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FURTHER RESULT ON A REPAIRABLE, STANDBY, HUMAN AND MACHINE SYSTEM

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Abstract: On the basis of our previous work, we study asymptotic behavior of the time-dependent solution of a repairable, standby, human & machine system and prove that its time-dependent solution exponentially converges to its steady-state solution.

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1. Introduction

According to Sridharan et al. [7], a repairable, standby, human & machine system can be described by the following system of partial differential equations:

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$$\begin{aligned}\frac{dp_0(t)}{dt} &= -(\lambda + \lambda_{c_0} + \lambda_{h_0} + \eta)p_0(t)\\ &\quad + \sum_{i=1}^{2} \mu_i p_i(t) + \sum_{i=3}^{5} \int_0^\infty \mu_i(x) p_i(x,t)dx, \qquad (1)\\ \frac{dp_1(t)}{dt} &= \lambda p_0(t) - (\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)p_1(t), \qquad (2)\\ \frac{dp_2(t)}{dt} &= \eta p_0(t) - (\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda)p_2(t), \qquad (3)\\ \frac{\partial p_i(x,t)}{\partial t} + \frac{\partial p_i(x,t)}{\partial x} &= -\mu_i(x)p_i(x,t), \quad i = 3,4,5, \qquad (4)\\ p_3(0,t) &= \lambda[p_1(t) + p_2(t)], \quad t > 0, \qquad (5)\\ p_4(0,t) &= \sum_{i=0}^{2} \lambda_{c_i} p_i(t), \quad t > 0, \qquad (6)\\ p_5(0,t) &= \sum_{i=0}^{2} \lambda_{h_i} p_i(t), \quad t > 0, \qquad (7)\\ p_0(0) &= \phi_0 \ge 0,\ p_i(0) = \phi_i \ge 0,\ i = 1,2;\\ p_j(x,0) &= \psi_j(x) \ge 0,\ j = 3,4,5. \qquad (8)\end{aligned}$$

Here $(x,t) \in [0,\infty) \times [0,\infty)$, $\sum_{i=0}^{2} \phi_i + \sum_{j=3}^{5} \int_0^\infty \psi_j(x)dx = 1$, $p_i(t)$ represents the probability that the system is in state i at time t $(i = 0,1,2)$; $p_j(x,t)$ represents the probability that at time t, the failed system is in state j $(j = 3,4,5)$ and has an elapsed repair time of x; λ_{c_i} represents common-cause failure rates from state i to state 4, $i = 0,1,2$; λ_{h_i} represent human-error rates from state i $(i = 0,1,2)$ to state 5; η represents hardware failure rate for standby unit; λ represents hardware failure rate for operating unit; μ_i represents repair rate of failed unit in state i ($i = 1,2$); $\mu_j(x)$ represents time-dependent system repair rate when system is in state j and has an elapsed repair time of x which satisfies

$$\mu_j(x) \ge 0, \qquad \int_0^\infty \mu_j(x)dx = \infty, \quad j = 3,4,5.$$

λ_{c_i} $(i = 0,1,2)$, λ_{h_i} $(i = 0,1,2)$, λ, μ_i $(i = 1,2)$ and η are positive constants.

In 1998, Sridharan et al. [7] established the above model by using the supplementary variable technique and studied its time-dependent availability by using Laplace transform, discovered the time-dependent availability decreases as time increases for exponential repair time distribution. The availability of the system depends on the solution of the model. Although Sridharan et al. [7]

used the steady-state solution and the time-dependent solution during calculating the system availability, they did not discuss the time-dependent solution of the system and its asymptotic behavior. In 2003, Gupur [1] firstly converted the model into an abstract Cauchy problem in a Banach space, then by using the Hille-Yosida theorem, the Phillips theorem and the Fattorini theorem proved that the above model has a unique positive time-dependent solution which satisfies the probability condition. In 2006, Gupur [2] studied asymptotic behavior of the time-dependent solution of the above model and obtained that the C_0-semigroup generated by the underlying operator, which corresponds to the model, is a quasi-compact operator, 0 is an eigenvalue of the underlying operator with algebraic multiplicity one and therefore, the C_0-semigroup converges exponentially to a projection operator. Hence, he deduced that the time-dependent solution of the system (1)-(8) converges strongly to its steady-state solution. When the repair rates $\mu_j(x)$ are constants, by using theorem 14 in Gupur et al. [5] (theorem 1.96 in Gupur [3]) Xu [8] studied the asymptotic behavior of the time-dependent solution of the above model and obtained that its time-dependent solution strongly converges to its steady-state solution. Until now, any other results have not been found in the literature.

In this paper, by using the results in Gupur [2] and the idea in Gupur [4] we show that the essential growth bound of the C_0- semigroup generated by the underlying operator, which corresponds to the system (1)-(8), is less than a negative number, thus we deduce that 0 is an isolated eigenvalue of the underlying operator and 0 is a pole of order 1. Finally, by using the residue theorem we determine the expression of the projection operator and conclude that the time-dependent solution of the system (1)-(8) exponentially converges to its steady-state solution. Our result implies that convergence of the time-dependent solutions of the reliability models which are described by finite partial differential equations with integral boundary conditions is quite different from reliability models which are described by infinite number of partial differential equations with integral boundary conditions (see Gupur [3]).

In this paper, we use the notations in Gupur [1, 2]. For simplicity, we introduce a notation as follows.

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \lambda & \lambda & 0 & 0 & 0 \\ \lambda_{c_0} & \lambda_{c_1} & \lambda_{c_2} & 0 & 0 & 0 \\ \lambda_{h_0} & \lambda_{h_1} & \lambda_{h_2} & 0 & 0 & 0 \end{pmatrix}.$$

Take state space X as follows:

$$X = \left\{ p \in \mathbb{R}^3 \times \left(L^1[0,\infty)\right)^3 \;\middle|\; \|p\| = \sum_{i=0}^{2} |p_i| + \sum_{i=3}^{5} \|p_i\|_{L^1[0,\infty)} \right\}.$$

It is obvious that X is a Banach space. Define operators and their domains as follows.

$$D(A) = \left\{ p \in X \;\middle|\; \begin{array}{l} \frac{dp_i(x)}{dx} \in L^1[0,\infty),\ p_i(x)\ (i = 3,4,5) \\ \text{is absolutely continuous and } p(0) = \Gamma p(x) \end{array} \right\},$$

$$A\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix} = \begin{pmatrix} -(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta) & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{dx}-\mu_3(x) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{d}{dx}-\mu_4(x) & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{d}{dx}-\mu_5(x) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix}.$$

$$B\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix}, \quad D(B) = X,$$

$$E\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix} = \begin{pmatrix} \sum_{i=3}^{5}\int_0^\infty \mu_i(x)p_i(x)dx \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & \mu_1 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix}, \; D(E) = X.$$

