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ON APPLICATION OF DECOMPOSITIONS OF THE KNOWN TWO VARIABLE POLYNOMIALS TO GENERATING SOME IDENTITIES OF TRIGONOMETRIC NATURE

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Abstract: In this paper certain decompositions of the Ma polynomials, classic Cauchy polynomials and Ferrers-Jackson polynomials are used to generating some identities of trigonometric nature. Moreover, the Authors discuss also some potential applications of these decompositions to generating some identities for general recurrence sequences of the second order.

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1. Introduction

In papers [14, 15] the Authors have discussed the applications of the presented below polynomials:

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– Ma Polynomials

$$M_n(x,y) = (x+y)^n\left(x^n+y^n\right) + (-x\,y)^n = \\ = \sum_{k=0}^{\lfloor n/3\rfloor} (-1)^k \frac{n}{n-2k}\binom{n-2k}{k}\left(x^2+x\,y+y^2\right)^{n-3k}\left(x\,y\,(x+y)\right)^{2k}, \quad (1)$$

– Cauchy polynomials

$$p_n(x,y) := (x+y)^{2n+1} - x^{2n+1} - y^{2n+1} = \\ = \sum_{k=0}^{\lfloor (n-1)/3\rfloor} \frac{2n+1}{n-k}\binom{n-k}{2k+1}\left(x\,y\,(x+y)\right)^{2k+1}\left(x^2+x\,y+y^2\right)^{n-1-3k} \quad (2)$$

– Ferrers-Jackson polynomials

$$q_n(x,y) := (x+y)^{2n} + x^{2n} + y^{2n} = \\ = \sum_{k=0}^{\lfloor n/3\rfloor} \frac{2n}{n-k}\binom{n-k}{2k}\left(x\,y\,(x+y)\right)^{2k}\left(x^2+x\,y+y^2\right)^{n-3k}, \quad (3)$$

and their decompositions to generating the limits of quotients of polynomials in two variables. Additionally, we note that Paolo Ribenboim in [8] has presented the other decompositions of two last polynomials (see chapter VII in [8]) together with their applications (for the solutions of some special cases of Fermat's Last Theorem).

Similarity of identity (1) to identities (2) and (3) is not a coincidence, the respective algebraic connections have been described in Theorem 1 of [14]. Among others, it has been proven there that

$$M_{2n+1}(x,y) = \left(x^{2n+1}+y^{2n+1}\right)p_n(x,y) + x^{2(2n+1)} + (x\,y)^{2n+1} + y^{2(2n+1)}, \quad (4)$$

$$M_{2n}(x,y) = \left(x^{2n}+y^{2n}\right)q_n(x,y) - x^{4n} - (x\,y)^{2n} - y^{4n}. \quad (5)$$

In this paper, in Sections 2 and 3 we focus on discussion concerning the application of formulae (1), (2) and (3) for generating the identities of trigonometric nature. Possibility of applying these formulae for elements of some selected recurrence sequences of the second order is also discussed here. Moreover, let us notice that these results essentially complete the Ma's paper [4] and paper [16], made by one of the Authors, where similar type identities have been applied for the powers of elements of some, so called, conjugate recurrence sequences.

2. Some trigonometric identities

In this section we will present some applications of identities (1), (2) and (3) in creating certain nonstandard trigonometric identities.

Setting $x = 4\sin^2\alpha$ and $y = 4\cos^2\alpha$ we can receive from (1), (2) and (3), respectively, the following trigonometric identities

$$\sin^{2n}\alpha + \left(-\sin^2\alpha\,\cos^2\alpha\right)^n + \cos^{2n}\alpha = \\ = \sum_{k=0}^{\lfloor n/3\rfloor} (-1)^k \frac{n}{n-2k}\binom{n-2k}{k}\left(1-\sin^2\alpha\,\cos^2\alpha\right)^{n-3k}(\sin\alpha\,\cos\alpha)^{4k},$$

$$1-\sin^{2(2n+1)}\alpha - \cos^{2(2n+1)}\alpha = \\ = \sum_{k=0}^{\lfloor (n-1)/3\rfloor} \frac{2n+1}{n-k}\binom{n-k}{2k+1}\left(1-\sin^2\alpha\,\cos^2\alpha\right)^{n-3k-1}(\sin\alpha\,\cos\alpha)^{4k+2},$$

$$1+\sin^{4n}\alpha + \cos^{4n}\alpha = \\ = \sum_{k=0}^{\lfloor n/3\rfloor} \frac{2n}{n-k}\binom{n-k}{2k}\left(1-\sin^2\alpha\,\cos^2\alpha\right)^{n-3k}(\sin\alpha\,\cos\alpha)^{4k},$$

