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# ON SLIGHTLY $\mathcal{G}$-SEMICONTINUOUS FUNCTIONS

M. Baskar[1 §], N. Rajesh[2]

[1]Department of Mathematics
KCG College of Technology
Karapakkam, Chennai, 600097, Tamilnadu, INDIA

[2]Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur, 613005, Tamilnadu, INDIA

**Abstract:** In this paper a new class of functions called slightly $\mathcal{G}$-semicontinuous functions has been defined and studied in grill topological spaces.

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**Key Words:** grill topological spaces, $\mathcal{G}$-semiopen sets, slightly $\mathcal{G}$-semicontinuity

## 1. Introduction

The idea of grills on a topological space was first introduced by Choquet [5]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [3], [4], [12] for details). In [10], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Mondal and Mukherjee [9] have defined new

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§Correspondence author

classes of sets in a grill topological space and obtained a new decomposition of continuity in terms of grills. The aim of this paper is to give a new class of functions called slightly $\mathcal{G}$-semicontinuous functions in grill topological space. Some characterizations and several basic properties of this class of functions are obtained.

## 2. Preliminaries

Let $(X, \tau)$ be a topological space with no separation properties assumed. For a subset $A$ of a topological space $(X, \tau)$, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and the interior of $A$ in $(X, \tau)$, respectively. The power set of X will be denoted by $\mathcal{P}(X)$. The definition of grill on a topological space, as given by Choquet [5], goes as follows: A non-null collection $\mathcal{G}$ of subsets of a topological space $(X, \tau)$ is said to be a grill on $X$ if

1. $\emptyset \notin \mathcal{G}$,
2. $A \in \mathcal{G}$ and $A \subset B$ implies that $B \in \mathcal{G}$,
3. $A, B \subset X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

**Definition 1.** [10] Let $(X, \tau)$ be a topological space and $\mathcal{G}$ a grill on $X$. A mapping $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: $\Phi(A) = \Phi_{\mathsf{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G}$ for every open set $U$ containing $x\}$ for each $A \in \mathcal{P}(X)$. The mapping $\Phi$ is called the operator associated with the grill $\mathcal{G}$ and the topology $\tau$.

**Definition 2.** [10] Let $\mathcal{G}$ be a grill on a topological space $(X, \tau)$. Then we define a map $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\Psi(A) = A \cup \Phi(A)$ for all $A \in \mathcal{P}(X)$. The map $\Psi$ is a Kuratowski closure axiom. Corresponding to a grill $\mathcal{G}$ on a topological space $(X, \tau)$, there exists a unique topology $\tau_{\mathsf{G}}$ on $X$ given by $\tau_{\mathsf{G}} = \{U \subset X : \Psi(X \backslash U) = X \backslash U\}$, where for any $A \subset X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathsf{G}}\,\mathrm{Cl}(A)$. For any grill $\mathcal{G}$ on a topological space $(X, \tau)$, $\tau \subset \tau_{\mathsf{G}}$. If $(X, \tau)$ is a topological space with a grill $\mathcal{G}$ on $X$, then we call it a grill topological space and denote it by $(X, \tau, \mathcal{G})$.

**Definition 3.** [9] A subset $S$ of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semiopen if $S \subset \Psi(\mathrm{Int}(S))$. The complement of a $\mathcal{G}$-semiopen set is called a $\mathcal{G}$-semiclosed set.

**Definition 4.** The intersection of all $\mathcal{G}$-semiclosed sets containing $S \subset X$ is called the $\mathcal{G}$-semiclosure of $S$ and is denoted by $s\,\mathrm{Cl}_{\mathsf{G}}(S)$. The family of all $\mathcal{G}$-semiopen (resp. $\mathcal{G}$-semiclosed) sets of $(X, \tau, \mathcal{G})$ is denoted by $\mathcal{G}SO(X)$ (resp. $\mathcal{G}SC(X)$). The family of all $\mathcal{G}$-semiopen (resp. $\mathcal{G}$-semiclosed) sets of $(X, \tau, \mathcal{G})$ containing a point $x \in X$ is denoted by $\mathcal{G}SO(X, x)$ (resp. $\mathcal{G}SC(X, x)$).

**Definition 5.** [8] A subset $A$ of a topological space $(X, \tau)$ is called a semiopen set if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semiopen set is called a semiclosed set. The intersection of all semiclosed sets of $(X, \tau)$ containing $A$ is called the semiclosure of $A$ and is denoted by $s\,\mathrm{Cl}(A)$.