Then the system of the above equations (1)–(8) can be written as an abstract Cauchy problem in the Banach space X which was given in Gupur [1]:

$$\begin{cases} \frac{dp(t)}{dt} = (A + B + E)p(t), & \forall t \in [0, \infty), \\ p(0) = (\phi_0, \phi_1, \phi_2, \psi_3, \psi_4, \psi_5). \end{cases} \tag{9}$$

In 2003, Gupur [1] have obtained the following results:

Theorem 1. *$A + B$ generates a positive contraction C_0- semigroup $S(t)$. $A + B + E$ generates a positive contraction C_0- semigroup $T(t)$ which is isometric for the initial value. Hence, the system (9) has a unique positive time-dependent solution $p(x, t) = T(t)p(0)$ satisfying*

$$\sum_{i=0}^{2} p_i(t) + \sum_{j=3}^{5}\int_0^\infty p_j(x, t)dx = 1, \quad \forall t \in [0, \infty).$$

For $\phi \in X$ we define two operators as follows.

$$(U(t)\phi)(x) = \begin{cases} 0 & x \in [0, t) \\ (S(t)\phi)(x) & x \in [t, \infty) \end{cases},$$

$$(V(t)\phi)(x) = \begin{cases} (S(t)\phi)(x) & x \in [0, t) \\ 0 & x \in [t, \infty) \end{cases},$$

then $S(t)\phi = U(t)\phi + V(t)\phi, \quad \forall \phi \in X$.

In 2006, Gupur [2] have proved the following results:

Theorem 2. *$V(t)$ is a compact operator on X. Assume that there exist two positive constants $\overline{\mu}$ and $\underline{\mu}$ such that $0 < \underline{\mu} \leq \mu_i(x) \leq \overline{\mu} < \infty$ for $i = 3, 4, 5$, then*

$$\|U(t)\| \leq e^{-\min\{\underline{\mu}, \lambda+\lambda_{c_0}+\lambda_{h_0}+\eta, \mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda, \mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda\}t} \times \left\{4 + \frac{2\lambda}{|\mu_1 + \lambda_{c_1} + \lambda_{h_1} - \lambda_{c_0} - \lambda_{h_0} - \eta|} + \frac{2\eta}{|\mu_2 + \lambda_{c_2} + \lambda_{h_2} - \lambda_{c_0} - \lambda_{h_0} - \eta|}\right\}.$$

Hence, $S(t)$ and $T(t)$ are quasi-compact operators on X. 0 is an eigenvalue of $A + B + E$ and $(A + B + E)^*$ with geometric multiplicity one. So, there exist a positive projection operator $\mathbb{P}$ with rank one and suitable positive constants $\delta > 0$ and $\mathbb{M} \geq 0$ such that

$$\|T(t) - \mathbb{P}\| \leq \mathbb{M}e^{-\delta t},$$

where $\mathbb{P} = \frac{1}{2\pi i} \int_{\overline{\Gamma}} (zI - A - B - E)^{-1} dz$, $\overline{\Gamma}$ is a circle with center 0 and sufficiently small radius. In particular, the time-dependent solution of the system (9) converges strongly to its steady-state solution as time tends to infinity, that is,

$$\lim_{t\to\infty} \|p(\cdot, t) - \alpha p(\cdot)\| = 0,$$

here $p(x)$ is an eigenvector corresponding to 0.

In this paper, we determine the expression of $\mathbb{P}$ and by using theorem 2 deduce our desired result.

2. Main Results

Lemma 3. *For $\gamma \in \rho(A + B + E)$ we have*

$$(\gamma I - A - B - E)^{-1} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}, \forall z \in X,$$

here

$$y_0 = \frac{|H_0(\gamma)|}{|H(\gamma)|}, \quad y_1 = \frac{|H_1(\gamma)|}{|H(\gamma)|}, \quad y_2 = \frac{|H_2(\gamma)|}{|H(\gamma)|},$$

$$y_3(x) = \left(\lambda \frac{|H_1(\gamma)|}{|H(\gamma)|} + \lambda \frac{|H_2(\gamma)|}{|H(\gamma)|}\right) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} + e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} \int_0^x z_3(\tau) e^{\gamma\tau + \int_0^\tau \mu_3(\xi) d\xi} d\tau,$$

$$y_4(x) = \left(\lambda_{c_0} \frac{|H_0(\gamma)|}{|H(\gamma)|} + \lambda_{c_1} \frac{|H_1(\gamma)|}{|H(\gamma)|} + \lambda_{c_2} \frac{|H_2(\gamma)|}{|H(\gamma)|}\right) \times e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} + e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} \int_0^x z_4(\tau) e^{\gamma\tau + \int_0^\tau \mu_4(\xi) d\xi} d\tau,$$

$$y_5(x) = \left(\lambda_{h_0} \frac{|H_0(\gamma)|}{|H(\gamma)|} + \lambda_{h_1} \frac{|H_1(\gamma)|}{|H(\gamma)|} + \lambda_{h_2} \frac{|H_2(\gamma)|}{|H(\gamma)|}\right) \times e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} + e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} \int_0^x z_5(\tau) e^{\gamma\tau + \int_0^\tau \mu_5(\xi) d\xi} d\tau,$$

$$\begin{aligned} |H(\gamma)| = \Big(&\gamma + \lambda + \lambda_{c_0} + \lambda_{h_0} + \eta \\ &- \lambda_{c_0} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\ &- \lambda_{h_0} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx\Big) \\ &\times (\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)(\gamma + \mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\ &- \eta(\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\ &\times \Big(\mu_2 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\ &+ \lambda_{c_2} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\ &+ \lambda_{h_2} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx\Big) \\ &- \lambda(\gamma + \mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\ &\times \Big(\mu_1 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \end{aligned}$$

$$+ \lambda_{c_1} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx$$
$$+ \lambda_{h_1} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Big),$$

$$\begin{aligned}|H_0(\gamma)| = &\Bigg(\sum_{i=3}^{5} \int_0^\infty \mu_i(x) e^{-\gamma x - \int_0^x \mu_i(\xi) d\xi} \int_0^x z_i(\tau) e^{\gamma\tau + \int_0^\tau \mu_i(\xi) d\xi} d\tau dx \\
&+ z_0 \Bigg) (\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\
&\times (\gamma + \mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&+ z_2 (\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\
&\times \Big(\mu_2 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\
&+ \lambda_{c_2} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&+ \lambda_{h_2} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Big) \\
&+ z_1 (\gamma + \mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&\times \Big(\mu_1 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\
&+ \lambda_{c_1} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&+ \lambda_{h_1} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Big),\end{aligned}$$

