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COMPARATIVE STUDY OF HAAR WAVELET WITH NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: A comparative study of Haar wavelet method with well known numerical methods (Schmidt method, Crank-Nicolson method and Du-Fort Frankel method)for numerical solutions of partial differential equations have been carried out. Numerical examples show that finite difference methods such as Schmidt method, Crank-Nicolson method and Du-Fort Frankel method are equivalent to Haar wavelet method.

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1. Introduction

Most physical phenomena and processes encountered in engineering problems are governed by partial differential equations (PDEs). Disciplines that use partial differential equations to describe the phenomena of interest include fluid mechanics such as flow in long distance pipelines, blood flow, ocean currents, atmospheric dynamics, air pollution, underground dispersion of contaminants, plasma reactors for semiconductor equipments, flow in gas turbine and internal combustion engines. In solid mechanics, problems encountered in vibrations, elasticity, plasticity and structure loading are governed by partial differential equations. The propagation of acoustic and electromagnetic waves and problems in heat and mass transfer are also governed by partial differential equations. Numerical simulations of partial differential equations is far more demanding than that of ordinary differential equations. Many numerical and wavelet methods are used to find the numerical solutions of partial differential equations. Wavelet analysis and wavelet transform are recently developed mathematical tool for solving the linear and non-linear ordinary differential equations, partial differential equations. Wavelets also applied in numerous disciplines such as image compression, data compression, denoising data, etc. Wavelet methods have been applied to solve different types of partial differential equations from beginning of the early 1990s. Haar, Legendre, Chebyshev wavelets are mathematical tools used to find the numerical solution of partial differential equations. Wavelet methods for the solution of differential equations are discussed in many papers (see, for example [2, 3, 4, 5, 6]). Wavelet, being a powerful mathematical tool, has been widely used in image digital processing, quantum field theory and numerical analysis. Most common wavelets used to solve differential and integral equations are Haar wavelets [1, 8, 10, 11, 14, 19, 20, 21, 22], Harmonic wavelets of successive approximation [18], Chebyshev wavelets [12], Legendre wavelets [13].

In Section 2, Haar wavelets method is presented. In Section 3, we describe function approximation. In Section 4, we present Schmidt method, In Section 5, we present Crank-Nicolson method and in Section 6, Du-Fort-Frankel method is presented. In Section 7, numerical examples have been solved using various methods and compared. In Section 8, conclusion is given.

2. Haar Wavelet Method

Haar wavelets have been applied extensively for signal processing in communications and research areas in physical science, mathematical science, chemical science and biological science. Beginning from 1980s, wavelets have been used for solution of partial differential equations. The wavelet algorithms for solving partial differential equations are based on the Galerkin techniques or on the collocation methods. Evidently all attempts to simplify the wavelet solutions for partial differential equations are welcome. One possibility for this is to make use of Haar wavelet family. Haar wavelets which are Daubechies of order 1, are consists of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. A drawback of Haar wavelets is their discontinuity. Since the derivatives do not exist in the breaking points, it is not possible to apply the Haar wavelets for solving partial differential equations directly. There are two possibilities to overcome these situations:

- One way is to regularize the Haar wavelets with interpolating Splines (e.g B-Splines),
- The other way is to make use of the integral method.

In the last few years, the field of Haar wavelets for solving partial differential equations has attracted interest of researchers in several areas of science and engineering. A survey of numerical solutions of differential equation is presented in Hariharan [15]. Wavelet analysis ia a new branch of mathematics and widely applied in differential equations. Several methods have been proposed to find the numerical solution of different linear and nonlinear partial differential equations. Wavelets have been applied extensively in mathematical problems related to scientific and engineering fields. There are many wavelet families such as Daubechies wavelet [17], Hermite-type trigonometric wavelet and many more. In 1910, Alfred Haar [16] introduced a function which presents a rectangular pulse pair. After that various generalizations were proposed. Among all these wavelet families, It is the simplest orthonormal wavelet with compact support. Hariharan et al. [10] presented the numerical solution of Fisher's equation. Also, Hariharan and Kannan [11] presented the numerical solution of Fitzhugh-Nagumo equation. Berwal et. al [8] presented the numerical solution of Telegraph equation. Celik [22] presented the numerical solution of generalized Burger-Huxley equation with Haar wavelet method. Lepik [19, 20, 21] presented the numerical solution of differential and integral equation with Haar wavelet method. The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another.