**Definition 6.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be slightly continuous [7] (resp. slightly semicontinuous [6]) if $f^{-1}(V)$ is open (resp. semiopen) in $X$ for every clopen set $V$ of $Y$. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma, \mathcal{G})$ is said to be $\mathcal{G}$-semiirresolute if $f^{-1}(V) \in S\mathcal{G}O(X)$ for every $V \in S\mathcal{G}O(Y)$.

## 3. Slightly $\mathcal{G}$-Semicontinuous Functions

**Definition 7.** A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is called :

1. slightly $\mathcal{G}$-semicontinuous at $x \in X$ if for each clopen subset $V$ of $Y$ containing $f(x)$, there exists $U \in S\mathcal{G}O(X, x)$ such that $f(U) \subset V$;
2. slightly $\mathcal{G}$-semicontinuous if it is slightly $\mathcal{G}$-semicontinuous at each point of $X$.

**Theorem 8.** *The following statements are equivalent for a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$:*

1. *$f$ is slightly $\mathcal{G}$-semicontinuous;*
2. *for every clopen subset $V$ of $Y$, $f^{-1}(V)$ is $\mathcal{G}$-semiopen in $X$;*
3. *for every clopen subset $V$ of $Y$, $f^{-1}(V)$ is $\mathcal{G}$-semiclosed in $X$;*
4. *for every clopen subset $V$ of $Y$, $f^{-1}(V)$ is $\mathcal{G}$-semiclopen in $X$.*

*Proof.* The proof is clear. □

**Proposition 9.** *Every slightly $\mathcal{G}$-semicontinuous function is slightly semicontinuous.*

*Proof.* It follows from Proposition 3.1 of [1]. □

The converse of Proposition 9 is need not be true as shown by the following example.

**Example 10.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and

$$\mathcal{G} = \mathcal{P}(X)\backslash\{\emptyset, \{b\}\}.$$

Then the function $f : (X, \tau, \mathcal{G}) \to (X, \tau)$ is defined by $f(a) = c$, $f(b) = a$ and $f(c) = b$ is slightly semicontinuous but not slightly $\mathcal{G}$-semicontinuous.

**Proposition 11.** *Every slightly continuous function slightly $\mathcal{G}$-semicontinuous.*

*Proof.* It follows from Remark 1 of [1]. □

The converse of Proposition 11 is need not be true as shown by the following example.

**Example 12.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and

$$\mathcal{G} = \mathcal{P}(X)\backslash\{\emptyset, \{b\}\}.$$

Then the function $f : (X, \tau, \mathcal{G}) \to (X, \tau)$ is defined by $f(a) = c$, $f(b) = a$ and $f(c) = b$ is slightly continuous but is not slightly $\mathcal{G}$-semicontinuous.

**Theorem 13.** *Let $f : (X, \tau, \mathcal{G}) \to (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \to (Z, \gamma)$ be functions, then the following properties hold:*

1. *If $f$ is slightly $\mathcal{G}$-semicontinuous and $g$ is slightly continuous, then $g \circ f : (X, \tau, \mathcal{G}) \to (Z, \gamma)$ is slightly $\mathcal{G}$-semicontinuous.*

2. *If $f$ is $\mathcal{G}$-semiirresolute and $g$ is slightly $\mathcal{J}$-semicontinuous, then $g \circ f$ is slightly $\mathcal{G}$-semicontinuous.*

3. *If $f$ is $\mathcal{G}$-semiirresolute and $g$ is slightly continuous, then $g \circ f$ is slightly $\mathcal{G}$-semicontinuous.*

*Proof.* The proof is clear. □

**Definition 14.** A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma, \mathcal{J})$ is said to be strongly $\mathcal{G}$-semiopen if $f(U) \in S\mathcal{J}O(Y)$ for every $U \in S\mathcal{G}O(X)$.

**Theorem 15.** *Let $f : (X, \tau, \mathcal{G}) \to (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \to (Z, \eta)$ be functions. Then the following properties hold:*

1. *If $f$ is strongly $\mathcal{G}$-semiopen surjection and $g \circ f$ is slightly $\mathcal{G}$-semicontinuous, then $g$ is slightly $\mathcal{J}$-semicontinuous.*
2. *Let $f$ be strongly $\mathcal{G}$-semiopen and $\mathcal{G}$-semiirresolute surjection. Then $g$ is slightly $\mathcal{J}$-semicontinuous if and only if $g \circ f$ is slightly $\mathcal{G}$-semicontinuous.*

*Proof.* The proof is clear. □

**Remark 16.** A subset $A$ of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semiopen if and only if for all $x \in A$, there exists $A_x \in S\mathcal{G}O(X)$ such that $x \in A_x \subset A$.

