 4: The preference relation of the decision organization $O_2$:.

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | [0.5,0.5] | [0.4,0.5] | [0.6,0.8] |
| $X_2$ | [0.5,0.6] | [0.5,0.5] | [0.3,0.4], [0.5,0.6] |
| $X_3$ | [0.2,0.4] | [0.4,0.5], [0.6,0.7] | [0.5,0.5] |

**The preference relation for the activity 2-4 (table 4,table 5)**

Table 5: The preference relation of the decision organization $O_1$:.

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | [0.5,0.5] | [0.2,0.3], [0.5,0.6] | [0.5,0.6], [0.7,0.9] |
| $X_2$ | [0.4,0.5], [0.7,0.8] | [0.5,0.5] | [0.5,0.8] |
| $X_3$ | [0.1,0.3], [0.4,0.5] | [0.2,0.5] | [0.5,0.5] |

Table 6: The preference relation of the decision organization $O_2$:.

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | [0.5,0.5] | [0.5,0.8] | [0.3,0.5], [0.6,0.7], [0.8,0.9] |
| $X_2$ | [0.2,0.5] | [0.5,0.5] | [0.4,0.5], [0.7,0.8] |
| $X_3$ | [0.1,0.2], [0.3,0.4], [0.5,0.7] | [0.2,0.3], [0.5,0.6] | [0.5,0.5] |

Step 1: Compute the average IVHFE $\tilde{a}_i^{(k)}$ of the alternative $X_i$ (i = 1,2,3) over all the other alternatives for the organization $O_k(k = 1, 2)$ by the IVHFA

**The preference relation for the activity 3-4 (table 6,table 7)**

Table 7: The preference relation of the decision organization $O_1$:.

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | [0.5,0.5] | [0.4,0.5], [0.7,0.8] | [0.6,0.7] |
| $X_2$ | [0.2,0.3], [0.5,0.6] | [0.5,0.5] | [0.4,0.6] |
| $X_3$ | [0.3,0.4] | [0.4,0.6] | [0.5,0.5] |

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $X_1$ | [0.5,0.5] | [0.4,0.6] | [0.7,0.8] |
| $X_2$ | [0.4,0.6] | [0.5,0.5] | [0.3,0.4], [0.5,0.7], [0.8,0.9] |
| $X_3$ | [0.2,0.3] | [0.1,0.2], [0.3,0.5], [0.6,0.7] | [0.5,0.5] |

Table 8: The preference relation of the decision organization $O_2$

operator.

$$\tilde{a}_i^{(k)} = IVHFA(\tilde{a}_{i1}^{(k)}, \tilde{a}_{i2}^{(k)}, \tilde{a}_{i3}^{(k)}).$$

For example

$$\begin{aligned}\tilde{a}_1^{(1)} =& IVHFA(\tilde{a}_{11}^{(1)}, \tilde{a}_{12}^{(1)}, \tilde{a}_{13}^{(1)})\\ =& IVHFA(\{[0.5,0.5],[0.4,0.5],[0.5,0.6]\},\{[0.5,0.5],[0.4,0.5],[0.8,0.9]\},\\ & \{[0.5,0.5],[0.7,0.9],[0.5,0.6]\},\{[0.5,0.5],[0.7,0.9],[0.8,0.9]\})\\ =& \{[0.4653,0.53226],[0.60482,0.70398],[0.57463,0.725],[0.68562,0.82596]\},\end{aligned}$$

Here we have considered $\frac{1}{3} = 0.33$.

The overall results are listed in table below:

$$O_1 \begin{pmatrix} \tilde{a}_1^{(1)} & \{[0.4653,0.53226],[0.60482,0.70398],[0.57463,0.725],[0.68562,0.82596]\}; \\ \tilde{a}_2^{(1)} & \{[0.35084,0.4374],[0.4653,0.53226]\}; \\ \tilde{a}_3^{(1)} & \{[0.38875,0.45379],[0.4653,0.53226]\}; \end{pmatrix}$$

$$O_2 \begin{pmatrix} \tilde{a}_1^{(2)} & \{[0..50326,0.6279]\}; \\ \tilde{a}_2^{(2)} & \{[0.4374,0.50326],[0.49652,0.56547]\}; \\ \tilde{a}_3^{(2)} & \{[0.37559,0.4653],[0.45379,0.54825]\}; \end{pmatrix}.$$

