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ISOMORPHISM CONDITIONS FOR CAYLEY DIGRAPHS OF DIFFERENT COMPLETELY SIMPLE SEMIGROUPS

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Abstract: The isomorphism problem for Cayley digraphs has been extensively investigated over the past 30 years. This paper we introduce the conditions for Cayley digraphs of different finite groups and of different completely simple semigroups can be isomorphic.

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1. Introduction

Arthur Cayley (1821-1895) introduced Cayley digraphs of groups in 1878. One of the first investigations on Cayley graphs of algebraic structures can be found in Maschkes Theorem from 1896 about groups of genus zero, that is, groups

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which possess a generating system such that the Cayley graph is planar. The result for groups originates from [10] and is meanwhile folklore, see for example [2]. After this it is natural to investigate Cayley digraphs for semigroups which are unions of groups, so-called completely regular semigroups, see for example [7]. In [1], [8] and [5] Cayley digraphs of right(left) groups, rectangular group and finite simple semigroups, respectively are characterized. In 2012 [5] introduced the conditions for Cayley digraphs of a completely simple semigroup with a one-element connection set to be isomorphic and connected. In this paper, we introduce the conditions for Cayley graphs of different finite groups and then we extent to completely simple semigroups with a one-element connection set to be isomorphic.

2. Basic Definitions

All sets in this paper are assumed to be finite. A non-empty set G is called a *semigroup* if a binary operation is defined on S that is associative, i.e. $a(bc) = (ab)c$ for all $a, b, c \in G$. A semigroup S is called a *monoid* if a binary operation is defined on G that has an *identity*, i.e. there exists an element e in G such that $ae = a = ea$ for all a in G. Let a be an element in a monoid G. An element b of G is called an *inverse* of a if $ab = e = ba$ where e is the identity. Obviously, if G is a monoid and $a \in G$ has an inverse in G, then that inverse is unique. A monoid G will be called a *group* if every elements of G has an inverse in G. A nonempty subset A of a semigroup S is called a *left ideal* if $SA \subseteq A$, a *right ideal* if $AS \subseteq A$, and a (two-sided) *ideal* if it is both a left and a right ideal. A semigroup S is called a *simple* if it has no proper ideals. An element a of a semigroup S is called a *completely regular* if there exists an element $x \in S$ such that $a = axa$ and $ax = xa$. A semigroup S is called a *completely regular* if all its elements are completely regular. A completely regular semigroup S is called a *completely simple* if it is a simple.

Suppose that G is a group, I and Λ are nonempty sets, and P is a $\Lambda \times I$ matrix over a group G. The *Rees matrix semigroup* $\mathcal{M}(G, I, \Lambda, P)$ *with sandwich matrix* P consists of all triples (g, i, λ), where $i \in I, \lambda \in \Lambda$, and $g \in G$ with multiplication defined by the rule $(g_1, i_1, \lambda_1)(g_2, i_2, \lambda_2) = (g_1 p_{\lambda_1 i_2} g_2, i_1, \lambda_2)$.

Theorem 1. *[9] A semigroup S is completely simple if and only if S is isomorphic to a Rees matrix semigroup.*

From now on, the Rees matrix semigroup, will represent the completely simple semigroup.

The *cardinality* of a set X, denoted by $|X|$, is the number of elements in X.

For any nonempty subset A of a semigroup S, let $\langle A\rangle$ denote the *subsemigroup generated by* A in S. Let G be a group and $a \in G$. The *order* of a is the cardinality of the cyclic subsemigroup $\langle\{a\}\rangle$ and is denoted by $|\langle a\rangle|$. For any subgroup K of G, G/K denotes the set of all distinct left cosets of K in G.

Let (V_1, E_1) and (V_2, E_2) be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a *digraph homomorphism* if $(u, v) \in E_1$ implies $(\varphi(u), \varphi(v)) \in E_2$, i.e. φ preserves arcs. We write $\varphi : (V_1, E_1) \to (V_2, E_2)$. A digraph homomorphism $\varphi : (V, E) \to (V, E)$ is called a *digraph endomorphism.* If $\varphi : (V_1, E_1) \to (V_2, E_2)$ is a bijective digraph homomorphism and φ^{-1} is also a digraph homomorphism, then φ is called a *digraph isomorphism.* A digraph isomorphism $\varphi : (V, E) \to (V, E)$ is calleded a *digraph automorphism.*

For any family of nonempty set $\{X_i | i \in I\}$, let $\dot\cup_{i\in I} X_i$ denote the disjoint union of $X_i, i \in I$.