Let us consider the interval $x\epsilon[A,B]$, where A and B are given constants. We shall define the quantity $M = 2^J$, where J is the maximal level of resolution. The interval $[A,B]$ is divided into $2M$ subintervals of equal length. The length of each subinterval is $\triangle x = (B-A)/2M$. Now, we introduced two parameters : dilatation parameter $j = 0,1,2,......,J$ and translation parameter $k = 0,1,2,.....,m-1$, where $m = 2^j$. The index i is calculated according the formula $i = m+k+1$. In the case of minimal values, $m=1$, $k=0$ we have $i=2$. The maximal value of i is $i=2M$.

$$h_i(x) = \begin{cases} 1, & \xi_1(i) \le x < \xi_2(i), \\ -1, & \xi_2(i) \le x < \xi_3(i), \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Where $\xi_1(i) = A + 2k\mu\triangle x$, $\xi_2(i) = A + (2k+1)\mu\triangle x$, $\xi_3(i) = A + 2(k+1)\mu\triangle x$, $\mu = M/m$. It is assumed that the value $i=1$, corresponding to the scaling function in $[A,B]$.

$$h_1(x) = \begin{cases} 1, & A \le x \le B, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

For solving rth partial differential equations, we need the integrals

$$P_{n,i}(x) = \int_A^x \int_A^x .....\int_A^x h_i(t)dt^n = \frac{1}{(n-1)!}\int_A^x (x-t)^{n-1} h_i(t)dt. \tag{3}$$

where $n = 1,2,3,.....,r$ and $i = 1,2,3,.....,2M$. The case $n=0$ corresponds to the function $h_i(t)$.

The collocation points are $x_l = 0.5(\bar{x_{l-1}} + \bar{x_l})$, $l = 1,2,....,2M$, where $\bar{x_l}$ denotes the lth grid point $\bar{x_l} = A + l\triangle x$, $l = 1,2,.....,2M$. The operational matrix of integration P, which is a $2M$ square matrix, is defined by the relations:

$$P_{i,1}(x) = \int_0^x h_i(t)dt. \tag{4}$$

$$P_{i,n+1}(x) = \int_0^x P_{i,n}(t)dt, \tag{5}$$

where $n = 1,2,3,4....$

$$P_{\alpha.i}(x) = \begin{cases} 0, & x < \xi_1(i) \\ \frac{1}{(\alpha)!}\{(x-\xi_1(i))^\alpha\} & x\epsilon[\xi_1(i),\xi_2(i)) \\ \frac{1}{(\alpha)!}\{(x-\xi_1(i))^\alpha - 2(x-\xi_2(i))^\alpha\} & x\epsilon[\xi_2(i),\xi_3(i)) \\ \frac{1}{(\alpha)!}\{(x-\xi_1(i))^\alpha - 2(x-\xi_2(i))^\alpha + (x-\xi_3(i))^\alpha\} & x > \xi_3(i) \end{cases} \tag{6}$$

Let $\dot{u}''(x,t)$ can be expanded in term of Haar wavelets as follows:

$$\dot{u}''(x,t) = \sum_{i=1}^{2M} a_i h_i(x), \qquad t\epsilon(t_s, t_{s+1}], \tag{7}$$

Integrating the above equation with respect to t from t_s to t, and twice with respect to x, from 0 to x, we get,

$$u''(x,t) = (t - t_s)\sum_{i=1}^{2M} a_i h_i(x) + u''(x, t_s), \tag{8}$$

$$u'(x,t) = (t - t_s)\sum_{i=1}^{2M} a_i P_{1,i}(x) + u'(x, t_s) - u'(0, t_s) + u'(0,t), \tag{9}$$

$$u(x,t) = (t - t_s)\sum_{i=1}^{2M} a_i P_{2,i}(x) + u(x, t_s) - u(0, t_s) - x[u'(0, t_s) - u'(0,t)] + u(0,t), \tag{10}$$

$$\dot{u}(x,t) = \sum_{i=1}^{2M} a_i P_{2,i}(x) + x\dot{u}'(0,t) + \dot{u}(0,t), \tag{11}$$

