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SOLUTIONS OF THE SPACE-TIME FRACTIONAL OF SOME NONLINEAR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS USING MODIFIED KUDRYASHOV METHOD

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Abstract: In this work, we handle the space-time fractional of classical Drinfeld's Sokolov-Wilson system, the fractional (2+1)-dimensional Davey-Stewartson system and the fractional generalized Hirota- Satsuma coupled KdV system to solve analytically. Firstly, we use fractional traveling wave transformations to convert fractional nonlinear partial differential equations to nonlinear ordinary differential equations. Next, the modified Kudryashov method is applied to find exact solutions of these nonlinear systems.

Key Words: space-time fractional of classical Drinfeld's Sokolov-Wilson system, space-time fractional (2+1)-dimensional Davey-Stewartson system and the space-time fractional generalized Hirota- Satsuma coupled KdV system, modified Kudryashov method

1. Introduction

The fractional derivatives construct a basis to describe the features of systems

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of mathematical modeling and simulation of systems that leads to nonlinear fractional differential equations (FDEs)and to solve such equations [Podlubny (1999)].

In recent years, numerous effective methods for solving these system have been found in the most useful works on nonlinear FDEs. For example the subequation method [2,5,6,14], the exp-function method [16], the first integral method [11,15], the complex transform method [12] and so on.The common of these methods is based on the homogenous balance principle.

In this study, firstly we will describe the modified Kudryashov [10] and applied in many studies to construct the exact analytical solutions of nonlinear differential equations. This method is also based on the homogenous balance principle. Therefore, it can be applied to solve the fractional order nonlinear equations.Then, we will apply the proposed method to the space-time fractional of classical Drinfeld's Sokolov-Wilson system, the fractional (2+1)-dimensional Davey-Stewartson system and the fractional generalized Hirota- Satsuma coupled KdV system by the help of Jumarie's modified Riemann-Liouville derivative.

2. Preliminaries and the Modified Kudryashov Method

Jumarie's modified Riemann-Liouville derivative is defined as[7,8]:

(1)

$$Where \;\; D_x^f(x) := lim_{h\to 0} h^{-\alpha} \sum_{k=o}^{\infty} (-1)^k f[x+(\alpha-k)h]. \tag{2}$$

In addition, some properties for the proposed modified Riemann-Liouvuille derivative are given in [7,8] as follows:

$$D_t^\alpha t^\gamma = \Gamma(1+\gamma)/\Gamma(1+\gamma-\alpha) t^{\gamma-\alpha}, \gamma > 0, \tag{3}$$

$$D_t^\alpha [f(t)g(t)] = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t), \tag{4}$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)] D_t^\alpha g(t) = D_g^\alpha f[g(t)][g'(t)]^\alpha, \tag{5}$$

which are the direct consequence of

$$D^\alpha X(t) = \Gamma(1+\alpha) DX(t). \tag{6}$$

We present the main steps of the modified Kudryashov method as follows [1,3,4,9,10]:

For a given nonlinear FDEs for a function u of independent variables, $X = (x, y, z, ..., t)$:

$$F(u, u_t, u_x, u_y, u_z, ..., D_t^\alpha u, D_x^\alpha u, D_y^\alpha u, D_z^\alpha u, ...) = 0, \quad (7)$$

where,$D_t^\alpha u, D_x^\alpha u, , D_y^\alpha u$ and $D_z^\alpha u$ are the modified Riemann-Liouville derivatives of u with respect to t, x, y and$z$.

F is a polynomial in $u = u(x, y, z, , t)$ and its various partial derivatives, in which the nonlinear terms and highest order derivatives are involved.

Step 1: We investigate the traveling wave solutions of equation (7) of the form:

$$\begin{gathered} u(x, y, z, , t) = u(\xi), \\ \xi = \omega x^\beta/\Gamma(1+\beta) + \varepsilon y^\gamma/\Gamma(1+\gamma) + \sigma z^\delta/\Gamma(1+\delta) + ... + \lambda t^\alpha/\Gamma(1+\alpha), \end{gathered} \quad (8)$$

where $\omega, \varepsilon, \sigma$ and λ are arbitrary constants. Then equation (7) reduces to a nonlinear ordinary differential equation of the form:

$$G(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, ...) = 0. \quad (9)$$

Step 2: We suppose that the exact solutions of equation (9) can be obtained in the following form:

$$u(\xi) = \sum_{i=0}^{N} a_i Q(\xi)^i, \quad (10)$$

where $Q = 1/(1e^\xi)$ and the function Q is the solution of equation

$$Q_\xi = Q^2 - Q. \quad (11)$$

Step 3: According to the method, we assume that the solution of equation (9) can be expressed in the form:

$$u(\xi) = a_N Q^N + \ldots. \quad (12)$$

Calculation of value N in formula (11) that is the pole order for the general solution of equation (9). In order to determine the value of N we balance the highest order nonlinear terms in equation (9) analogously as in the classical Kudryashov method. Supposing $u^l(\xi)u^{(s)}(\xi)$ and $[u^p(\xi)]^r$ are the highest order nonlinear terms of equation (9) and balancing the highest order nonlinear terms we have:

$$N = (s - rp)/(r - l - 1). \quad (13)$$

Step 4: Substituting equation (10) into equation (9) and equating the coefficients of Q^i to zero, we get a system of algebraic equations. By solving this system, we obtain the exact solutions of equation (9).

3. Applications

3.1. The Space-Time Fractional of Classical Drinfeld's Sokolov-Wilson System

We first apply the method to the space-time fractional of classical Drinfeld's Sokolov-Wilson system in the form:

$$\begin{aligned} \partial^\alpha u/\partial t^\alpha + pv(\partial^\beta v/\partial x^\beta) = 0, \\ \partial^\alpha v/\partial t^\alpha + q(\partial^{3\beta} v/\partial x^{3\beta}) + ru(\partial^\beta v/\partial x^\beta) + sv(\partial^\beta u/\partial x^\beta) = 0, \end{aligned} \tag{14}$$

where $0 < \alpha, \beta \leq 1, x > 0$ and u and v are the functions of (x, t).

By considering the traveling wave transformation:

$$\begin{aligned} u(x,t) = u(\xi), v(x,t) = v(\xi), \\ \xi = \omega x^\beta/\Gamma(1+\beta) - \lambda t^\alpha/\Gamma(1+\alpha), \end{aligned} \tag{15}$$

where ω and λ are constants. Then equation (14) can be reduced to the following ordinary differential equations:

$$-\lambda u' + p\omega v v' = 0, \ \Rightarrow u' = (p\omega/\lambda) v v', \tag{16}$$

$$-\lambda v' + q\omega^3 v''' + r\omega u v' + s\omega v u' = 0, \tag{17}$$

where the prime denotes the derivative with respect to ξ. Integrating (16), we obtain:

$$u = (p\omega/2\lambda) v^2 + c_1, \tag{18}$$

where c_1 is an arbitrary integration constant.

Substituting u and u' into (17) yields:

$$q^3 v''' + p\omega^2((r+2s)/2\lambda) v^2 v' + (r\omega c_1 - \lambda) v' = 0. \tag{19}$$

Integrating (19), we get:

$$q\omega^3 v'' + p\omega^2((r+2s)/6\lambda) v^3 + (r\omega c_1 - \lambda) v = c_2, \tag{20}$$

where c_2 is an arbitrary integration constant.

Let $q\omega^3 = g$, $(p\omega^2(r+2s)/6\lambda) = k$, $(r\omega c_1 - \lambda) = h$, $\Rightarrow$

$$gv'' + kv^3 + hv - c_2 = 0. \tag{21}$$

Balancing the linear term of the highest order with the highest order nonlinear term in equation (21), we compute:

$$N = 1. \tag{22}$$

Thus, we have

$$v(\xi) = a_0 + a_1 Q, \tag{23}$$

and taking the derivatives of $v(\xi)$ with respect to ξ, we obtain:

$$v_\xi = a_1 Q^2 - a_1 Q, \tag{24}$$

$$v_{\xi\xi} = 2a_1 Q^3 - 3a_1 Q^2 + a_1 Q. \tag{25}$$

Substituting equation (23) and equation (25) into equation (21) and collecting the coefficient of each power of Q^i setting each of coefficient to zero, solving the resulting system of algebraic equations we obtain the following solutions:

$$a_1 = \sqrt{(2g/k)}\ i, \quad a_0 = -\sqrt{(g/2k)}\ i, \quad \Rightarrow \quad a_1 = -2a_0. \tag{26}$$

Inserting equation (26) into equation (23) we obtain the following solutions of equation (21)

$$v_1 = a_0[1 - 2/(1 + e^\xi)], \tag{27}$$

so,the travelling wave solution is given by:

$$u_1 = A[a_0(1 - 2/(1 + e^\xi))]^2 + c_1, \tag{28}$$

where $p\omega/2\lambda = A$.

$$v_2 = a_0[1 - 2/(1 - e^\xi)], \tag{29}$$

so,the travelling wave solution is given by:

$$u_2 = A[a_0(1 - 2/(1 - e^\xi))]^2 + c_1, \tag{30}$$

thus, two solutions have been obtained for the system (14).