Let $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ be digraphs such that $V_i \cap V_j = \emptyset$ for all $i \neq j$. The *disjoint union* of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ is defined as $\dot\cup_{i=1}^{n}(V_i, E_i) = \left(\dot\cup_{i=1}^{n} V_i, \dot\cup_{i=1}^{n} E_i\right)$.

Let S be a semigroup (group) and $A \subseteq S$. We define the *Cayley digraph* $Cay(S, A)$ as follows: S is the vertex set and (u, v), $u, v \in S$, is an arc in $Cay(S, A)$ if there exists an element $a \in A$ such that $v = ua$. The set A is called the *connection set* of $Cay(S, A)$.

A digraph (V, E) is called a *semigroup (group) digraph* or *digraph of a semigroup (group)* if there exists a semigroup (group) S and a connection set $A \subseteq S$ such that (V, E) is isomorphic to the Cayley digraph $Cay(S, A)$. For any $v \in V$, the number of arcs incident to v is the *indegree* of v and is denoted by $\overrightarrow{d}(v)$.

A subdigraph F of a digraph G is called a *strong subdigraph* of G if and only if whenever u and v are vertices of F and (u, v) is an arc in G, then (u, v) is an arc in F as well.

A *path* from a vertex u_0 to some vertex u_n in a graph (V, E) is a sequence of vertices $u_0, u_1, u_2, \ldots, u_n$, where (u_{i-1}, u_i) for all i, is an arc in (V, E). If (u_{i-1}, u_i) or (u_i, u_{i-1}) is an arc in (V, E), then we say that there is *semipath* between u_0 and u_n. A graph (V, E) is *connected* if there is a semipath between any two vertices.

3. Cayley Digraphs of Groups with One-Element Connection Set

In this section, we shall give the conditions for two Cayley digraphs of finite groups $Cay(G, \{a\})$ and $Cay(G', \{b\})$ being isomorphic. The following lemmas will be used in the proof of the main theorem.

Lemma 2. *Let G be a group, $a \in G$, $G/\langle a\rangle = \{g_1\langle a\rangle, g_2\langle a\rangle, ..., g_t\langle a\rangle\}$ and $(g_i\langle a\rangle, E_i)$ a strong subdigraphs of $Cay(G, \{a\})$. Then*

$$Cay(G, \{a\}) = \dot\cup_{i=1}^{t}(g_i\langle a\rangle, E_i).$$

Proof. It easy to see that

$$G = \dot\cup_{i=1}^{t} g_i\langle a\rangle.$$

To show that

$$(g_1\langle a\rangle, E_1), (g_2\langle a\rangle, E_2), \dots, (g_t\langle a\rangle, E_t)$$

are disjoint strong subdigraphs of $Cay(G, \{a\})$. Suppose that there exist $x = g_j a^{j'}, y = g_k a^{k'}$ where $j \neq k$ and (x, y) is an arc in $Cay(G, \{a\})$. Then $y = xa = g_j a^{j'} a = g_j a^{j'+1}$. So $x, y \in g_j\langle a\rangle$ which is a contradiction. So

$$(g_1\langle a\rangle, E_1), (g_2\langle a\rangle, E_2), \dots, (g_t\langle a\rangle, E_t)$$

are disjoint strong subdigraphs of $Cay(G, \{a\})$.