$$\begin{aligned}|H_1(\gamma)| = z_1 &\Big(\gamma + \lambda + \lambda_{c_0} + \lambda_{h_0} + \eta \\
&- \lambda_{c_0} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&- \lambda_{h_0} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Big) \\
&\times (\gamma + \mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&+ z_2 \lambda \Big(\mu_2 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_{c_2} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&+ \lambda_{h_2} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Bigg) \\
&- z_1 \eta \Bigg(\mu_2 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\
&+ \lambda_{c_2} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&+ \lambda_{h_2} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Bigg) \\
&+ \lambda \Bigg(\sum_{i=3}^{5} \int_0^\infty \mu_i(x) e^{-\gamma x - \int_0^x \mu_i(\xi) d\xi} \int_0^x z_i(\tau) e^{\gamma \tau + \int_0^\tau \mu_i(\xi) d\xi} d\tau dx \\
&+ z_0 \Bigg) (\gamma + \mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda),
\end{aligned}
$$

$$
\begin{aligned}
|H_2(\gamma)| = {} & z_2 (\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\
&\times \Bigg(\gamma + \lambda + \lambda_{c_0} + \lambda_{h_0} + \eta \\
&- \lambda_{c_0} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&- \lambda_{h_0} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Bigg) \\
&+ z_1 \eta \Bigg(\mu_1 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\
&+ \lambda_{c_1} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&+ \lambda_{h_1} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Bigg) \\
&+ \eta \Bigg(\sum_{i=3}^{5} \int_0^\infty \mu_i(x) e^{-\gamma x - \int_0^x \mu_i(\xi) d\xi} \int_0^x z_i(\tau) e^{\gamma \tau + \int_0^\tau \mu_i(\xi) d\xi} d\tau dx \\
&+ z_0 \Bigg) (\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\
&- z_2 \lambda \Bigg(\mu_1 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx
\end{aligned}
$$

$$+\lambda_{c_1}\int_0^\infty \mu_4(x)e^{-\gamma x-\int_0^x \mu_4(\xi)d\xi}dx$$
$$+\lambda_{h_1}\int_0^\infty \mu_5(x)e^{-\gamma x-\int_0^x \mu_5(\xi)d\xi}dx\Big).$$

Proof. For any given $z \in X$ we consider the equation $(\gamma I - A - B - E)y = z$ which is equivalent to

$$(\gamma+\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)y_0 = \sum_{i=1}^{2}\mu_i y_i + \sum_{i=3}^{5}\int_0^\infty \mu_i(x)y_i(x)dx + z_0, \tag{10}$$
$$(\gamma+\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)y_1 = \lambda y_0 + z_1, \tag{11}$$
$$(\gamma+\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)y_2 = \eta y_0 + z_2, \tag{12}$$
$$\frac{dy_i(x)}{dx} = -(\gamma+\mu_i(x))y_i(x) + z_i(x), \quad i = 3,4,5, \tag{13}$$
$$y_3(0) = \lambda y_1 + \lambda y_2, \tag{14}$$
$$y_4(0) = \lambda_{c_0}y_0 + \lambda_{c_1}y_1 + \lambda_{c_2}y_2, \tag{15}$$
$$y_5(0) = \lambda_{h_0}y_0 + \lambda_{h_1}y_1 + \lambda_{h_2}y_2. \tag{16}$$

By solving (13) we have

$$y_i(x) = d_i e^{-\gamma x-\int_0^x \mu_i(\xi)d\xi} + e^{-\gamma x-\int_0^x \mu_i(\xi)d\xi}\int_0^x z_i(\tau)e^{\gamma\tau+\int_0^\tau \mu_i(\xi)d\xi}d\tau,\ i = 3,4,5. \tag{17}$$

By combining (17) with (14), (15) and (16) we deduce

$$y_3(x) = (\lambda y_1+\lambda y_2)e^{-\gamma x-\int_0^x \mu_3(\xi)d\xi} + e^{-\gamma x-\int_0^x \mu_3(\xi)d\xi}\int_0^x z_3(\tau)e^{\gamma\tau+\int_0^\tau \mu_3(\xi)d\xi}d\tau, \tag{18}$$
$$y_4(x) = (\lambda_{c_0}y_0+\lambda_{c_1}y_1+\lambda_{c_2}y_2)e^{-\gamma x-\int_0^x \mu_4(\xi)d\xi} + e^{-\gamma x-\int_0^x \mu_4(\xi)d\xi}\int_0^x z_4(\tau)e^{\gamma\tau+\int_0^\tau \mu_4(\xi)d\xi}d\tau, \tag{19}$$
$$y_5(x) = (\lambda_{h_0}y_0+\lambda_{h_1}y_1+\lambda_{h_2}y_2)e^{-\gamma x-\int_0^x \mu_5(\xi)d\xi} + e^{-\gamma x-\int_0^x \mu_5(\xi)d\xi}\int_0^x z_5(\tau)e^{\gamma\tau+\int_0^\tau \mu_5(\xi)d\xi}d\tau. \tag{20}$$

By inserting (18), (19) and (20) into (10) and rearranging (11) and (12) we obtain