**Theorem 17.** *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is slightly $\mathcal{G}$-semicontinuous if and only if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$ is slightly $\mathcal{G}$-semicontinuous.*

*Proof.* Let $x \in X$ and let $W$ be a clopen subset of $X \times Y$ containing $g(x)$. Then $W \cap (\{x\} \times Y)$ is clopen in $\{x\} \times Y$ containing $g(x)$. Also $\{x\} \times Y$ is homeomorphic to $Y$. Hence $\{y \in Y | (x, y) \in W\}$ is a clopen subset of $Y$. Since $f$ is slightly $\mathcal{G}$-semicontinuous, $\cup \{f^{-1}(y) | (x, y) \in W\}$ is a $\mathcal{G}$-semiopen subset of$(X, \tau, \mathcal{G})$. Further, $x \in \cup \{f^{-1}(y) \mid (x, y) \in W\} \subset g^{-1}(W)$. Hence $g^{-1}(W)$ is $\mathcal{G}$-semiopen. Then $g$ is slightly $\mathcal{G}$-semicontinuous. Conversely, Let $F$ be a clopen subset of $Y$. Then $X \times F$ is a clopen subset of $X \times Y$. Since $g$ is slightly $\mathcal{G}$-semicontinuous, $g^{-1}$ $(X \times F)$ is a $\mathcal{G}$-semiopen subset of $X$. Also, $g^{-1}$ $(X \times F) = f^{-1}$ $(F)$. Hence $f$ is slightly $\mathcal{G}$-semicontinuous. □

**Definition 18.** A grill topological space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$-semiconnected if $X$ is not the union of two disjoint nonempty $\mathcal{G}$-semiopen sets of $X$.

The proof of the following theorems are easy and hence omitted.

**Theorem 19.** *If $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is slightly $\mathcal{G}$-semicontinuous surjection and $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semiconnected, then $Y$ is connected.*

**Theorem 20.** *If $f$ is a slightly $\mathcal{G}$-semicontinuous function from a $\mathcal{G}$-semiconnected space $(X, \tau, \mathcal{G})$ onto space $(Y, \sigma)$, then $Y$ is not a discrete space.*

**Theorem 21.** *A grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semiconnected if for every slightly $\mathcal{G}$-semicontinuous function from a space $(X, \tau, \mathcal{G})$ into any $T_0$-space $Y$ is constant.*

Let $\{X_\alpha : \alpha \in \Lambda\}$ and $\{Y_\alpha : \alpha \in \Lambda\}$ be two families of topological spaces with the same index set $\Lambda$. The product space of $\{X_\alpha : \alpha \in \Lambda\}$ is denoted by $\Pi$ $\{X_\alpha{:}\alpha \in \Lambda\}$ (or simply $\Pi X_\alpha$). Let $f_\alpha$: $X_\alpha \to Y_\alpha$ be a function for each $\alpha \in \Lambda$. The product function $f$: $\Pi X_\alpha \to \Pi Y_\alpha$ is defined by $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ for each $\{x_\alpha\} \in \Pi X_\alpha$.

**Theorem 22.** *If a function $f : (X, \tau, \mathcal{G}) \to \Pi Y_\alpha$ is slightly $\mathcal{G}$-semicontinuous, then $P_\alpha \circ f$: $(X, \tau, \mathcal{G}) \to Y_\alpha$ is slightly $\mathcal{G}$-semicontinuous for each $\alpha \in \Lambda$, where $P_\alpha$ is the projection of $\Pi Y_\alpha$ onto $Y_\alpha$.*

*Proof.* Let $V_\alpha$ be any clopen set of $Y_\alpha$. Then, $P_\alpha^{-1}(V_\alpha)$ is clopen in $\Pi Y_\alpha$ and hence $(P_\alpha \circ f)^{-1}$ $(V_\alpha) = f^{-1}(P_\alpha^{-1}(V_\alpha))$ is $\mathcal{G}$-semiopen in $X$. Therefore, $P_\alpha \circ f$ is slightly $\mathcal{G}$-semicontinuous. □

## 4. Separation Axioms

**Definition 23.** A grill topological space $(X, \tau, \mathcal{G})$ is said to be:

1. $\mathcal{G}$-semi-$T_1$ [2] if for each pair of distinct points $x$ and $y$ of $X$, there exist *beta*-$\mathcal{G}$-open sets $U$ and $V$ of $X$ such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.

2. $\mathcal{G}$-semi-$T_2$ [2] if for each pair of distinct points $x$ and $y$ in $X$, there exists disjoint $\mathcal{G}$-semiopen sets $U$ and $V$ in $X$ such that $x \in U$ and $y \in V$.