Since $E(g_i\langle a\rangle, E_i) \subseteq E(Cay(G, \{a\}))$ for all $i \in \{1, 2, \dots, t\}$, then

$$E(\dot\cup_{i=1}^{t}(g_i\langle a\rangle, E_i)) \subseteq E(Cay(G, \{a\})).$$

Let $x = g_j a^{j'}, y = g_k a^{k'}$ and (x, y) is an arc in $Cay(G, \{a\})$. Then $y = xa = g_j a^{j'} a = g_j a^{j'+1}$. So $x, y \in g_j\langle a\rangle$. Hence (x, y) is an arc in $(g_j\langle a\rangle, E_j)$ and then $E(Cay(G, \{a\})) \subseteq E(\dot\cup_{i=1}^{t}(g_i\langle a\rangle, E_i))$. So we have $E(Cay(G, \{a\})) = E(\dot\cup_{i=1}^{t}(g_i\langle a\rangle, E_i))$. Therefore $Cay(G, \{a\}) = \dot\cup_{i=1}^{t}(g_i\langle a\rangle, E_i)$. □

Lemma 3. *Let G be a group, $a \in G$, $G/\langle a\rangle = \{g_1\langle a\rangle, g_2\langle a\rangle, ..., g_t\langle a\rangle\}$ and $(g_i\langle a\rangle, E_i)$ a strong subdigraphs of $Cay(G, \{a\})$. Then $(g_j\langle a\rangle, E_j) \cong (g_k\langle a\rangle, E_k)$ for all $j, k \in \{1, ..., t\}$.*

Proof. We define $f : g_j\langle a\rangle \to g_k\langle a\rangle$ by $f(g_j a^l) = g_k a^l$. Clearly that f is well defined and bijection. We show that f and f^{-1} are digraph homomorphisms. Let $x = g_j a^l, y = g_j a^m \in g_j\langle a\rangle$ and (x, y) is an arc in $g_j\langle a\rangle$. Then $y = xa$. So $f(y) = f(g_j a^l a) = f(g_j a^{l+1}) = g_k a^{l+1} = g_k a^l a = f(x)a$. Hence $(f(x), f(y))$ is an arc in $(g_k\langle a\rangle, E_k)$. This mean that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Therefore $(g_j\langle a\rangle, E_j) \cong (g_k\langle a\rangle, E_k)$. □

Lemma 4. *Let G be a group, $a \in G$, $G/\langle a\rangle = \{g_1\langle a\rangle, g_2\langle a\rangle, ..., g_t\langle a\rangle\}$ and $(g_i\langle a\rangle, E_i)$ a strong subdigraphs of $Cay(G, \{a\})$. Then $(g_i\langle a\rangle, E_i)$ is connected for all $i \in \{1, ..., t\}$.*

Proof. To show that $(g_i\langle a\rangle, E_i)$ is connected. Let $x = g_ia^j, y = g_ia^k$ where $j \neq k$ and $j + l = k$ for some $l \in \mathbb{N}$. For $1 \leq m \leq l$, $g_ia^{j+m} = g_ia^{j+m-1}a$. So $(g_ia^{j+m-1}, g_ia^{j+m})$ is an arc in $Cay(G, \{a\})$ and then $(g_ia^{j+m-1}, g_ia^{j+m})$ is an arc in $(g_i\langle a\rangle, E_i)$. So we have $x = g_ia^j, g_ia^{j+1}, g_ia^{j+2}, ..., g_ia^{j+l-1}, g_ia^{j+l} = y$ is a path from x to y. Therefore $(g_i\langle a\rangle, E_i)$ is connected. □

Theorem 5. *Let G and G' be finite groups, $a \in G$ and $b \in G'$. Then $Cay(G, \{a\}) \cong Cay(G', \{b\})$ if and only if the following conditions hold:*

(1) $|G| = |G'|$;

(2) $|\langle a\rangle| = |\langle b\rangle|$.

Proof. Let $G/\langle a\rangle = \{g_1\langle a\rangle, g_2\langle a\rangle, ..., g_t\langle a\rangle\}$ and $G'/\langle b\rangle = \{g'_1\langle b\rangle, g'_2\langle b\rangle, ..., g'_{t'}\langle b\rangle\}$

($\Rightarrow$) Suppose that $Cay(G, \{a\}) \cong Cay(G', \{b\})$. We thus get $|G| = |G'|$. By Lemma 2, we have $\dot\bigcup_{i=1}^{t}(g_i\langle a\rangle, E_i) \cong \dot\bigcup_{j=1}^{t'}(g'_j\langle b\rangle, E_j)$. By Lemma 3 and 4, we get that $t = t'$. Since $t = |G/\langle a\rangle|$ and $t' = |G'/\langle b\rangle|$,

$$\begin{aligned} |G/\langle a\rangle| &= |G'/\langle b\rangle| \\ |G|/|\langle a\rangle| &= |G'|/|\langle b\rangle| \\ |\langle a\rangle| &= |\langle b\rangle|. \end{aligned}$$

($\Leftarrow$) Let $|G| = |G'|$ and $|\langle a\rangle| = |\langle b\rangle|$. Then we thus get $t = t'$.