3.2. The Fractional (2+1)-Dimensional Davey-Stewartson System

We, second apply consider the fractional (2+1)- Dimensional Davey-Stewartson System defines as:

$$\begin{gathered} i\partial^\alpha u/\partial t^\alpha + \partial^{2\beta} u/\partial x^{2\beta} - \partial^{2\gamma} u/\partial y^{2\gamma} - 2|u|^2 u - 2uv = 0, \\ \partial^{2\beta} v/\partial x^{2\beta} + \partial^{2\gamma} v/\partial y^{2\gamma} + 2\partial^{2\beta}(|u|^2)/\partial x^{2\beta} = 0, \end{gathered} \tag{31}$$

where $0 < \alpha, \beta, \gamma \leq 1, x, y > 0$, and u and v are the functions of (x, y, t). By considering the traveling wave transformation:

$$\begin{gathered} u(x, y, t) = e^{i\theta} u(\xi), \quad v(x, y, t) = v(\xi), \\ \theta = (px^\beta/\Gamma(1+\beta) + qy^\gamma/\Gamma(1+\gamma) + ct^\alpha/\Gamma(1+\alpha)) + r, \\ \xi = (\omega x^\beta/\Gamma(1+\beta) + \varepsilon y^\gamma/\Gamma(1+\gamma) + \lambda t^\alpha/\Gamma(1+\alpha)) + \sigma, \end{gathered} \tag{32}$$

where $p, q, c, \omega, \varepsilon$ and λ are constants. Then equation (31) can be reduced to the following ordinary differential equations:

$$(q^2 - p^2 - c)u + (\omega^2 - \varepsilon^2)u'' - 2u^3 - 2uv = 0, \tag{33}$$

$$(\omega^2 + \varepsilon^2)v'' + 2(u^2)'' = 0. \tag{34}$$

Integrating (34) in the system and neglecting constants of integration,we have found:

$$v = -2u^2/(\omega^2 + \varepsilon^2). \tag{35}$$

Substituting (35) into (33) of the system and integrating we find:

$$(q^2 - p^2 - c)u + (\omega^2 - \varepsilon^2)u'' - 2((\omega^2 + \varepsilon^2 - 2)/(\omega^2 + \varepsilon^2))u^3 = 0$$

. Let $(\omega^2 - \varepsilon^2) = k$, $((\omega^2 + \varepsilon^2 - 2)/(\omega^2 + \varepsilon^2)) = h$, $(q^2 - p^2 - c) = s$, $\Rightarrow$

$$su + ku'' - 2hu^3 = 0. \tag{36}$$

Balancing the order of u^3 with the order of u'' in equation (36),we find $N = 1$, so, the solution takes the form:

$$u(\xi) = a_0 + a_1 Q. \tag{37}$$

Inserting equation (37) into equation (36) and making use of equation (11) and collecting the coefficient of each power of setting each of coefficient to zero , solving the resulting system of algebraic equations we obtain the following solutions:

$$a_1 = \sqrt{k/h}, \quad a_0 = -(1/2)\sqrt{k/h}, \quad \Rightarrow, \quad a_1 = -2a_0, \tag{38}$$

$$u_1(\xi) = a_0[1 - 2/(1 + e^{\xi})], \tag{39}$$

so,the travelling wave solution is given by:

$$v_1(\xi) = A(a_0[1 - 2/(1 + e^{\xi})])^2, \tag{40}$$

where $(-2/(\omega^2 + \varepsilon^2)) = A$,

$$u_2(\xi) = a_0[1 - 2/(1 - e^{\xi})], \tag{41}$$

so,the travelling wave solution is given by

$$v_2(\xi) = A(a_0[1 - 2/(1 - e^{\xi})])^2. \tag{42}$$

Thus, two solutions have been obtained for the system (31).

3.3. The Fractional Generalized Hirota-Satsuma Coupled KdV System

We next consider the fractional generalized Hirota-Satsuma coupled KdV system defines as:

$$\partial^\alpha u/\partial t^\alpha = (1/4)\partial^{3\beta} u/\partial x^{3\beta} + 3u\partial^\beta u/\partial x^\beta + 3\partial^\beta (w - v^2)/\partial x^\beta, \qquad (43)$$

$$\partial^\alpha v/\partial t^\alpha = -(1/2)\partial^{3\beta} v/\partial x^{3\beta} - 3u\partial^\beta v/\partial x^\beta, \qquad (44)$$

$$\partial^\alpha w/\partial t^\alpha = -(1/2)\partial^{3\beta w}/\partial x^{3\beta} - 3u\partial^\beta w/\partial x^\beta, \qquad (45)$$

where $0 < \alpha, \beta \leq 1, x > 0$, and u, v and w are the function of (x, t).