or in equivalent form

$$1+(-\sin^2\alpha)^n + \tan^{2n}\alpha = \\ = \sum_{k=0}^{\lfloor n/3\rfloor} (-1)^k \frac{n}{n-2k}\binom{n-2k}{k}\left(1+\sin^2\alpha\tan^2\alpha\right)^{n-3k}\left(\sin\alpha\,\tan\alpha\right)^{2k} \\ = \sum_{k=0}^{\lfloor n/3\rfloor} (-1)^k \frac{n}{n-2k}\binom{n-2k}{k}\left(1-\sin^2\alpha+\tan^2\alpha\right)^{n-3k}\left(-\sin^2\alpha+\tan^2\alpha\right)^{k},$$

$$\left(1+\tan^2\alpha\right)^{2n+1} - 1 - \tan^{4n+2}\alpha = \\ = \sum_{k=0}^{\lfloor (n-1)/3\rfloor} \frac{2n+1}{n-k}\binom{n-k}{2k+1}\left(1+\sin^2\alpha\tan^2\alpha\right)^{n-3k-1}\left(\sin\alpha\,\tan\alpha\right)^{2k+1} \\ = \sum_{k=0}^{\lfloor (n-1)/3\rfloor} \frac{2n+1}{n-k}\binom{n-k}{2k+1}\cos^{2n-4k-1}\alpha\left(1+\tan^2\alpha+\tan^4\alpha\right)^{n-3k-1}\tan^{4k+2}\alpha,$$

$$\begin{aligned}
&\big(1+\tan^2\alpha\big)^{2n}+1+\tan^{4n}\alpha=\\
&\qquad=\sum_{k=0}^{\lfloor n/3\rfloor}\frac{2n}{n-k}\binom{n-k}{2k}\big(1+\sin^2\alpha\tan^2\alpha\big)^{n-3k}\big(\sin\alpha\,\tan\alpha\big)^{4k}\\
&\qquad=\sum_{k=0}^{\lfloor n/3\rfloor}\frac{2n}{n-k}\binom{n-k}{2k}\cos^{2n-4k}\alpha\big(1+\tan^2\alpha+\tan^4\alpha\big)^{n-3k}\tan^{4k}\alpha.
\end{aligned}$$

Taking now $x=1$ and $y=e^{i\varphi}$ we obtain from (1) the identity

$$\begin{aligned}
&2^{n+1}\Big(\cos\frac{\varphi}{2}\Big)^n\cos\frac{n\varphi}{2}+(-1)^n=\\
&\qquad=\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k\frac{n}{n-2k}\binom{n-2k}{k}\big(2\cos\varphi+1\big)^{n-3k}\Big(2\cos\frac{\varphi}{2}\Big)^{2k}. \qquad (6)
\end{aligned}$$

Hence we get, for example

– for $\varphi=\frac{\pi}{2}$:

$$2^{\frac{n+2}{2}}\cos\Big(n\frac{\pi}{4}\Big)+(-1)^n=(1\pm i)^n\big(1+(\pm i)^n\big)+(-1)^n=\sum_{k=0}^{\lfloor n/3\rfloor}(-2)^k\frac{n}{n-2k}\binom{n-2k}{k},$$

– for $\varphi=\frac{\pi}{3}$:

$$2\Big(\frac{\sqrt{3}}{2}\Big)^n\cos\Big(n\frac{\pi}{6}\Big)+\Big(-\frac{1}{2}\Big)^n=\sum_{k=0}^{\lfloor n/3\rfloor}\frac{n}{n-2k}\binom{n-2k}{k}\Big(-\frac{3}{8}\Big)^k, \qquad (7)$$

– for $\varphi=\frac{\pi}{4}$:

$$2\Big(\frac{\sqrt{2}}{1+\sqrt{2}}\Big)^{\frac{n}{2}}\cos\Big(n\frac{\pi}{8}\Big)+\Big(-\frac{1}{1+\sqrt{2}}\Big)^n=\sum_{k=0}^{\lfloor n/3\rfloor}\frac{n}{n-2k}\binom{n-2k}{k}\Big(\frac{-\sqrt{2}}{(1+\sqrt{2})^2}\Big)^k.$$