We define $f : G \to G'$ by $f(g_i\langle a\rangle) = g'_i\langle b\rangle$ with $f(g_ia^m) = g'_ib^m$. Clearly, f is well defined. Since $|\langle a\rangle| = |\langle b\rangle|$, f is a bijection. To show that f and f^{-1} are digraph homomorphisms. Let $x = g_ia^l, y = g_ia^k \in g_i\langle a\rangle$ and (x, y) be an arc in $Cay(G, \{a\})$. Then $y = xa = g_ia^la$. Therefore $f(y) = f(g_ia^la) = f(g_ia^{l+1}) = g'_ib^{l+1} = g'_ib^lb = f(x)b$. Hence $(f(x), f(y))$ is an arc in $Cay(G', \{b\})$.

This means that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Hence $Cay(G, \{a\}) \cong Cay(G', \{b\})$. □

Example 1. Let $\mathbb{Z}_6$ and S_3 be finite groups, where $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$, $S_3 = \{(1), \sigma, \sigma^2, \tau, \tau\sigma^2, \tau\sigma\}$ is the symmetric group with (1) an identity, $\sigma = (123)$, $\sigma^2 = (132), \tau = (12), \tau\sigma^2 = (13), \tau\sigma = (23)$, and let $a = \bar{2} \in \mathbb{Z}_6$, $b = \sigma \in S_3$.

It is easily seen that $|G| = |G'| = 6$, $|\langle\bar{2}\rangle| = |\langle\sigma^2\rangle| = 3$ and $Cay(\mathbb{Z}_6, \{\bar{2}\}) \cong Cay(S_3, \{\sigma^2\})$ (see Figures 1 and 2).

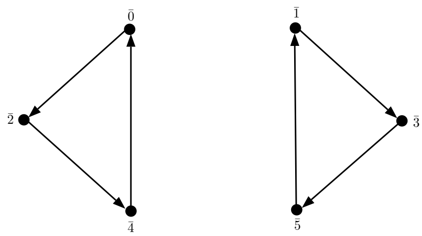

Figure 1: Cayley digraph $Cay(\mathbb{Z}_6, \{\bar{2}\})$.

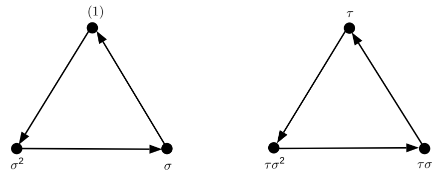

Figure 2: Cayley digraph $Cay(S_3, \{\sigma^2\})$.

4. Cayley Digraphs of Completely Simple Semigroups with One - Element Connection Set

In this section, we shall give the conditions for two Cayley digraphs of finite completely simple semigroups $Cay(S, \{a\})$ and $Cay(S', \{b\})$ being isomorphic. The following lemmas will be used in the proof of the main theorem.

Proposition 6. *[5] Let G be a group, $S = \mathcal{M}(G, \{i\}, \Lambda, P)$. Take $a = (g, i, \beta) \in S$. Then $((g_1, i, \lambda_1), (g_2, i, \lambda_2))$ is an arc in $Cay(S, \{a\})$ if and only if $g_2 = g_1 p_{\lambda_1 i} g$ and $\lambda_2 = \beta$.*

Lemma 7. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup, $a = (g, j, \beta) \in S$ and $v \in V(Cay(S, \{a\}))$. Then $\overrightarrow{d}(v) \neq 0$ if and only if $v \in S_\beta$ where $S_\beta = \{(g, i, \beta) \mid g \in G, i \in I\}$.*

Proof. ($\Rightarrow$) Let $v = (h, i, \alpha)$, assume that $\overrightarrow{d}(v) \neq 0$, then there exists $u \in S$ such that $v = ua$. By Proposition 6, we get that $\beta = \alpha$. Therefore $v \in S_\beta$.