When w=0 , (43)-(45) reduce to be the well-known Hirota-Satsuma coupled KdV system. By considering the traveling wave transformation:

$$\begin{gathered} u(x,t) = u(\xi), \quad v(x,t) = v(\xi), \quad w(x,t) = w(\xi), \\ \xi = kx^\beta/\Gamma(1+\beta) - \lambda t^\alpha/\Gamma(1+\alpha), \end{gathered} \qquad (46)$$

where $kand\lambda$ are constants. Then Eq' s. (43)- (45) can be reduced to the following ordinary differential equations:

$$-\lambda u' = (1/4)k^3 u''' + 3kuu' + 3k(w - v^2)', \qquad (47)$$

$$-\lambda v' = -(1/2)k^3 v''' - 3kuv', \qquad (48)$$

$$-\lambda w' = -(1/2)k^3 w''' - 3kuw'. \qquad (49)$$

Integrating (47) we get

$$-\lambda u = (1/4)k^3 u'' + (3/2)ku^2 + 3k(w - v^2), \qquad (50)$$

from equation (48)and equation (49) we get

$$(1/2)k^3 v''' = (\lambda - 3ku)v', \qquad (51)$$

$$(1/2)k^3 w''' = (\lambda - 3ku)w'. \qquad (52)$$

By divided equation (51) on (52) we get

$$v'''/w''' = v'/w', \quad \Rightarrow w'/w''' = v'/v''', \quad \Rightarrow w' = A_0 v'. \qquad (53)$$

Integrating (53) we get

$$w = A_0 v + B_0. \qquad (54)$$

$$Let \quad u = Av^2 + Bv + C, \qquad (55)$$

where A_0, B_0, A, B, and C are constants.

From equation (55), we have

$$u' = 2Avv' + Bv' = (2Av + B)v', \quad u'' = (2Av + B)v'' + 2A(v')^2,$$

multiply u'' by k^3 We get

$$k^3u'' = (2Av + B)k^3v'' + 2Ak^3(v')^2. \tag{56}$$

From equation (50):

$$k^3u'' = -6ku^2 - 4\lambda u + 12k(v^2 - w). \tag{57}$$

Substituting (54)and (55) into (57) we get

$$\begin{aligned} k^3u'' = -6k[A^2v^4 + 2ABv^3 + (B^2 + 2AC)v^2 + 2BCv + C^2] \\ - 4\lambda(Av^2 + Bv + C) + 12kv^2 - 12kA_0v - 12kB_0, \end{aligned} \tag{58}$$

$\Rightarrow$

$$\begin{aligned} k^3u'' = -6kA^2v^4 - 12kABv^3 + (-6kB^2 - 12kAC - 4\lambda A + 12k)v^2 \\ +(-12kBC - 4\lambda B - 12kA_0)v + (-6kC^2 - 4\lambda C - 12kB_0). \end{aligned} \tag{59}$$

Substituting (55) into (51)we get:

$$k^3v''' = -6kAv^2v' - 6kBvv' + 2(\lambda - 3kC)v'. \tag{60}$$

Integrating (59) we get

$$k^3v'' = -2kAv^3 - 3kBv^2 + 2(\lambda - 3kC)v + c_1, \tag{61}$$

where c_1 is an integration constant.

Integrating (60) once again and multiply by 2 we haves:

$$k^3(v')^2 = -kAv^4 - 2kBv^3 + 2(\lambda - 3kC)v^2 + 2c_1v + c_2, \tag{62}$$

where c_2 is an integration constant.

Substituting (58) and (61) onto (56) we get

$$\begin{aligned} (2Av + B)k^3v'' = -4kA^2v^4 - 8kABv^3 + (-8\lambda A - 6kB^2 + 12k)v^2 \\ +(-12kBC - 4\lambda B - 12kA_0 - 4Ac_1)v + (-2Ac_2 - 6kC^2 - 4\lambda C - 12kB_0), \end{aligned} \tag{63}$$

let $-4kA^2 = k_0$, $-8kAB = k_1$, $(-8\lambda A - 6kB^2 + 12k) = k_2$, $(-4Ac_1 - 12kBC - 4\lambda B - 12kA_0) = k_3$, $(-2Ac_2 - 6kC^2 - 4\lambda C - 12kB_0) = k_4$, $2Ak^3 = k_5$, $Bk^3 = k_6$, $\Rightarrow$

$$(k_5 v + k_6)v'' = k_0 v^4 + k_1 v^3 + k_2 v^2 + k_3 v + k_4. \quad (64)$$

Balacing the order of v^4 with the order of vv'' in equation (62),gives: $N = 1$.