For activity 2-4 the result of Step 1:

$$O_1 \begin{pmatrix} \tilde{a}_1^{(1)} & \{[0.41205,0.47734],[0.50326,0.66922],[0.49652,0.56547],[0.57463,0.725]\}; \\ \tilde{a}_2^{(1)} & \{[0.4653,0.6279],[0.57463,0.725]\}; \\ \tilde{a}_3^{(1)} & \{[0.28619,0.4374],[0.37559,0.49652]\}; \end{pmatrix}$$

$$O_2 \left( \begin{array}{ll} \tilde{a}_1^{(2)} & \{[0.4374, 0.6279], [0.53226, 0.68562], [0.6279, 0.78122]\}; \\ \tilde{a}_2^{(2)} & \{[0.37559, 0.49652], [0.50326, 0.6279]\}; \\ \tilde{a}_3^{(2)} & \{[0.28619, 0.343], [0.38875, 0.45379], [0.343, 0.40251], \\ & [0.4374, 0.50326], [0.41205, 0.52467], [0.49652, 0.60482]\}; \end{array} \right).$$

For activity 3-4 the result of step 1:

$$O_1 \left( \begin{array}{ll} \tilde{a}_1^{(1)} & \{[0.50326, 0.57463], [0.60482, 0.68562]\}; \\ \tilde{a}_2^{(1)} & \{[0.37559, 0.47734]\}; \\ \tilde{a}_3^{(1)} & \{[0.40251, 0.50326]\}; \end{array} \right)$$

$$O_2 \left( \begin{array}{ll} \tilde{a}_1^{(2)} & \{[0.54825, 0.65432]\}; \\ \tilde{a}_2^{(2)} & \{[0.40251, 0.50326], [0.4653, 0.60482], [0.60482, 0.725]\}; \\ \tilde{a}_3^{(2)} & \{[0.28619, 0.343], [0.343, 0.4374], [0.45379, 0.5246]\}; \end{array} \right).$$

Step 2: Derive the weights of the decision organizations. We first use equation (2) to compute $d(\tilde{a}_{ij}^{(l)}, \tilde{a}_{ij}^{(k)})$, $i, j = 1, 2, 3$ and $l, k = 1, 2$.

The difference matrix:

$$D_{lk} = \left(\tilde{d}_{ij}^{(lk)}\right)_{n\times n} = (d(\tilde{a}_{ij}^{(l)}, \tilde{a}_{ij}^{(k)}))_{n\times n}$$

can be thus obtained:

$$D_{12} = D_{21} = \begin{pmatrix} 0 & 0.2 & 0.2 \\ 0.2 & 0 & 0.75 \\ 0.2 & 0.75 & 0 \end{pmatrix},$$

$$d_{12} = d_{21} = 0.105556,$$

$$D_1 = D_2 = d_{12} = 0.105556,$$

$$W_1 = W_2 = 0.5.$$

Similarly for the activity 2-4

$$D_{12} = D_{21} = \begin{pmatrix} 0 & 0.325 & 0.216667 \\ 0.325 & 0 & 0.225 \\ 0.216667 & 0.225 & 0 \end{pmatrix},$$

$$d_{12} = d_{21} = 0.17037,$$

$$D_1 = D_2 = d_{12} = 0.17037,$$

$$W_1 = W_2 = 0.5.$$

Similarly for the activity 3-4

$$D_{12} = D_{21} = \begin{pmatrix} 0 & 0.2 & 0.1 \\ 0.2 & 0 & 0.16667 \\ 0.1 & 0.16667 & 0 \end{pmatrix},$$

$$d_{12} = d_{21} = 0.103704,$$

$$D_1 = D_2 = d_{12} = 0.103704,$$

$$W_1 = W_2 = 0.5.$$

Step 3: Compute the score function using the new operator defined in Section 5. for activity 1-2, 1-3, 4-5:

$$S(a_1) = \big[1 - \frac{1}{4\times 1}[\{(1-.4653)^{.5} + (1-.60482)^{.5} + (1-.57463)^{.5} + (1-.68562)^{.5}\} \times \{(1-.50326)^{.5}\}], 1 - \frac{1}{4\times 1}[\{(1-.53226)^{.5} + (1-.70398)^{.5} + (1-.725)^{.5} + (1-.82596)^{.5}\} \times \{(1-.6279)^{.5}\}]\big] = [0.54668, 0.66914].$$