($\Leftarrow$) Assume that $v \in S_\beta$, then $v = (h, i, \beta)$ for some $h \in G, i \in I$. There exists $w = (hg^{-1}p_{\beta j}^{-1}, i, \beta) \in S$ such that $wa = (hg^{-1}p_{\beta j}^{-1}, i, \beta)\,(g, j, \beta) = (hg^{-1}p_{\beta j}^{-1}p_{\beta j}g, i, \beta) = (h, i, \beta) = v$, then (w, v) is an arc in $Cay(S, \{a\})$. So $\overrightarrow{d}(v) \neq 0$. $\square$

Corollary 8. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup and $a = (g, j, \beta) \in S$. Then the numbers of vertices with nonzero indegree of $Cay(S, \{a\})$ is $|G||I|$.*

Corollary 9. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup, $a = (g, j, \beta) \in S$ and $v \in V(Cay(S, \{a\}))$. Then $\overrightarrow{d}(v) = 0$ if and only if $v \notin S_\beta$ where $S_\beta = \{(g, i, \beta) \mid g \in G, i \in I\}$.*

Lemma 10. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup, $I = \{1, 2, \ldots, m\}$, $\Lambda = \{1, 2, \ldots, n\}$ and $a = (g, j, \beta) \in S$. Then for all $v \in V(Cay(S, \{a\}))$, $\overrightarrow{d}(v) = |\Lambda|$ if and only if $v \in S_\beta$ where $S_\beta = \{(g, i, \beta) \mid g \in G, i \in I\}$.*

Proof. ($\Rightarrow$) Let $v = (h, i, \alpha)$ and $\overrightarrow{d}(v) = |\Lambda|$, so $\overrightarrow{d}(v) = n \neq 0$. By Lemma 7, we get that $v = (h, i, \beta) \in S_\beta$.

($\Leftarrow$) Let $v = (h, i, \beta) \in S_\beta$. Since $|\Lambda| = n$, there are distinct of $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $(hg^{-1}p^{-1}_{\alpha_1 j}, i, \alpha_1), (hg^{-1}p^{-1}_{\alpha_2 j}, i, \alpha_2), \ldots, (hg^{-1}p^{-1}_{\alpha_n j}, i, \alpha_n) \in S$. We put $u_1 = (hg^{-1}p^{-1}_{\alpha_1 j}, i, \alpha_1), u_2 = (hg^{-1}p^{-1}_{\alpha_2 j}, i, \alpha_2), \ldots, u_n = (hg^{-1}p^{-1}_{\alpha_n j}, i, \alpha_n)$. Since $u_1 a = (hg^{-1}p^{-1}_{\alpha_1 j}, i, \alpha_1)(g, j, \beta) = (hg^{-1}p^{-1}_{\alpha_1 j}p_{\alpha_1 j}g, i, \beta) = (h, i, \beta) = v$, (u_1, v) is an arc in $Cay(S, \{a\})$. Similarly, $(u_2, v), (u_3, v), \ldots, (u_n, v)$ are arcs in $Cay(S, \{a\})$. We will show that there is no arc to the vertex v, unless the vertices $u_1, u_2, \ldots, u_n$. Suppose that there exists $u = (g', j', \alpha_l) \in S$ such that $u \neq u_k$ for all $k \in \{1, 2, \ldots, n\}$ and (u, v) is an arc in $Cay(S, \{a\})$. Then $v = ua$, so $(h, i, \beta) = (g', j', \alpha_l)(g, j, \beta) = (g'p_{\alpha_l j}g, j', \beta)$. Therefore $h = g'p_{\alpha_l j}g, i = j'$. Then $hg^{-1}p'_{\alpha_l j} = g'$ and so $u = (g', j', \alpha_l) = (hg^{-1}p^{-1}_{\alpha_l j}, i, \alpha_l)$. Since $\alpha_l \in \Lambda$, $u = u_k$ for some $k \in \{1, 2, \ldots, n\}$. That is a contradiction. Hence $\overrightarrow{d}(v) = n = |\Lambda|$. □