$$
\begin{aligned}
&(\gamma+\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)y_0\\
&=\mu_1y_1+\mu_2y_2+\int_0^\infty \mu_3(x)(\lambda y_1+\lambda y_2)e^{-\gamma x-\int_0^x\mu_3(\xi)d\xi}dx\\
&\quad+\int_0^\infty \mu_3(x)e^{-\gamma x-\int_0^x\mu_3(\xi)d\xi}\int_0^x z_3(\tau)e^{\gamma\tau+\int_0^\tau\mu_3(\xi)d\xi}d\tau dx\\
&\quad+\int_0^\infty \mu_4(x)(\lambda_{c_0}y_0+\lambda_{c_1}y_1+\lambda_{c_2}y_2)e^{-\gamma x-\int_0^x\mu_4(\xi)d\xi}dx\\
&\quad+\int_0^\infty \mu_4(x)e^{-\gamma x-\int_0^x\mu_4(\xi)d\xi}\int_0^x z_4(\tau)e^{\gamma\tau+\int_0^\tau\mu_4(\xi)d\xi}d\tau dx\\
&\quad+\int_0^\infty \mu_5(x)(\lambda_{h_0}y_0+\lambda_{h_1}y_1+\lambda_{h_2}y_2)e^{-\gamma x-\int_0^x\mu_5(\xi)d\xi}dx\\
&\quad+\int_0^\infty \mu_5(x)e^{-\gamma x-\int_0^x\mu_5(\xi)d\xi}\int_0^x z_5(\tau)e^{\gamma\tau+\int_0^\tau\mu_5(\xi)d\xi}d\tau dx+z_0\\
&\Longrightarrow\\
&\bigg(\gamma+\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta-\lambda_{c_0}\int_0^\infty \mu_4(x)e^{-\gamma x-\int_0^x\mu_4(\xi)d\xi}dx\\
&\quad-\lambda_{h_0}\int_0^\infty \mu_5(x)e^{-\gamma x-\int_0^x\mu_5(\xi)d\xi}dx\bigg)y_0\\
&\quad-\bigg(\mu_1+\lambda\int_0^\infty \mu_3(x)e^{-\gamma x-\int_0^x\mu_3(\xi)d\xi}dx\\
&\quad+\lambda_{c_1}\int_0^\infty \mu_4(x)e^{-\gamma x-\int_0^x\mu_4(\xi)d\xi}dx\\
&\quad+\lambda_{h_1}\int_0^\infty \mu_5(x)e^{-\gamma x-\int_0^x\mu_5(\xi)d\xi}dx\bigg)y_1\\
&\quad-\bigg(\mu_2+\lambda\int_0^\infty \mu_3(x)e^{-\gamma x-\int_0^x\mu_3(\xi)d\xi}dx\\
&\quad+\lambda_{c_2}\int_0^\infty \mu_4(x)e^{-\gamma x-\int_0^x\mu_4(\xi)d\xi}dx\\
&\quad+\lambda_{h_2}\int_0^\infty \mu_5(x)e^{-\gamma x-\int_0^x\mu_5(\xi)d\xi}dx\bigg)y_2\\
&=z_0\\
&\quad+\sum_{i=3}^{5}\int_0^\infty \mu_i(x)e^{-\gamma x-\int_0^x\mu_i(\xi)d\xi}\int_0^x z_i(\tau)e^{\gamma\tau+\int_0^\tau\mu_i(\xi)d\xi}d\tau dx, \qquad (21)
\end{aligned}
$$

$$-\lambda y_0 + (\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) y_1 = z_1, \tag{22}$$
$$-\eta y_0 + (\gamma + \mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) y_2 = z_2. \tag{23}$$

(21), (22) and (23) together with the Cramer rule give

$$y_0 = \frac{|H_0(\gamma)|}{|H(\gamma)|}, \quad y_1 = \frac{|H_1(\gamma)|}{|H(\gamma)|}, \quad y_2 = \frac{|H_2(\gamma)|}{|H(\gamma)|}, \tag{24}$$

where

$$\begin{aligned}
|H(\gamma)| = \Bigg(&\gamma + \lambda + \lambda_{c_0} + \lambda_{h_0} + \eta \\
&- \lambda_{c_0} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&- \lambda_{h_0} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Bigg) \\
&\times (\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\
&\times (\gamma + \mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&- \eta(\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\
&\times \Bigg(\mu_2 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\
&+ \lambda_{c_2} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&+ \lambda_{h_2} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Bigg) \\
&- \lambda(\gamma + \mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&\times \Bigg(\mu_1 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\
&+ \lambda_{c_1} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&+ \lambda_{h_1} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Bigg),
\end{aligned}$$

$$\begin{aligned}
|H_0(\gamma)| = \Bigg(&\sum_{i=3}^{5} \int_0^\infty \mu_i(x) e^{-\gamma x - \int_0^x \mu_i(\xi) d\xi} \int_0^x z_i(\tau) e^{\gamma \tau + \int_0^\tau \mu_i(\xi) d\xi} d\tau dx \\
&+ z_0 \Bigg) (\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)
\end{aligned}$$

$$
\begin{aligned}
&\times (\gamma+\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)\\
&+ z_2(\gamma+\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)\\
&\times \left(\mu_2+\lambda\int_0^\infty \mu_3(x)e^{-\gamma x-\int_0^x \mu_3(\xi)d\xi}dx\right.\\
&+ \lambda_{c_2}\int_0^\infty \mu_4(x)e^{-\gamma x-\int_0^x \mu_4(\xi)d\xi}dx\\
&\left.+ \lambda_{h_2}\int_0^\infty \mu_5(x)e^{-\gamma x-\int_0^x \mu_5(\xi)d\xi}dx\right)\\
&+ z_1(\gamma+\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)\\
&\times \left(\mu_1+\lambda\int_0^\infty \mu_3(x)e^{-\gamma x-\int_0^x \mu_3(\xi)d\xi}dx\right.\\
&+ \lambda_{c_1}\int_0^\infty \mu_4(x)e^{-\gamma x-\int_0^x \mu_4(\xi)d\xi}dx\\
&\left.+ \lambda_{h_1}\int_0^\infty \mu_5(x)e^{-\gamma x-\int_0^x \mu_5(\xi)d\xi}dx\right),
\end{aligned}
$$

$$
\begin{aligned}
|H_1(\gamma)| = z_1\Bigg(&\gamma+\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta\\
&- \lambda_{c_0}\int_0^\infty \mu_4(x)e^{-\gamma x-\int_0^x \mu_4(\xi)d\xi}dx\\
&- \lambda_{h_0}\int_0^\infty \mu_5(x)e^{-\gamma x-\int_0^x \mu_5(\xi)d\xi}dx\Bigg)\\
&\times (\gamma+\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)\\
&+ z_2\lambda\Bigg(\mu_2+\lambda\int_0^\infty \mu_3(x)e^{-\gamma x-\int_0^x \mu_3(\xi)d\xi}dx\\
&+ \lambda_{c_2}\int_0^\infty \mu_4(x)e^{-\gamma x-\int_0^x \mu_4(\xi)d\xi}dx\\
&+ \lambda_{h_2}\int_0^\infty \mu_5(x)e^{-\gamma x-\int_0^x \mu_5(\xi)d\xi}dx\Bigg)\\
&- z_1\eta\Bigg(\mu_2+\lambda\int_0^\infty \mu_3(x)e^{-\gamma x-\int_0^x \mu_3(\xi)d\xi}dx\\
&+ \lambda_{c_2}\int_0^\infty \mu_4(x)e^{-\gamma x-\int_0^x \mu_4(\xi)d\xi}dx\\
&+ \lambda_{h_2}\int_0^\infty \mu_5(x)e^{-\gamma x-\int_0^x \mu_5(\xi)d\xi}dx\Bigg)
\end{aligned}
$$