Similarly $S(a_2) = [0.43916, 0.51102]$, $S(a_3) = [0.42154, 0.50077]$ using equation (1):

$$S(a_1) > S(a_2) > S(a_3) \;\Rightarrow\; X_1 > X_2 > X_3.$$

For activity 2-4

$$S(a_1) = [0.51739, 0.66116],\; S(a_2) = [0.48294, 0.62577],\; S(a_3) = [0.29364, 0.3736]$$

$$S(a_1) > S(a_2) > S(a_3) \;\Rightarrow\; X_1 > X_2 > X_3.$$

For activity 3-4

$$S(a_1) = [0..55188, 0.64344],\; S(a_2) = [0.45917, 0.57203],\; S(a_3) = [0.38304, 0.4714]$$

$$S(a_1) > S(a_2) > S(a_3) \;\Rightarrow\; X_1 > X_2 > X_3.$$

According to the preference matrix out of the three alternatives first alternative is optimal for all the activities.
So we will redraw the previous network in fig.3, taking the optimal values.

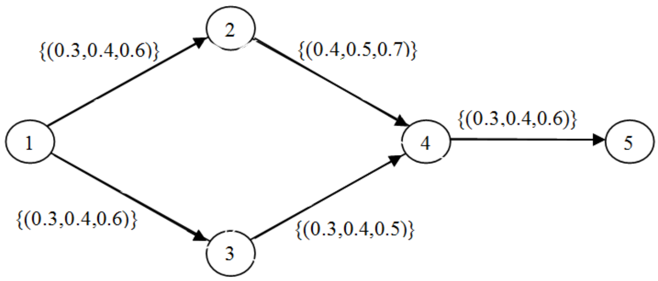


Figure 3: A Fuzzy Project Network.

Step 4. Forward pass method: Here (fig.3) in both paths 1-2-4-5 and 1-3-4-5 the crisp activity duration are same given in Table 1.Since the activity times in fig. 3 are represented in terms of their membership grade,by using addition operation between two membership grades of two crisp activity timings,we represent the membership grade of the sum of that two crisp activity times.

$E_1 = 0$. Here we start with 0 as reference or starting point. We can take any set as reference or starting point except 1. If we take 1, addition and subtraction will have no meaning in hesitant fuzzy. For any other reference, we will reach the same starting reference set after backward pass calculation which is a main criteria in critical path calculation.

Using addition operation of two hesitant fuzzy sets given in Section 2

$$E_2 = E_1 + t_{12} = (0.3, 0.4, 0.6),$$

$$E_3 = E_1 + t_{13} = (0.3, 0.4, 0.6),$$

$$\begin{aligned} E_4 &= \max(E_3 + t_{34}, E_2 + t_{24}) \\ &= \max\{(0.3, 0.4, 0.6) + (0.3, 0.4, 0.5), (0.3, 0.4, 0.6) + (0.4, 0.5, 0.7)\} \\ &= \max\{(0.51, 0.64, 0.8), (0.58, 0.7, 0.88)\} = (0.58, 0.7, 0.88), \end{aligned}$$

as

$$S(0.51, 0.64, 0.8) < S(0.58, 0.7, 0.88),$$

$$E_5 = E_4 + t_{45} = (0.706, 0.82, 0.952).$$

Backward pass method:

$$L_5 = E_5 = (0.706, 0.82, 0.952),$$

using subtraction operation of two hesitant fuzzy sets given in Section 2

$$L_4 = L_5 - t_{45} = (0.58, 0.7, 0.88),$$

$$L_3 = L_4 - t_{34} = (0.4, 0.5, 0.76),$$

$$L_2 = L_4 - t_{24} = (0.3, 0.4, 0.6),$$

$$L_1 = \min\{L_2 - t_{12}, L_3 - t_{13}\} = \min\{0, (0.143, 0.167, 0.4)\} = 0.$$

So critical path is 1-2-4-5 and in this path $E_1 = L_1, E_2 = L_2, E_4 = L_4$ and $E_5 = L_5$.

## 7. Conclusion

In this paper at first we have proposed a new operator to calculate score function for interval valued hesitant preference relation. This operator will be very useful and make calculation easier to find out the optimal and best alternative amongst many given by DMs. Secondly, we introduce how hesitant fuzzy, the most and recent useful technique to express people's hesitancy, can be used to determine critical path.

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