So, the solution takes the form:

$$v = a_0 + a_1 Q. \quad (65)$$

Inserting equation (63) into equation (62) and making use of equation (11), we get:

$$\begin{gathered} k_5[a_0a_1Q + (a_1^2 - 3a_0a_1)Q^2 + (2a_0a_1 - 3a_1^2)Q^3 + 2a_1^2Q^4] \\ +k_6[2a_1Q^3 - 3a_1Q^2 + a_1Q] = k_0[a_0^4 + 4a_0^3a_1Q + 6a_0^2a_1^2Q^2 + 4a_0a_1^3Q^3 + a_1^4Q^4] \\ +k_1[a_0^3 + 3a_0^2a_1Q + 3a_0a_1^2Q^2 + a_1^3Q^3] \\ +k_2[a_0^2 + 2a_0a_1Q + a_1^2Q^2] + k_3[a_0 + a_1Q] + k_4. \end{gathered} \quad (66)$$

We solve the obtained system of algebraic equations give the following

$$a_1 = \sqrt{2k_5/k_0}, \quad a_0 = (2k_0k_6 - 2k_1k_5 - 3\sqrt{2}k_5\sqrt{k_0k_5})/(6k_0k_5), \quad (67)$$

$$v_1 = [(2k_0k_6 - 2k_1k_5 - 3\sqrt{2}k_5\sqrt{k_0k_5})/(6k_0k_5)] + (\sqrt{(2k_5/k_0)})[1/(1+e^{\xi})], \quad (68)$$

so,the travelling wave solution is given by

$$w_1 = A_0[(2k_0k_6 - 2k_1k_5 - 3\sqrt{2}k_5\sqrt{k_0k_5})/(6k_0k_5) + (\sqrt{(2k_5/k_0)})[1/(1+e^{\xi})]] + B_0, \quad (69)$$

so,the travelling wave solution is given by

$$\begin{gathered} u_1 = A[(2k_0k_6 - 2k_1k_5 - 3\sqrt{2}k_5\sqrt{k_0k_5})/(6k_0k_5) + (\sqrt{(2k_5/k_0)})[1/(1 + e^{\xi})]]^2 \\ +B[(2k_0k_6 - 2k_1k_5 - 3\sqrt{2}k_5\sqrt{k_0k_5})/(6k_0k_5) + \sqrt{(2k_5/k_0)})[1/(1 + e^{\xi})]] + C. \end{gathered} \quad (70)$$

$$v_2 = (2k_0k_6 - 2k_1k_5 - 3\sqrt{2}k_5\sqrt{k_0k_5})/(6k_0k_5) + (\sqrt{(2k_5/k_0)})[1/(1 - e^{\xi})], \quad (71)$$

so,the travelling wave solution is given by

$$w_2 = A_0[(2k_0k_6 - 2k_1k_5 - 3\sqrt{2}k_5\sqrt{k_0k_5})/(6k_0k_5) + (\sqrt{(2k_5/k_0)})[1/(1 - e^{\xi})]] + B_0, \quad (72)$$

so,the travelling wave solution is given by

$$\begin{gathered} u_2 = A[(2k_0k_6 - 2k_1k_5 - 3\sqrt{2}k_5\sqrt{k_0k_5})/(6k_0k_5) + (\sqrt{(2k_5/k_0)})[1/(1 - e^{\xi})]]^2 \\ +B[(2k_0k_6 - 2k_1k_5 - 3\sqrt{2}k_5\sqrt{k_0k_5})/(6k_0k_5) + (\sqrt{(2k_5/k_0)})[1/(1 - e^{\xi})]] + C. \end{gathered} \quad (73)$$

4. Conclusion

In this work, we find the analytical solutions of the space-time fractional of classical Drinfeld's Sokolov-Wilson system, the fractional (2+1)-dimensional Davey-Stewartson system and the fractional generalized Hirota- Satsuma coupled KdV system,by using the modified Kudryashov method. Also, we use Jumarie's modified Riemann-Liouvaille derivation formulas and properties to reduce the fractional order differential equations into Riccati type equations. It can be seen clearly that the method is suitable for solving Riccati equations since it is also based on the homogenous balance principle. The obtained solutions are rational function solutions whose structures are in the traveling wave form. This method is effective, useful and easily computable with the help of computer algebra system Mathematica. Therefore, it can be applied to other nonlinear systems.

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