Corollary 11. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup, $a \in S$ and $v \in V(Cay(S, \{a\}))$. Then either $\overrightarrow{d}(v) = |\Lambda|$ or $\overrightarrow{d}(v) = 0$.*

Lemma 12. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ and $S' = \mathcal{M}(G', I', \Lambda', P')$ be finite simple semigroups, $a = (g, j, \beta) \in S, b = (h, i, \lambda) \in S'$. If $Cay(S, \{a\}) \cong Cay(S', \{b\})$ then $|\Lambda| = |\Lambda'|$.*

Proof. Let $Cay(S, \{a\}) \cong Cay(S', \{b\})$, then $|S| = |S'|$ and so $|G||I||\Lambda| = |G'||I'||\Lambda'|$. We have that the numbers of vertices with nonzero indegree of both $Cay(S, \{a\})$ and $Cay(S', \{b\})$ are equal. By Corollary 8 , we get that $|G||I| = |G'||I'|$. Therefore $|\Lambda| = |\Lambda'|$. □

Corollary 13. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ and $S' = \mathcal{M}(G', I', \Lambda', P')$ be finite simple semigroups, $a \in S$, $b \in S'$. If $Cay(S, \{a\}) \cong Cay(S', \{b\})$ then $|G||I| = |G'||I'|$.*

Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a finite simple semigroup and $a = (g, j, \beta) \in S$. In [5], we have Cayley digraph $Cay(S, \{a\})$ is the disjoint union of $|I|t$ copies of Cayley digraph of right group $Cay\big(\langle p_{\beta j} g\rangle \times R_{|\Lambda|}, \{(p_{\beta j}g, r_\beta)\}\big)$, where $R_{|\Lambda|}$ is a right zero semigroup and $t = |G/\langle p_{\beta j} g\rangle|$. Moreover, we have Cayley digraph $Cay\big(\langle p_{\beta j} g\rangle \times R_{|\Lambda|}, \{(p_{\beta j}g, r_\beta)\}\big)$ is connected. So we get that $|I|t$ is the number of connected components of $Cay(S, \{a\})$. The next theorem, give the conditions for two cayley digraphs of different completely simple semigroups $Cay(S, \{a\})$ and $Cay(S', \{b\})$ to be isomorphic to each other.

Theorem 14. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ and $S' = \mathcal{M}(G', I', \Lambda', P')$ be finite simple semigroups, $a = (g, j, \beta) \in S$ and $b = (h, i, \lambda) \in S'$. Then $Cay(S, \{a\}) \cong Cay(S', \{b\})$ if and only if the following conditions hold:*

(1) $|\Lambda| = |\Lambda'|$;

(2) $|G||I| = |G'||I'|$;

(3) $|\langle p_{\beta j} g\rangle| = |\langle p_{\lambda i} h\rangle|$.

Proof. Let $t = |G/\langle p_{\beta j} g\rangle|$ and $t' = |G'/\langle p_{\lambda i} h\rangle|$.

($\Rightarrow$) Suppose that $Cay(S, \{a\}) \cong Cay(S', \{b\})$. By Lemma 12 and Corollary 13, we have $|\Lambda| = |\Lambda'|$ and $|G||I| = |G'||I'|$. Because $|I|t$ and $|I'|t'$ are the numbers of connected components of $Cay(S, \{a\})$ and $Cay(S', \{b\})$ respectively, then $|I|t = |I'|t'$. Since $t = |G/\langle p_{\beta j} g\rangle|$ and $t' = |G'/\langle p_{\lambda i} h\rangle|$,

$$\begin{aligned} |I||G|/|\langle p_{\beta j} g\rangle| &= |I'||G'|/|\langle p_{\lambda i} h\rangle| \\ |\langle p_{\beta j} g\rangle| &= |\langle p_{\lambda i} h\rangle|. \end{aligned}$$