$$+ \lambda\bigg(\sum_{i=3}^{5} \int_0^\infty \mu_i(x) e^{-\gamma x - \int_0^x \mu_i(\xi) d\xi} \int_0^x z_i(\tau) e^{\gamma\tau + \int_0^\tau \mu_i(\xi) d\xi} d\tau dx$$
$$+ z_0 \bigg) (\gamma + \mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda),$$

$$\begin{aligned}|H_2(\gamma)| = {} & z_2(\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\ & \bigg(\gamma + \lambda + \lambda_{c_0} + \lambda_{h_0} + \eta \\ & - \lambda_{c_0} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\ & - \lambda_{h_0} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \bigg) \\ & + z_1 \eta \bigg(\mu_1 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\ & + \lambda_{c_1} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\ & + \lambda_{h_1} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \bigg) \\ & + \eta \bigg(\sum_{i=3}^{5} \int_0^\infty \mu_i(x) e^{-\gamma x - \int_0^x \mu_i(\xi) d\xi} \int_0^x z_i(\tau) e^{\gamma\tau + \int_0^\tau \mu_i(\xi) d\xi} d\tau dx \\ & + z_0 \bigg) (\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\ & - z_2 \lambda \bigg(\mu_1 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\ & + \lambda_{c_1} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\ & + \lambda_{h_1} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \bigg).\end{aligned}$$

(24) together with (18), (19) and (20) imply

$$\begin{aligned}y_3(x) = {} & \left(\lambda \frac{|H_1(\gamma)|}{|H(\gamma)|} + \lambda \frac{|H_2(\gamma)|}{|H(\gamma)|} \right) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} \\ & + e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} \int_0^x z_3(\tau) e^{\gamma\tau + \int_0^\tau \mu_3(\xi) d\xi} d\tau, \end{aligned} \tag{25}$$

$$y_4(x) = \left(\lambda_{c_0}\frac{|H_0(\gamma)|}{|H(\gamma)|} + \lambda_{c_1}\frac{|H_1(\gamma)|}{|H(\gamma)|} + \lambda_{c_2}\frac{|H_2(\gamma)|}{|H(\gamma)|}\right) e^{-\gamma x - \int_0^x \mu_4(\xi)d\xi}$$
$$+ e^{-\gamma x - \int_0^x \mu_4(\xi)d\xi} \int_0^x z_4(\tau) e^{\gamma\tau + \int_0^\tau \mu_4(\xi)d\xi} d\tau, \quad (26)$$
$$y_5(x) = \left(\lambda_{h_0}\frac{|H_0(\gamma)|}{|H(\gamma)|} + \lambda_{h_1}\frac{|H_1(\gamma)|}{|H(\gamma)|} + \lambda_{h_2}\frac{|H_2(\gamma)|}{|H(\gamma)|}\right) e^{-\gamma x - \int_0^x \mu_5(\xi)d\xi}$$
$$+ e^{-\gamma x - \int_0^x \mu_5(\xi)d\xi} \int_0^x z_5(\tau) e^{\gamma\tau + \int_0^\tau \mu_5(\xi)d\xi} d\tau. \quad (27)$$

(24)∼(27) are just the result of this lemma. □

Theorem 4. *If $\mu(x)$ is Lipschitz continuous and there exist two positive constants $\overline{\mu}$ and $\underline{\mu}$ such that $0 < \underline{\mu} \le \mu(x) \le \overline{\mu} < \infty$, then the time-dependent solution of the system (9) exponentially converges to its steady-state solution, i.e.,*

$$\|p(\cdot,t) - p(\cdot)\| \le \mathbb{M}e^{-\delta t}, \quad \forall t \ge 0.$$

Proof. Theorem 2 implies

$$\begin{aligned}
\|S(t) - V(t)\| &= \|U(t)\| \\
&\le e^{-\min\{\underline{\mu}, \lambda+\lambda_{c_0}+\lambda_{h_0}+\eta, \mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda, \mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda\}t} \\
&\quad \times \left\{4 + \frac{2\lambda}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|} \right. \\
&\quad \left. + \frac{2\eta}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\right\} \\
&\Longrightarrow \\
&\lim_{t\to\infty} \frac{\ln\|S(t)-V(t)\|}{t} \\
&\le -\min\{\underline{\mu}, \lambda+\lambda_{c_0}+\lambda_{h_0}+\eta, \\
&\quad \mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda, \mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda\}
\end{aligned}$$

From which together with Proposition 2.10 in Engel and Nagel [2], p.258 we know that $\omega_{ess}(S(t))(i.e., \omega_{ess}(A+B))$, the essential growth bound of S(t)(i.e., (A+B)), satisfies

$$\omega_{ess}(S(t)) \le -\min\{\underline{\mu}, \lambda+\lambda_{c_0}+\lambda_{h_0}+\eta, \mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda, \mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda\}.$$

Since $E : X \to \mathbb{R}^6$ is compact, by Proposition 2.12 in [2], p.258 we deduce

$$\begin{aligned}
&\omega_{ess}(A+B+E) = \omega_{ess}(T(t)) = \omega_{ess}(S(t)) \\
&\le -\min\{\underline{\mu}, \lambda+\lambda_{c_0}+\lambda_{h_0}+\eta, \\
&\quad \mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda, \mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda\}.
\end{aligned} \tag{28}$$

This together Theorem 2 and $0 \in \sigma(A+B+E) \bigcap \{\gamma \in \mathbb{C} \mid \Re\gamma > \omega_{ess}(A+B+E)\}$ with Corollary 2.11 in [2], p.258 imply that 0 is an isolated eigenvalue of A+B+E with algebraic multiplicity one, that is to say, 0 is a pole of $(\gamma I - A - B - E)^{-1}$ of order 1. Hence, Theorem 2 and the residue theorem give, for the initial value $p(0) = (\phi_0, \phi_1, \phi_2, \psi_3, \psi_4, \psi_5) \in D(A^2)$,

$$\mathbb{P}p(0) = \lim_{\gamma\to 0} \gamma(\gamma I - A - B - E)^{-1} p(0). \tag{29}$$