From the initial and boundary conditions, we have the following equations as:

$$u(x,0) = f(x),\ u(0,t) = g_0(t),\ u(1,t) = g_1(t),\ u(0,t_s) = g_0(t_s),$$

$$u(1,t_s) = g_1(t_s),\ \dot{u}(0,t) = \dot{g}_0(t),\ \dot{u}(1,t) = \dot{g}_1(t).$$

At $x = 1$ in the formulae (10) and (11) and by using conditions, we have

$$u'(0,t) - u'(0,t_s) = -(t - t_s)\sum_{i=1}^{2M} a_i P_{2,i}(1) + g_1(t) - g_1(t_s) + g_0(t_s) - g_0(t), \tag{12}$$

$$\dot{u}'(0,t) = -\sum_{i=1}^{2M} a_i P_{2,i}(1) - \dot{g}_0(t) + \dot{g}_1(t). \tag{13}$$

If the equations (12) and (13) are substituted into equations $(8)-(11)$, and the results are discretized by assuming $x \to x_l$ and $t \to t_{s+1}$, we obtain

$$u''(x_l, t_{s+1}) = (t_{s+1} - t_s)\sum_{i=1}^{2M} a_i h_i(x_l) + u''(x_l, t_s), \tag{14}$$

$$u'(x_l, t_{s+1}) = (t_{s+1} - t_s)\sum_{i=1}^{2M} a_i P_{1,i}(x_l) - (t_{s+1} - t_s)\sum_{i=1}^{2M} a_i P_{2,i}(1) + u'(x_l, t_s) + g_1(t_{s+1}) - g_1(t_s) + g_0(t_s) - g_0(t_{s+1}), \quad (15)$$

$$u(x_l, t_{s+1}) = (t_{s+1} - t_s)\sum_{i=1}^{2M} a_i P_{2,i}(x_l) + x_l[(t_{s+1} - t_s)\sum_{i=1}^{2M} a_i P_{2,i}(1) - g_1(t_{s+1}) + g_1(t_s) - g_0(t_s) + g_0(t_{s+1})] + u(x_l, t_s) + g_0(t_{s+1}) - g_0(t_s), \quad (16)$$

$$\dot{u}(x_l, t_{s+1}) = \sum_{i=1}^{2M} a_i P_{2,i}(x_l) + x_l[\sum_{i=1}^{2M} a_i P_{2,i}(1) + \dot{g_0}(t_{s+1}) - \dot{g_1}(t_{s+1})] + \dot{g_0}(t_{s+1}), \quad (17)$$

But, we know that

$$P_{2,i}(1) = \begin{cases} 0.5, & i = 1, \\ \frac{1}{4m^2}, & i > 1. \end{cases} \quad (18)$$

In the given scheme, which leads us from the time layer t_s to t_{s+1} is used. From here, wavelet coefficients are calculated and solution of given partial differential equation is obtained.

3. Function Approximation

We know that all the Haar wavelets are orthogonal to each other:

$$\int_0^1 h_i(x)h_l(x)dx = \begin{cases} 2^{-j} & i = l = 2^j + k \\ 0 & i \neq l \end{cases} \quad (19)$$

Therefore, they construct a very good transform basis. Any square integrable function $y(x)$ in the interval $[0, 1]$ can be expanded by a Haar series of infinite terms:

$$y(x) = \sum_{i=1}^{\infty} c_i h_i(x), \quad (20)$$

where the Haar coefficients c_i are determined as:

$$c_0 = \int_0^1 y(x)h_0(x)dx \tag{21}$$

$$c_i = 2^j \int_0^1 y(x)h_i(x)dx \tag{22}$$

where $i = 2^j + k,\ j\geq 0$ and $0\leq k < 2^j,\ x\epsilon[0,1]$ such that the following integral square error ε is minimized:

$$\varepsilon = \int_0^1 [y(x) - \sum_{i=1}^{m} c_i h_i(x)]^2 dx \tag{23}$$

where $m = 2^j$ and $j = 0, 1, 2, 3, ...$ Usually the series expansion of (20) contains infinite terms. If $y(x)$ is piecewise constant by itself or may be approximated as piecewise constant during each subinterval, then $y(x)$ will be terminated at finite m terms. This means

$$y(x) \cong \sum_{i=1}^{m} c_i h_i(x) = {c_m}^T h_m(x) \tag{24}$$

where the coefficients ${c_m}^T$ and the Haar function vectors $h_m(x)$ are defined as: ${c_m}^T = [c_1, c_2, c_3, ..., c_m]$ and $h_m(x) = [h_1(x), h_2(x), h_3(x), ..., h_m(x)]^T$,

4. Schmidt Method

Consider a rectangular mesh in the $x-t$ plane with spacing $\triangle x$ along x direction and $\triangle t$ along time t direction. Denoting a mesh point $(x,t) = (ih, jk)$, we have

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\triangle t} \tag{25}$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{\triangle x} \tag{26}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\triangle x)^2} \tag{27}$$

Substituting these values in the given partial differential equation. The relation obtained here, enables us to determine the value of u at the $(i, j+1)$th mesh point in terms of the known function values at the points $x_(i-1)$, x_i and $x_(i+1)$ at the instant t_j.

5. Crank-Nicolson Method

According to this method, $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ are replaced by the average of its central-difference approximations on the jth and $(j+1)$th time rows.

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\triangle t} \tag{28}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\{\frac{u_{i+1,j+1} - u_{i-1,j+1}}{\triangle x} + \frac{u_{i+1,j} - u_{i-1,j}}{\triangle x}\} \tag{29}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\{\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{(\triangle x)^2} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\triangle x)^2}\} \tag{30}$$

Substituting these values in the given partial differential equation. The relation obtained here, enables us to determine the value of u at the $(i, j+1)$th mesh point in terms of the known function values at the points $x_(i-1)$, x_i and $x_(i+1)$ at the instant t_j.

6. Du-Fort-Frankel Method

If we replace the derivatives in the given partial differential equation by the central difference approximations,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2\triangle t} \tag{31}$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{\triangle x} \tag{32}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\triangle x)^2} \tag{33}$$

Substituting these values in the given partial differential equation. In the obtained relation, if we replace

$$u_{i,j} = \frac{u_{i,j+1} + u_{i,j-1}}{2} \tag{34}$$

This difference scheme is called Du-Fort-Frankel method.

7. Numerical Examples

Example 1: Solve the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{35}$$

subject to the initial and boundary conditions $u(x, 0) = sin(\pi x)$, $u(0, t) = u(1, t) = 0$. Table 1 shows the comparison of numerical results for $t = 0.01$ and Table 2 shows the comparison of absolute errors for $t = 0.01$. Numerical results obtained for $\triangle t = 0.001$. Graph 1 shows the comparison of different numerical solutions for $t = 0.001, 0.01, 0.015, 0.02, 0.1$.

xL/16	Exact sol.	Haar sol.	Schmidt sol.	CN sol.	DFF sol.
1	0.1768	0.1768	0.1768	0.1768	0.1768
3	0.5034	0.5036	0.5033	0.5035	0.5034
5	0.7533	0.7537	0.7532	0.7536	0.7534
7	0.8886	0.8891	0.8885	0.8889	0.8887
9	0.8886	0.8891	0.8885	0.8889	0.8887
11	0.7533	0.7537	0.7532	0.7536	0.7534
13	0.5034	0.5036	0.5033	0.5035	0.5034
15	0.1768	0.1768	0.1768	0.1768	0.1768

Table 1: Comparison of numerical solutions of example 1 for t=0.01

xL/16	Haar method	Schmidt method	CN method	DFF method
1	0.00000	0.00000	0.00000	0.00000
3	2.0E-04	1.0E-04	1.0E-04	0.00000
5	4.0E-04	1.0E-04	3.0E-04	1.0E-04
7	5.0E-04	1.0E-04	3.0E-04	1.0E-04
9	5.0E-04	1.0E-04	3.0E-04	1.0E-04
11	4.0E-04	1.0E-04	3.0E-04	1.0E-04
13	2.0E-04	1.0E-04	1.0E-04	0.00000
15	0.00000	0.00000	0.00000	0.00000