3. clopen-$T_1$ [11] if for each pair of distinct points $x$ and $y$ of $X$, there exist clopen sets $U$ and $V$ of $X$ such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.

4. clopen-$T_2$ [11] if for each pair of distinct points $x$ and $y$ in $X$, there exist disjoint clopen sets $U$ and $V$ in $X$ such that $x \in U$ and $y \in V$.

**Theorem 24.** *If $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is a slightly $\mathcal{G}$-semicontinuous injection and $Y$ is a clopen-$T_1$ space, then $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-$T_1$ space.*

*Proof.* Suppose that $Y$ is clopen-$T_1$. For any two distinct points $x$ and $y$ in $X$, there exist clopen sets $V$ and $W$ of $Y$ such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since $f$ is slightly $\mathcal{G}$-semicontinuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $\mathcal{G}$-semiopen subsets of $(X, \tau, \mathcal{G})$ such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-$T_1$ space. □

**Theorem 25.** *If $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is a slightly $\mathcal{G}$-semicontinuous injection and $Y$ is a clopen-$T_2$ space, then $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-$T_2$ space.*

*Proof.* For any pair of distinct points $x$ and $y$ in $X$, there exist disjoint clopen sets $U$ and $V$ in $Y$ such that $f(x) \in U$ and $f(y) \in V$. Since $f$ is slightly $\mathcal{G}$-semicontinuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\mathcal{G}$-semiopen sets in$(X, \tau, \mathcal{G})$ containing $x$ and $y$, respectively. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ because $U \cap V = \emptyset$. This shows that the space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-$T_2$. □

**Definition 26.** A grill topological space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$-semiregular if for each closed set $F$ and each point $x \notin F$, there exist disjoint $\mathcal{G}$-semiopen sets $U$ and $V$ of $X$ such that $F \subset U$ and $x \in V$.

**Definition 27.** A grill topological space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$-seminormal if for any pair of disjoint closed subsets $F_1$ and $F_2$ of $X$, there exist disjoint $\mathcal{G}$-semiopen sets $U$ and $V$ of $X$ such that $F_1 \subset U$ and $F_2 \subset V$.

**Definition 28.** A topological space $(X, \tau)$ is said to be:

1. ultra Hausdorff [11] if every two distinct points of $X$ can be separated by disjoint clopen sets.

2. ultra regular [11] if each pair of a point and a closed set not containing the point can be separated by disjoint clopen sets.

3. ultra normal [11] if every two disjoint closed sets of $X$ can be separated by clopen sets.

**Theorem 29.** *Let $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ be a slightly $\mathcal{G}$-semicontinuous injection. Then*

1. *if $(Y, \sigma)$ is ultra Hausdorff, then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-$T_2$,*
2. *if $(Y, \sigma)$ is ultra regular and $f$ is open or closed, then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semiregular,*
3. *if $(Y, \sigma)$ is ultra normal and $f$ is closed, then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-seminormal.*

*Proof.* (1) Let $x_1$, $x_2$ be two distinct points of $X$. Then since $f$ is injective and $Y$ is ultra Hausdorff, there exist clopen sets $V_1$ and $V_2$ of $Y$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$, and $V_1 \cap V_2 = \emptyset$. By Theorem 8, $x_i \in f^{-1}(V_i) \in S\mathcal{G}O(X)$ for $i = 1,2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus, $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-$T_2$.
(2) (i) Suppose that $f$ is open. Let $x \in X$ and $U$ be an open set containing $x$. Then $f(U)$ is an open set of $Y$ containing $f(x)$. Since $Y$ is ultra regular, there exists a clopen set $V$ such that $f(x) \in V \subset f(U)$. Since $f$ is a slightly $\mathcal{G}$-semicontinuous injection, by Definition 7 $x \in f^{-1}(V) \subset U$ and $f^{-1}(V)$ is $\mathcal{G}$-semiclopen in $X$. Therefore, $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semiregular. (ii) Suppose that $f$ is closed. Let $x \in X$ and $F$ be any closed set of $X$ not containing $x$. Since $f$ is injective and closed, $f(x) \notin f(F)$ and $f(F)$ is closed in $Y$. By the ultra regularity of $Y$, there exists a clopen set $V$ such that $f(x) \in V \subset Y \backslash f(F)$. Therefore, $x \in f^{-1}(V)$ and $F \subset X \backslash f^{-1}(V)$. By Theorem 8, $f^{-1}(V)$ is an $\mathcal{G}$-semiclopen set in $(X, \tau, \mathcal{G})$. Thus, $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semiregular.
(3). Similar to the proof of (2). □

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