In the following we will determine the above limit. By using the l'Hospital rule and

$$\int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi)d\xi} dx = -e^{-\int_0^x \mu(\xi)d\xi}\Big|_0^\infty = 1.$$

we calculate

$$\begin{aligned}
&\lim_{\gamma\to 0} \frac{\gamma}{|H(\gamma)|} \\
&= \lim_{\gamma\to 0} 1 \Big/ \Bigg\{ \Bigg(1 + \lambda_{c_0} \int_0^\infty x\mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi)d\xi} dx \\
&\quad + \lambda_{h_0} \int_0^\infty x\mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi)d\xi} dx \Bigg) \\
&\quad \times (\gamma+\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(\gamma+\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda) \\
&\quad + \Bigg(\gamma+\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta - \lambda_{c_0} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi)d\xi} dx \\
&\quad - \lambda_{h_0} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi)d\xi} dx \Bigg) (\gamma+\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda) \\
&\quad + \Bigg(\gamma+\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta - \lambda_{c_0} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi)d\xi} dx \\
&\quad - \lambda_{h_0} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi)d\xi} dx \Bigg) (\gamma+\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda) \\
&- \eta \Bigg(\mu_2 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi)d\xi} dx \\
&\quad + \lambda_{c_2} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi)d\xi} dx
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_{h_2} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Big) \\
&+ \eta(\gamma + \mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\
&\times \Big(\lambda \int_0^\infty x\mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\
&+ \lambda_{c_2} \int_0^\infty x\mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&+ \lambda_{h_2} \int_0^\infty x\mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Big) \\
&+ \lambda(\gamma + \mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \Big(\lambda \int_0^\infty x\mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\
&+ \lambda_{c_1} \int_0^\infty x\mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&+ \lambda_{h_1} \int_0^\infty x\mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Big) \\
&- \lambda \Big(\mu_1 + \lambda \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx \\
&+ \lambda_{c_1} \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx \\
&+ \lambda_{h_1} \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx \Big) \Big\} \\
= 1 \Big/ \Big\{ &\Big(1 + \lambda_{c_0} \int_0^\infty x\mu_4(x) e^{- \int_0^x \mu_4(\xi) d\xi} dx \\
&+ \lambda_{h_0} \int_0^\infty x\mu_5(x) e^{- \int_0^x \mu_5(\xi) d\xi} dx \Big) \\
&\times (\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&+ \Big(\lambda + \lambda_{c_0} + \lambda_{h_0} + \eta - \lambda_{c_0} \int_0^\infty \mu_4(x) e^{- \int_0^x \mu_4(\xi) d\xi} dx \\
&- \lambda_{h_0} \int_0^\infty \mu_5(x) e^{- \int_0^x \mu_5(\xi) d\xi} dx \Big) (\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&+ \Big(\lambda + \lambda_{c_0} + \lambda_{h_0} + \eta - \lambda_{c_0} \int_0^\infty \mu_4(x) e^{- \int_0^x \mu_4(\xi) d\xi} dx \\
&- \lambda_{h_0} \int_0^\infty \mu_5(x) e^{- \int_0^x \mu_5(\xi) d\xi} dx \Big) \\
&\times (\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)
\end{aligned}
$$

$$
\begin{aligned}
&- \eta\bigg(\mu_2 + \lambda \int_0^\infty \mu_3(x) e^{-\int_0^x \mu_3(\xi)d\xi} dx \\
&+ \lambda_{c_2} \int_0^\infty \mu_4(x) e^{-\int_0^x \mu_4(\xi)d\xi} dx \\
&+ \lambda_{h_2} \int_0^\infty \mu_5(x) e^{-\int_0^x \mu_5(\xi)d\xi} dx\bigg) \\
&+ \eta(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)\bigg(\lambda \int_0^\infty x\mu_3(x) e^{-\int_0^x \mu_3(\xi)d\xi} dx \\
&+ \lambda_{c_2} \int_0^\infty x\mu_4(x) e^{-\int_0^x \mu_4(\xi)d\xi} dx \\
&+ \lambda_{h_2} \int_0^\infty x\mu_5(x) e^{-\int_0^x \mu_5(\xi)d\xi} dx\bigg) \\
&+ \lambda(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda)\bigg(\lambda \int_0^\infty x\mu_3(x) e^{-\int_0^x \mu_3(\xi)d\xi} dx \\
&+ \lambda_{c_1} \int_0^\infty x\mu_4(x) e^{-\int_0^x \mu_4(\xi)d\xi} dx \\
&+ \lambda_{h_1} \int_0^\infty x\mu_5(x) e^{-\int_0^x \mu_5(\xi)d\xi} dx\bigg) \\
&- \lambda\bigg(\mu_1 + \lambda \int_0^\infty \mu_3(x) e^{-\int_0^x \mu_3(\xi)d\xi} dx \\
&+ \lambda_{c_1} \int_0^\infty \mu_4(x) e^{-\int_0^x \mu_4(\xi)d\xi} dx \\
&+ \lambda_{h_1} \int_0^\infty \mu_5(x) e^{-\int_0^x \mu_5(\xi)d\xi} dx\bigg)\bigg\} \\
= 1 \bigg/ \bigg\{&\bigg(1 + \lambda_{c_0} \int_0^\infty x\mu_4(x) e^{-\int_0^x \mu_4(\xi)d\xi} dx \\
&+ \lambda_{h_0} \int_0^\infty x\mu_5(x) e^{-\int_0^x \mu_5(\xi)d\xi} dx\bigg) \\
&\times (\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&+ (\lambda + \lambda_{c_0} + \lambda_{h_0} + \eta - \lambda_{c_0} - \lambda_{h_0})(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&+ (\lambda + \lambda_{c_0} + \lambda_{h_0} + \eta - \lambda_{c_0} - \lambda_{h_0})(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\
&+ \eta(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)\bigg(\lambda \int_0^\infty x\mu_3(x) e^{-\int_0^x \mu_3(\xi)d\xi} dx \\
&+ \lambda_{c_2} \int_0^\infty x\mu_4(x) e^{-\int_0^x \mu_4(\xi)d\xi} dx
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda_{h_2} \int_0^\infty x\mu_5(x)e^{-\int_0^x \mu_5(\xi)d\xi}dx\Big) \\
&- \eta(\mu_2 + \lambda + \lambda_{c_2} + \lambda_{h_2}) \\
&+ \lambda(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda)\Big(\lambda \int_0^\infty x\mu_3(x)e^{-\int_0^x \mu_3(\xi)d\xi}dx \\
&+ \lambda_{c_1} \int_0^\infty x\mu_4(x)e^{-\int_0^x \mu_4(\xi)d\xi}dx \\
&+ \lambda_{h_1} \int_0^\infty x\mu_5(x)e^{-\int_0^x \mu_5(\xi)d\xi}dx\Big) \\
&- \lambda(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)\Big\} \\
= 1\Big/\Big\{&\Big(1 + \lambda_{c_0} \int_0^\infty x\mu_4(x)e^{-\int_0^x \mu_4(\xi)d\xi}dx \\
&+ \lambda_{h_0} \int_0^\infty x\mu_5(x)e^{-\int_0^x \mu_5(\xi)d\xi}dx\Big) \\
&\times (\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&+ (\lambda + \eta)(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&+ (\lambda + \eta)(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) \\
&+ \eta(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)\Big(\lambda \int_0^\infty x\mu_3(x)e^{-\int_0^x \mu_3(\xi)d\xi}dx \\
&+ \lambda_{c_2} \int_0^\infty x\mu_4(x)e^{-\int_0^x \mu_4(\xi)d\xi}dx \\
&+ \lambda_{h_2} \int_0^\infty x\mu_5(x)e^{-\int_0^x \mu_5(\xi)d\xi}dx\Big) \\
&- \eta(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) + \lambda(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) \\
&\times \Big(\lambda \int_0^\infty x\mu_3(x)e^{-\int_0^x \mu_3(\xi)d\xi}dx \\
&+ \lambda_{c_1} \int_0^\infty x\mu_4(x)e^{-\int_0^x \mu_4(\xi)d\xi}dx \\
&+ \lambda_{h_1} \int_0^\infty x\mu_5(x)e^{-\int_0^x \mu_5(\xi)d\xi}dx\Big) \\
&- \lambda(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)\Big\} := \frac{1}{Q}. \qquad (30)
\end{aligned}
$$