Table 2: Comparison of absolute errors of example 1 for t=0.01

Example 2: Solve the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{36}$$

subject to the initial and boundary conditions $u(x,0) = sin\pi x + sin2\pi x$, $u(0,t) = 0, u(1,t) = 0$. Table 3 shows the comparison of numerical results for $t = 0.01$ and Table 4 shows the comparison of absolute errors for $t = 0.01$. Numerical results obtained for for $\triangle t = 0.001$. Graph 2 shows the comparison of different numerical solutions for $t = 0.001, 0.01, 0.015, 0.02, 0.05$.

xL/16	Exact sol.	Haar sol.	Schmidt sol.	CN sol.	DFF sol.
1	0.4346	0.4373	0.4339	0.4360	0.4350
3	1.1259	1.1323	1.1241	1.1292	1.1269
5	1.3759	1.3824	1.3740	1.3792	1.3768
7	1.1465	1.1495	1.1456	1.1480	1.1469
9	0.6307	0.6286	0.6313	0.6297	0.6305
11	0.1308	0.1250	0.1324	0.1279	0.1300
13	-0.1192	-0.1251	-0.1176	-0.1221	-0.1200
15	-0.0811	-0.0836	-0.0804	-0.0823	-0.0815

Table 3: Comparison of numerical solutions of example 2 for t=0.01

xL/16	Haar wavelet	Schmidt method	CN method	DFF method
1	2.7E-03	7.0E-04	1.4E-03	4.0E-04
3	6.4E-03	1.8E-03	3.3E-03	1.0E-03
5	6.6E-03	1.9E-03	3.3E-03	9.0E-04
7	3.0E-03	9.0E-04	1.5E-03	4.0E-04
9	2.1E-03	6.0E-04	1.0E-03	2.0E-04
11	5.8E-03	1.6E-03	2.9E-03	8.0E-04
13	5.9E-03	1.6E-03	2.9E-03	8.0E-04
15	2.5E-03	7.0E-04	1.2E-03	4.0E-04

Table 4: Comparison of absolute errors of example 2 for t=0.01

Example 3: Solve the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{37}$$

subject to the initial and boundary conditions $u(x,0) = 1 + 2x + 3sin\pi x$, $u(0,t) = 1, u(1,t) = 3$. Table 5 shows the comparison of numerical results for $t = 0.01$ and Table 6 shows the comparison of absolute errors for $t = 0.01$. Numerical results obtained for $\triangle t = 0.001$. Graph 3 shows the comparison of

xL/16	Exact sol.	Haar sol.	Schmidt sol.	CN sol.	DFF sol.
1	1.6553	1.6555	1.6552	1.6554	1.6553
3	2.8851	2.8859	2.8848	2.8855	2.8852
5	3.8850	3.8862	3.8846	3.8857	3.8851
7	4.5408	4.5422	4.5404	4.5417	4.5410
9	4.7908	4.7922	4.7904	4.7917	4.7910
11	4.6350	4.6362	4.6346	4.6357	4.6351
13	4.1351	4.1359	4.1348	4.1355	4.1352
15	3.4053	3.4055	3.4052	3.4054	3.4053

Table 5: Comparison of numerical solutions of example 3 for t=0.01

different numerical solutions for $t = 0.001, 0.01, 0.015, 0.02, 0.1$.

xL/16	Haar wavelet	Schmidt method	CN method	DFF method
1	3.0E-04	1.0E-04	1.0E-04	0.00000
3	8.0E-04	3.0E-04	4.0E-04	1.0E-04
5	1.2E-03	4.0E-04	7.0E-04	1.0E-04
7	1.4E-03	4.0E-04	9.0E-04	2.0E-04
9	1.4E-03	4.0E-04	9.0E-04	2.0E-04
11	1.2E-03	4.0E-04	7.0E-04	1.0E-04
13	8.0E-04	3.0E-04	4.0E-04	1.0E-04
15	3.0E-04	1.0E-04	1.0E-04	0.00000

Table 6: Comparison of absolute errors of example 3 for t=0.01

8. Conclusion

It is concluded that numerical results obtained by finite difference methods such as Schmidt method, Crank-Nicolson method and Du-Fort-Frankel methods are very similar to Haar wavelet method. Absolute errors of numerical examples show that Schmidt method, Crank-Nicolson method and Du-Fort-Frankel methods are equivalent to Haar wavelet method.