In the same way, we receive from (3):

$$\begin{aligned}
&\Big(2\cos\frac{\varphi}{2}\Big)^{2n}+2\cos(n\varphi)=\\
&\qquad=\sum_{k=0}^{\lfloor n/3\rfloor}\frac{2n}{n-k}\binom{n-k}{2k}\big(2\cos\varphi+1\big)^{n-3k}\Big(2\cos\frac{\varphi}{2}\Big)^{2k}. \qquad (8)
\end{aligned}$$

From this we obtain, for example

– for $\varphi = \frac{\pi}{2}$:

$$2^{n-1} + \cos\left(n\frac{\pi}{2}\right) = \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{n}{n-k}\binom{n-k}{2k}2^k,$$

– for $\varphi = \frac{\pi}{3}$:

$$\frac{1}{2}\left(\frac{3}{2}\right)^n + \frac{1}{2^n}\cos\left(n\frac{\pi}{3}\right) = \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{n}{n-k}\binom{n-k}{2k}\left(\frac{3}{8}\right)^k$$

(compare with formula (7)),

– for $\varphi = \frac{\pi}{4}$:

$$2^{\frac{n-2}{2}} + \frac{\cos(n\frac{\pi}{4})}{(1+\sqrt{2})^n} = \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{n}{n-k}\binom{n-k}{2k}\left(\frac{\sqrt{2}}{(1+\sqrt{2})^2}\right)^k.$$

Remark 1. If we set in (1) and (3) $x = e^{i\alpha}$ and $y = e^{i\beta}$, then we deduce again the formulae (6) and (8), respectively, for $\varphi := \alpha - \beta$.

At last, if we take $x = \cos\varphi$ and $y = i\sin\varphi$ in (2), we get

$$e^{i\,n\,\varphi}\left(\cos^n\varphi + (i\sin\varphi)^n\right) + \left(-\frac{i}{2}\sin 2\varphi\right)^n =$$
$$= \sum_{k=0}^{\lfloor n/3 \rfloor} \frac{n}{n-2k}\binom{n-2k}{k}\left(\cos 2\varphi + \frac{i}{2}\sin 2\varphi\right)^{n-3k}\left(\frac{1}{2}\sin 2\varphi\right)^{2k} e^{i\,2k\,\varphi},$$

since $i^{2k} = (-1)^k$. Hence we obtain

– for $\varphi = \frac{\pi}{4}$ (after some manipulations by using relations $e^{i\pi} = -1$ and $i = e^{i\pi/2}$):

$$\sum_{k=0}^{\lfloor n/3 \rfloor} \frac{n}{n-2k}\binom{n-2k}{k}(-2)^k = 2^{\frac{n}{2}+1}\cos\left(\frac{\pi}{4}n\right) + (-1)^n, \quad (9)$$

which implies

$$\sum_{k=1}^{\lfloor (4n+2)/3 \rfloor} \frac{2n+1}{2n+1-k}\binom{4n+2-2k}{k}(-2)^k = 0, \quad (10)$$

– for $\varphi = \frac{\pi}{3}$ and $n := 3n$:

$$(-8)^n \left(1 + \left(-i\sqrt{27}\right)^n\right) + \left(\sqrt{27}i\right)^n = \\ = \sum_{k=0}^{n} \frac{3n}{3n-2k}\binom{3n-2k}{k}\left(10 + i\,9\sqrt{3}\right)^{n-k}\left(6\left(-1+i\sqrt{3}\right)\right)^k, \quad (11)$$

which is a really attractive identity,

– for $\varphi = \frac{\pi}{8}$ we obtain the identities (see also [9, 12, 13]):

$$\left(2\sqrt{2}\right)^n e^{i\frac{\pi}{8}n}\left(\cos^n\frac{\pi}{8} + i^n \sin^n\frac{\pi}{8}\right) + (-i)^n = \\ = \sum_{k=0}^{\lfloor n/3\rfloor} \frac{n}{n-2k}\binom{n-2k}{k} 2^{3k/2}(2+i)^{n-3k} e^{i\frac{\pi}{4}k} \quad (12)$$

and

$$\left(2\sqrt{2}\right)^n e^{i\frac{\pi}{8}n} = \left(\sqrt{4+2\sqrt{2}} + i\sqrt{4-2\sqrt{2}}\right)^n,$$

since $\cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}}$, $\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$.

Remark 2. Identity (12) suggests, however false, supposition that either $\cos\frac{\pi}{8} \in \mathbb{Q}(\sqrt{2})$ or $\sin\frac{\pi}{8} \in \mathbb{Q}(\sqrt{2})$.