($\Leftarrow$) Let $|G||I| = |G'||I'|$, $|\Lambda| = |\Lambda'| = m$ and $|\langle p_{\beta j} g\rangle| = |\langle p_{\lambda i} h\rangle| = l$. Then $|S| = |G||I||\Lambda| = |G'||I'||\Lambda'| = |S'|$. We know that $Cay(S, \{a\})$ is the disjoint union of $|I|t$ copies of $Cay\big(\langle p_{\beta j} g\rangle \times R_m, \{(p_{\beta j} g, r_\beta)\}\big)$ and $Cay(S', \{b\})$ is the disjoint union of $|I'|t'$ copies of $Cay\big(\langle p_{\lambda i} h\rangle \times R_m, \{(p_{\lambda i} h, r_\lambda)\}\big)$ where $R_m = \{r_1, r_2, \ldots, r_m\}$ is a right zero semigroup. Therefore

$$\begin{aligned} |I|t &= |I||G/\langle p_{\beta j} g\rangle| \\ &= |I||G|/|\langle p_{\beta j} g\rangle| \\ &= |I'||G'|/|\langle p_{\beta j} g\rangle| \\ &= |I'|(t'|\langle p_{\lambda i} h\rangle|)/|\langle p_{\beta j} g\rangle| \end{aligned}$$

$$= \quad |I'|t'.$$

So we need only to show that $Cay\big(\langle p_{\beta j}g\rangle\times R_m,\{(p_{\beta j}g,r_\beta)\}\big)\cong Cay\big(\langle p_{\lambda i}h\rangle\times R_m,\{(p_{\lambda i}h,r_\lambda)\}\big)$. We define

$$f:\langle p_{\beta j}g\rangle\times R_m\rightarrow\langle p_{\lambda i}h\rangle\times R_m$$

$$\text{by } f\big((p_{\beta j}g)^d,r_\mu\big)=\begin{cases}\big((p_{\lambda i}h)^d,r_\lambda\big) & if \quad \mu=\beta\\ \big((p_{\lambda i}h)^d,r_\beta\big) & if \quad \mu=\lambda\\ \big((p_{\lambda i}h)^d,r_\alpha\big) & \text{otherwise}\end{cases}$$

Clearly, f is well defined. Since $|\langle p_{\beta j}g\rangle|=|\langle p_{\lambda i}h\rangle|$, f is a bijection. To show that f and f^{-1} are digraph homomorphisms. Let $x,y\in\langle p_{\beta j}g\rangle\times R_m$ and (x,y) be an arc in $Cay\big(\langle p_{\beta j}g\rangle\times R_m,\{(p_{\beta j}g,r_\beta)\}\big)$. Hence $x=\big((p_{\beta j}g)^{d_1},r_{k_1}\big),y=\big((p_{\beta j}g)^{d_2},r_{k_2}\big)$ for some $d_1,d_2\in\{1,2,...,|\langle p_{\beta j}g\rangle|\}$, and so $\big((p_{\beta j}g)^{d_2},r_{k_2}\big)=\big((p_{\beta j}g)^{d_1},\ r_{k_1}\big)(p_{\beta j}g,r_\beta)=\big((p_{\beta j}g)^{d_1+1},r_\beta\big)$. Hence $d_2\equiv d_1+1\ (mod\ l)$ and $k_2=\beta$. We will show that $\big(f(x),f(y)\big)$ is an arc in $Cay\big(\langle p_{\lambda i}h\rangle\times R_m,\{(p_{\lambda i}h,r_\lambda)\}\big)$. We consider three cases:

(case1) If $k_1=\beta$, then

$$\begin{aligned}f(y)&=f\big((p_{\beta j}g)^{d_2},r_\beta\big)\\&=\big((p_{\lambda i}h)^{d_2},r_\lambda\big)\\&=\big((p_{\lambda i}h)^{d_1}p_{\lambda i}h,r_\lambda\big)\\&=\big((p_{\lambda i}h)^{d_1},r_\lambda\big)(p_{\lambda i}h,r_\lambda)\\&=f\big((p_{\beta j}g)^{d_1},r_\beta\big)(p_{\lambda i}h,r_\lambda)\\&=f(x)(p_{\lambda i}h,r_\lambda).\end{aligned}$$