By Lemma 3, the Fubini theorem and

$$\int_0^\infty \mu(x)e^{-\int_0^x \mu(\xi)d\xi}dx = 1, \sum_{i=0}^{2}\phi_i + \sum_{i=3}^{5}\int_0^\infty \varphi_i(x)dx = 1, \tag{31}$$

$$\begin{aligned}
&\int_0^\infty \mu(x)e^{-\int_0^x \mu(\xi)d\xi}\int_0^x \psi_i(\tau)e^{\int_0^\tau \mu(\xi)d\xi}d\tau dx \\
&= \int_0^\infty \psi_i(\tau)e^{\int_0^\tau \mu(\xi)d\xi}\int_\tau^\infty \mu(x)e^{-\int_0^x \mu(\xi)d\xi}dxd\tau \\
&= \int_0^\infty \psi_i(\tau)e^{\int_0^\tau \mu(\xi)d\xi}\left(-e^{-\int_0^\tau \mu(\xi)d\xi}\Big|_{x=\tau}^{x=\infty}\right)d\tau \\
&= \int_0^\infty \psi_i(\tau)d\tau = \int_0^\infty \psi_i(x)dx, \quad i = 3, 4, 5
\end{aligned} \tag{32}$$

it is not difficult to determine, by replacing

$$(z_0, z_1, z_2, z_3(x), z_4(x), z_5(x))$$

in $|H_i(\gamma)|$ $(i = 0, 1, 2)$ by $(\phi_0, \phi_1, \phi_2, \psi_3(x), \psi_4(x), \psi_5(x))$,

$$\begin{aligned}
\lim_{\gamma\to 0}|H_0(\gamma)| &= \prod_{i=1}^{2}(\mu_i + \lambda_{c_i} + \lambda_{h_i} + \lambda)\left(\sum_{i=0}^{2}\phi_i + \sum_{i=3}^{5}\int_0^\infty \psi_i(x)dx\right) \\
&= \prod_{i=1}^{2}(\mu_i + \lambda_{c_i} + \lambda_{h_i} + \lambda) := Q_0,
\end{aligned} \tag{33}$$

$$\begin{aligned}
\lim_{\gamma\to 0}|H_1(\gamma)| &= \lambda(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda)\left(\sum_{i=0}^{2}\phi_i + \sum_{i=3}^{5}\int_0^\infty \psi_i(x)dx\right) \\
&= \lambda(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) := Q_1,
\end{aligned} \tag{34}$$

$$\begin{aligned}
\lim_{\gamma\to 0}|H_2(\gamma)| &= \eta(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)\left(\sum_{i=0}^{2}\phi_i + \sum_{i=3}^{5}\int_0^\infty \psi_i(x)dx\right) \\
&= \eta(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) := Q_2.
\end{aligned} \tag{35}$$

By combining (30), (33), (34) and (35) with Lemma 3 we derive

$$\lim_{\gamma\to 0}\gamma y_0 = \lim_{\gamma\to 0}\frac{\gamma}{|H(\gamma)|}|H_0(\gamma)| = \frac{Q_0}{Q}, \tag{36}$$

$$\lim_{\gamma\to 0}\gamma y_1 = \lim_{\gamma\to 0}\frac{\gamma}{|H(\gamma)|}|H_1(\gamma)| = \frac{Q_1}{Q}, \tag{37}$$

$$\lim_{\gamma\to 0}\gamma y_2=\lim_{\gamma\to 0}\frac{\gamma}{|H(\gamma)|}|H_2(\gamma)|=\frac{Q_2}{Q}, \tag{38}$$

$$\begin{aligned}
\lim_{\gamma\to 0}\gamma y_3(x) &= \lim_{\gamma\to 0}\gamma\Bigg\{\left(\lambda\frac{|H_1(\gamma)|}{|H(\gamma)|}+\lambda\frac{|H_2(\gamma)|}{|H(\gamma)|}\right)e^{-\gamma x-\int_0^x\mu_3(\xi)d\xi}\\
&\quad + e^{-\gamma x-\int_0^x\mu_3(\xi)d\xi}\int_0^x\psi_3(\tau)e^{\gamma\tau+\int_0^\tau\mu_3(\xi)d\xi}d\tau\Bigg\}\\
&= \lambda\left\{\lim_{\gamma\to 0}\frac{\gamma}{|H(\gamma)|}|H_1(\gamma)|+\lim_{\gamma\to 0}\frac{\gamma}{|H(\gamma)|}|H_2(\gamma)|\right\}\\
&\quad \times\lim_{\gamma\to 0}e^{-\gamma x-\int_0^x\mu_3(\xi)d\xi}\\
&\quad + \lim_{\gamma\to 0}\gamma e^{-\gamma x-\int_0^x\mu_3(\xi)d\xi}\int_0^x\psi_3(\tau)e^{\gamma\tau+\int_0^\tau\mu_3(\xi)d\xi}d\tau\\
&= \lambda\left[\frac{Q_1}{Q}+\frac{Q_2}{Q}\right]e^{-\int_0^x\mu_3(\xi)d\xi}\\
&= \frac{\lambda(Q_1+Q_2)}{Q}e^{-\int_0^x\mu_3(\xi)d\xi},
\end{aligned} \tag{39}$$