Similarly, we observe that neither $\cos\frac{\pi}{2^{n+1}} \in \mathbb{Q}\left(\cos\frac{\pi}{2^n}, \sin\frac{\pi}{2^n}\right)$ nor $\sin\frac{\pi}{2^{n+1}} \in \mathbb{Q}\left(\cos\frac{\pi}{2^n}, \sin\frac{\pi}{2^n}\right)$, which easily follows from identities $\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$. More precisely, the minimal polynomials of $\cos\frac{\pi}{2^n}$ and $\sin\frac{\pi}{2^n}$ possess the degree equal to 2^{n-1} (see papers [1, 11], for additional discussion see also papers [3, 5, 6]).

3. Applications to the recurrence sequences

Identities (1), (2) and (3) can be successfully applied for the successive elements of the following linear recurrence sequence of second order (more precisely, for $x = aR_n$, $y = bR_{n-1}$):

$$\begin{aligned} R_0, R_1 &\in \mathbb{C}, \\ R_{n+1} &= aR_n + bR_{n-1}, \qquad n = 1, 2, \ldots, \end{aligned} \quad (13)$$

where $a, b \in \mathbb{C}$. Certainly, to ensure the effective application of these identities in calculations one has to take care of the "gentle" form of expression

$$x^2 + xy + y^2 = a^2 R_n^2 + abR_n R_{n-1} + b^2 R_{n-1}^2 = a^2 R_n^2 + bR_{n-1}R_{n+1}. \qquad (14)$$

By using the Binet formula for R_n (for simplicity of discussion let us assume that the characteristic polynomial of equation (13) possesses two different complex roots α and β), that is the formula

$$R_n = A\alpha^n + B\beta^n,$$

we find

$$\begin{aligned} a^2 R_n^2 + bR_{n-1}R_{n+1} &= (a^2+b)R_n^2 + b(2AB(\alpha\beta)^n + (\alpha^2+\beta^2)AB(\alpha\beta)^{n-1}) \\ &= (a^2+b)R_n^2 + bAB(-2b+\alpha^2+\beta^2)(-b)^{n-1} \\ &= (a^2+b)R_n^2 - a^2 AB(-b)^n. \qquad (15) \end{aligned}$$

Let us also notice that from the system of equations

$$\begin{cases} A + B = R_0, \\ \alpha\, A + \beta\, B = R_1, \end{cases}$$

we get the system

$$\begin{cases} (\beta - \alpha)\, B = R_1 - \alpha\, R_0, \\ (\alpha - \beta)\, A = R_1 - \beta\, R_0, \end{cases}$$

from which, by multiplying both equations, we obtain

$$AB = \frac{-(R_1^2 - aR_0R_1 - bR_0^2)}{(\alpha-\beta)^2} = \frac{bR_0^2 + aR_0R_1 - R_1^2}{a^2+4b}.$$

From formula (15) it results that if $a^2 + b = 0$ then formulae (1)-(3) are numerically effective, otherwise these identities do not represent any significant numerical value.

Of course one can also use the substitution of type $x = (a^2+b)R_n$, $y = abR_{n-1}$ (then $x + y = R_{n+2}$) which, for instance, in case of sequences of Fibonacci or Lucas type generate the gentle form of expression $x^2 + xy + y^2$, however we will omit them here.

Remark 3. Additionally let us notice that identities of type (1) - (3) for the Fibonacci and Lucas polynomials, i.e. for polynomials

$$F_{n+1}(x) = x\,F_n(x) + F_{n-1}(x), \qquad n \in \mathbb{N},$$

$$F_0(x) = 0, \qquad F_1(x) = 1,$$

and

$$L_{n+1}(x) = x\,L_n(x) + L_{n-1}(x), \quad n \in \mathbb{N},$$
$$L_0(x) = 2, \qquad L_1(x) = x,$$

respectively, where $x \in \mathbb{C}$, have also a trigonometric nature, like in Section 4, with respect to the following connections of $F_n(x)$ and $L_n(x)$ with the n-th Chebyshev polynomials $U_n(x)$ and $T_n(x)$ of the second and first kind, respectively (see for instance [2], [9]):

$$i^{n-1} F_n(x) = U_{n-1}\Big(\frac{1}{2}\,i\,x\Big),$$

where $U_{n-1}(\cos\varphi) = \frac{\sin(n\varphi)}{\sin\varphi}$, for $n = 1, 2, ...$, and

$$i^n L_n(x) = 2\,T_n\Big(\frac{1}{2}\,i\,x\Big),$$

where $T_n(\cos\varphi) = \cos(n\varphi)$, for $n = 1, 2, ...$

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