(case2) If $k_1=\lambda$, then

$$\begin{aligned}f(y)&=f\big((p_{\beta j}g)^{d_2},r_\beta\big)\\&=\big((p_{\lambda i}h)^{d_2},r_\lambda\big)\\&=\big((p_{\lambda i}h)^{d_1}p_{\lambda i}h,r_\lambda\big)\\&=\big((p_{\lambda i}h)^{d_1},r_\beta\big)(p_{\lambda i}h,r_\lambda)\\&=f\big((p_{\beta j}g)^{d_1},r_\lambda\big)(p_{\lambda i}h,r_\lambda)\\&=f(x)(p_{\lambda i}h,r_\lambda).\end{aligned}$$

(case3) If $k_1 = \alpha$ where $\alpha \neq \beta, \lambda$, then

$$\begin{aligned} f(y) &= f\big((p_{\beta j}g)^{d_2}, r_\beta\big) \\ &= \big((p_{\lambda i}h)^{d_2}, r_\lambda\big) \\ &= \big((p_{\lambda i}h)^{d_1} p_{\lambda i}h, r_\lambda\big) \\ &= \big((p_{\lambda i}h)^{d_1}, r_\alpha\big)(p_{\lambda i}h, r_\lambda) \\ &= f\big((p_{\beta j}g)^{d_1}, r_\alpha\big)(p_{\lambda i}h, r_\lambda) \\ &= f(x)(p_{\lambda i}h, r_\lambda). \end{aligned}$$

By the above three cases we conclude that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Hence $Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big) \cong Cay\big(\langle p_{\lambda i}g'\rangle \times R_m, \{(p_{\lambda i}g', r_\lambda)\}\big)$. Therefore $Cay(S, \{a\}) \cong Cay(S, \{b\})$. □

Example 2. Let $S = \mathcal{M}(\mathbb{Z}_3, I, \Lambda, P)$ and $S' = \mathcal{M}(S_3, I', \Lambda', P')$ be finite simple semigroups, where $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$, $I = \{1, 2\}$, $\Lambda = \{1, 2\}$, $S_3 = \{(1), \sigma, \sigma^2, \tau, \tau\sigma^2, \tau\sigma\}$ is the symmetric group as in Example 1, $I' = \{1\}$, $\Lambda' = \{1, 2\}$,

$$P = \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{2} & \bar{1} \end{bmatrix}, \; P' = \begin{bmatrix} (1) & \sigma \\ \tau & (1) \end{bmatrix},$$

and let $a = (\bar{2}, 1, 1) \in S$, $b = (\sigma, 2, 1) \in S'$.

It is easily seen that $Cay(S, \{a\}) \cong Cay(S, \{b\})$ (see Figures 3 and 4), $|\Lambda| = |\Lambda'| = 2$ and $|\mathbb{Z}_3||I| = |S_3||I'| = 6$. Moreover, we see that $|\langle p_{11}\bar{2}\rangle| = |\langle p'_{12}\sigma\rangle|$ because $|\langle p_{11}\bar{2}\rangle| = |\langle \bar{0}\bar{2}\rangle| = |\langle \bar{2}\rangle| = 3$ and $|\langle p'_{12}\sigma\rangle| = |\langle \sigma^2\rangle| = 3$.

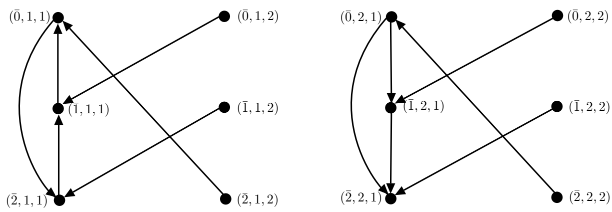

Figure 3: Cayley digraph $Cay(S, \{a\})$.

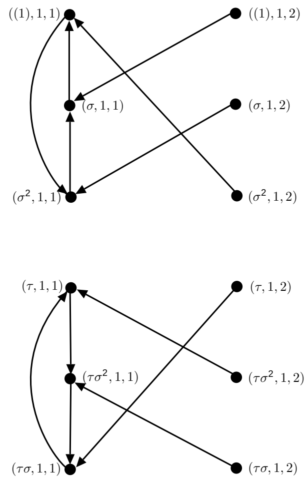

Figure 4: Cayley digraph $Cay(S', \{b\})$.

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