$$\begin{aligned}
\lim_{\gamma\to 0}\gamma y_4(x) &= \lim_{\gamma\to 0}\gamma\Bigg\{\Bigg(\lambda_{c_0}\frac{|H_0(\gamma)|}{|H(\gamma)|}+\lambda_{c_1}\frac{|H_1(\gamma)|}{|H(\gamma)|}\\
&\quad +\lambda_{c_2}\frac{|H_2(\gamma)|}{|H(\gamma)|}\Bigg)e^{-\gamma x-\int_0^x\mu_4(\xi)d\xi}\\
&\quad + e^{-\gamma x-\int_0^x\mu_4(\xi)d\xi}\int_0^x\psi_4(\tau)e^{\gamma\tau+\int_0^\tau\mu_4(\xi)d\xi}d\tau\Bigg\}\\
&= \Bigg\{\lambda_{c_0}\lim_{\gamma\to 0}\frac{\gamma}{|H(\gamma)|}|H_0(\gamma)|+\lambda_{c_1}\lim_{\gamma\to 0}\frac{\gamma}{|H(\gamma)|}|H_1(\gamma)|\\
&\quad +\lambda_{c_2}\lim_{\gamma\to 0}\frac{\gamma}{|H(\gamma)|}|H_2(\gamma)|\Bigg\}\lim_{\gamma\to 0}e^{-\gamma x-\int_0^x\mu_4(\xi)d\xi}\\
&\quad + \lim_{\gamma\to 0}\gamma e^{-\gamma x-\int_0^x\mu_4(\xi)d\xi}\int_0^x\psi_4(\tau)e^{\gamma\tau+\int_0^\tau\mu_4(\xi)d\xi}d\tau\\
&= \left[\lambda_{c_0}\frac{Q_0}{Q}+\lambda_{c_1}\frac{Q_1}{Q}+\lambda_{c_2}\frac{Q_2}{Q}\right]e^{-\int_0^x\mu_4(\xi)d\xi}\\
&= \frac{\lambda_{c_0}Q_0+\lambda_{c_1}Q_1+\lambda_{c_2}Q_2}{Q}e^{-\int_0^x\mu_4(\xi)d\xi},
\end{aligned} \tag{40}$$

$$\begin{aligned}\lim_{\gamma\to 0}\gamma y_5(x) &= \lim_{\gamma\to 0}\gamma\Bigg\{\Bigg(\lambda_{h_0}\frac{|H_0(\gamma)|}{|H(\gamma)|}+\lambda_{h_1}\frac{|H_1(\gamma)|}{|H(\gamma)|}\\ &\quad+\lambda_{h_2}\frac{|H_2(\gamma)|}{|H(\gamma)|}\Bigg)e^{-\gamma x-\int_0^x\mu_5(\xi)d\xi}\\ &\quad+e^{-\gamma x-\int_0^x\mu_5(\xi)d\xi}\int_0^x\psi_5(\tau)e^{\gamma\tau+\int_0^\tau\mu_5(\xi)d\xi}d\tau\Bigg\}\\ &=\Bigg\{\lambda_{h_0}\lim_{\gamma\to 0}\frac{\gamma}{|H(\gamma)|}|H_0(\gamma)|+\lambda_{h_1}\lim_{\gamma\to 0}\frac{\gamma}{|H(\gamma)|}|H_1(\gamma)|\\ &\quad+\lambda_{h_2}\lim_{\gamma\to 0}\frac{\gamma}{|H(\gamma)|}|H_2(\gamma)|\Bigg\}\lim_{\gamma\to 0}e^{-\gamma x-\int_0^x\mu_5(\xi)d\xi}\\ &\quad+\lim_{\gamma\to 0}\gamma e^{-\gamma x-\int_0^x\mu_5(\xi)d\xi}\int_0^x\psi_5(\tau)e^{\gamma\tau+\int_0^\tau\mu_5(\xi)d\xi}d\tau\\ &=\left[\lambda_{h_0}\frac{Q_0}{Q}+\lambda_{h_1}\frac{Q_1}{Q}+\lambda_{h_2}\frac{Q_2}{Q}\right]e^{-\int_0^x\mu_5(\xi)d\xi}\\ &=\frac{\lambda_{h_0}Q_0+\lambda_{h_1}Q_1+\lambda_{h_2}Q_2}{Q}e^{-\int_0^x\mu_5(\xi)d\xi}.\end{aligned}\tag{41}$$

By inserting (36)∼(41) into (29) we determine

$$\begin{aligned}&\mathbb{P}p(0)\\ &=\lim_{\gamma\to 0}\gamma(y_0,y_1,y_2,y_3(x),y_4(x),y_5(x))\\ &=\Big(\lim_{\gamma\to 0}\gamma y_0,\lim_{\gamma\to 0}\gamma y_1,\lim_{\gamma\to 0}\gamma y_2,\lim_{\gamma\to 0}\gamma y_3(x),\lim_{\gamma\to 0}\gamma y_4(x),\lim_{\gamma\to 0}\gamma y_5(x)\Big)\\ &=\Big(\frac{Q_0}{Q},\frac{Q_1}{Q},\frac{Q_2}{Q},\frac{\lambda(Q_0+Q_1)}{Q}e^{-\int_0^x\mu_3(\xi)d\xi},\\ &\quad\frac{\lambda_{c_0}Q_0+\lambda_{c_1}Q_1+\lambda_{c_2}Q_2}{Q}e^{-\int_0^x\mu_4(\xi)d\xi},\\ &\quad\frac{\lambda_{h_0}Q_0+\lambda_{h_1}Q_1+\lambda_{h_2}Q_2}{Q}e^{-\int_0^x\mu_5(\xi)d\xi}\Big):=p(x).\end{aligned}\tag{42}$$

This is the steady-state solution of the system (9) which was given in Gupur [2].

(42), (31) and Theorem 2 give our desired result:

$$\|p(\cdot,t)-p(\cdot)\|=\|T(t)p(0)-\mathbb{P}p(0)\|\le\|T(t)-\mathbb{P}\|\|p(0)\|$$

$$
\begin{aligned}
&\leq \mathbb{M}e^{-\delta t}\|p(0)\| \\
&= \mathbb{M}e^{-\delta t}\left(\sum_{i=0}^{2}\phi_i + \sum_{i=3}^{5}\int_0^\infty \psi_i(x)dx\right) \\
&= \mathbb{M}e^{-\delta t}, \quad \forall t \geq 0.
\end{aligned}
$$

This means that the time-dependent solution of the system (9) exponentially converges to its steady-state solution. □

The time-dependent solutions of reliability models, which are described by infinite number of partial differential equations with integral boundary conditions, at most strongly converge to their steady-state solutions (see Gupur [3]). Hence, theorem 4 shows the difference between convergence of the time-dependent solution of reliability models which are described by finite partial differential equations and convergence of the time-dependent solutions of reliability models which are described by infinite number of partial differential equations with integral boundary conditions.

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