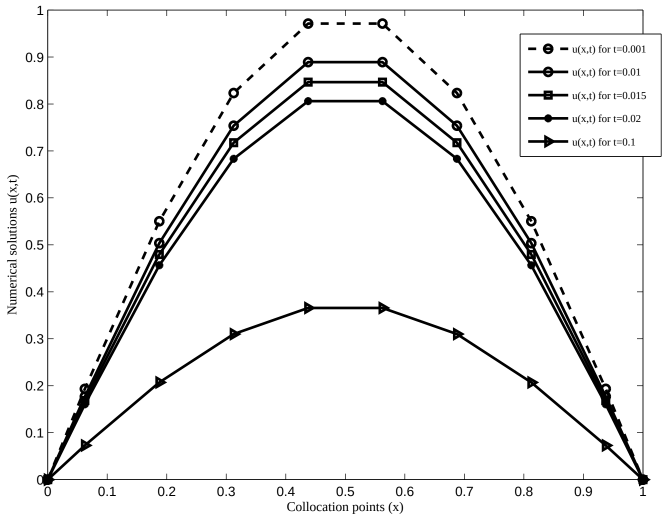

Figure 1: Comparison of numerical solutions of Example 1 for different t

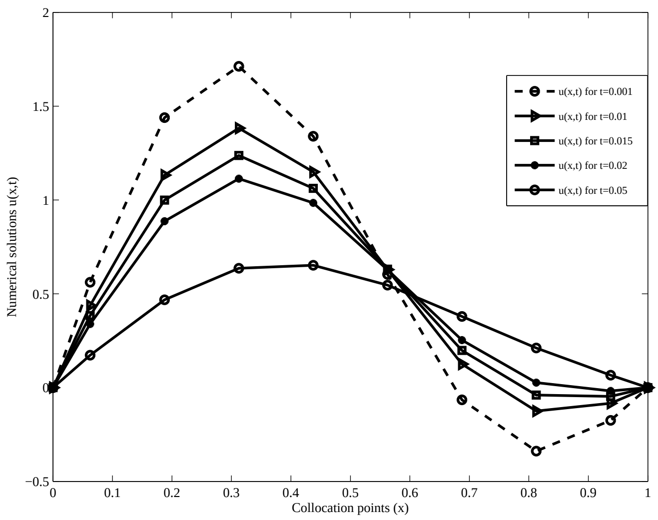

Figure 2: Comparison of numerical solutions of Example 2 for different t

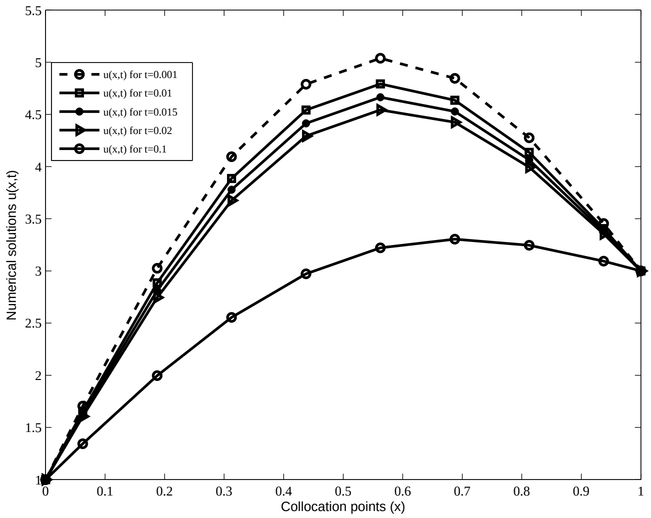

Figure 3: Comparison of numerical solutions of Example 3 for different t

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