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# GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER MANIFOLD

Dae Ho Jin[1], Jae Won Lee[2 §]

[1]Department of Mathematics
Dongguk University
Gyeongju, 780-714, REPUBLIC OF KOREA

[2]Department of Mathematics Education
Busan National University of Education
Busan, 611-736, REPUBLIC OF KOREA

**Abstract:** In this paper, we study generic lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$. We provide several new results on such a generic lightlike submanifold $M$ equipped with an induced structure tensor $F$ on $M$ induced by the almost complex structure tensor $J$ of $\bar{M}$.

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**Key Words:** generic lihtlike submanifold, indefinite Kaehler manifold, indefinite complex space form

## 1. Introduction

In the classical theory of non-degenerate submanifolds, there exists an important class of submanifolds of an almost complex manifold: A submanifold $(M, g)$ of an almost complex manifold $(\bar{M}, J, \bar{g})$ is called a *generic* (or, *anti-holomorphic*) *submanifold* [16, 17] if its normal bundle $TM^{\perp}$ is mapped into its tangent bundle $TM$ by action of the structure tensor $J$ of $\bar{M}$, i.e., $J(TM^{\perp}) \subset TM$. Any real hypersurface of an almost complex manifold is a

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§Correspondence author

generic submanifold.

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics, especially in the general relativity since lightlike submanifolds can be models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons *etc*). The study of such notion was initiated by Duggal-Bejancu [2] and later studied by many authors (see up-to date results in two books [4, 6]). For any lightlike submanifold $M$ equipped with a screen distribution $S(TM)$, since $TM$ is degenerate while $S(TM)$ is non-degenerate, we newly define the generic lightlike submanifold as follow: A lightlike submanifold $M$ of an indefinite almost complex manifold $\bar{M}$ equipped with an indefinite almost complex structure tensor $J$ is called a *generic lightlike submanifold* [5, 13, 14] if there exists a screen distribution $S(TM)$ of $M$ such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{1}$$

where $S(TM)^{\perp}$ is the orthogonal complement of $S(TM)$ in $T\bar{M}$. The geometry of generic lightlike submanifold is an extension of the geometry of lightlike hypersurface or half lightlike submanifold of codimension 2 of any indefinite almost complex manifold. Much of its theory will be immediately generalized in a formal way to general lightlike (*i.e.*, $r$-lightlike and coisotropic) submanifolds.

In this paper, we study generic lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$. In Section 2, we recall the structure equations of lightlike submanifolds which will be used in the sequel. In Section 3, using the above definition, we provide several new results on generic $r$-lightlike submanifolds of $\bar{M}$ subject such that (1) the structure tensor $F$ of $M$ induced by the almost complex structure tensor $J$ of $\bar{M}$ is parallel with respect to $\nabla$, or (2) either $M$ or $S(TM)$ is totally umbilical.

## 2. Lightlike Submanifolds

Let $(M, g)$ be an $m$-dimensional lightlike submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is a subbundle of the tangent bundle $TM$ and the normal bundle $TM^{\perp}$, of rank $r\,(1 \leq r \leq \min\{m, n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in $TM$ and $TM^{\perp}$ respectively, called the *screen* and *co-screen distributions* on $M$, such that

$$TM = Rad(TM) \oplus_{orth} S(TM)\,, \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \qquad (2)$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$. Denote by $F(M)$ the algebra of smooth functions on $M$, by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$ and by $(*\,.\,*)_i$ the $i$-th equation of the equations $(*\,.\,*)$. We use the same notations for any others. We use the following range of indices:

$$i,\, j,\, k,\, ... \in \{1,\, ...\, ,\, r\}, \qquad \alpha,\, \beta,\, \gamma,\, ... \in \{r+1,\, ...\, ,\, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to $TM$ in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively and let $\{N_1, \cdots, N_r\}$ be a lightlike basis of $ltr(TM)$ consisting of smooth sections of $S(TM)^{\perp}$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \cdots, \xi_r\}$ is a lightlike basis of $\Gamma(Rad\,(TM))$. Then we have

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \qquad (3)\\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

We say that a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is

(1) *$r$-lightlike* if $1 \le r < \min\{m,\, n\}$;
(2) *co-isotropic* if $1 \le r = n < m$;
(3) *isotropic* if $1 \le r = m < n$;
(4) *totally lightlike* if $1 \le r = m = n$.

The last three classes (2)∼(4) of the above definition are particular cases of the class (1) as follows $S(TM^{\perp}) = \{0\}$, $S(TM) = \{0\}$ and $S(TM) = S(TM^{\perp}) = \{0\}$ respectively. The geometry of $r$-lightlike submanifolds is more general form than that of the other submanifolds. For this reason, in this paper we consider only $r$-lightlike submanifolds with following local quasi-orthonormal field of frames:

$$\{\xi_1, \cdots, \xi_r\,,\ N_1, \cdots, N_r\,,\ F_{r+1}, \cdots, F_m\,,\ E_{r+1}, \cdots, E_n\},$$

where the sets $\{F_{r+1}, \cdots, F_m\}$ and $\{E_{r+1}, \cdots, E_n\}$ are orthonormal basis of $S(TM)$ and $S(TM^{\perp})$ respectively. In the following, let $X$, $Y$, $Z$ and $W$ be the vector fields on $M$, unless otherwise specified.

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and $P$ the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to (2). For an $r$-lightlike submanifold, the local Gauss-Weingartan formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{\alpha=r+1}^{n} h_{\alpha}^{s}(X,Y)E_{\alpha}, \tag{4}$$

$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X)E_{\alpha}, \tag{5}$$

$$\bar{\nabla}_X E_{\alpha} = -A_{E_{\alpha}}X + \sum_{i=1}^{r} \phi_{\alpha i}(X)N_i + \sum_{\beta=r+1}^{n} \sigma_{\alpha\beta}(X)E_{\beta}, \tag{6}$$

$$\nabla_X PY = \nabla_X^{*} PY + \sum_{i=1}^{r} h_i^{*}(X,PY)\xi_i, \tag{7}$$

$$\nabla_X \xi_i = -A_{\xi_i}^{*}X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \quad \forall\, X,\, Y \in \Gamma(TM), \tag{8}$$

where $\nabla$ and $\nabla^*$ are induced linear connections on $TM$ and $S(TM)$ respectively, the bilinear forms $h_i^{\ell}$ and $h_{\alpha}^{s}$ on $M$ are called the *local lightlike* and *screen second fundamental forms* on $TM$ respectively, $h_i^*$ are called the *local radical second fundamental forms* on $S(TM)$. $A_{N_i}$, $A_{\xi_i}^*$ and $A_{E_\alpha}$ are linear operators on $\Gamma(TM)$ and $\tau_{ij}$, $\rho_{i\alpha}$, $\phi_{\alpha i}$ and $\sigma_{\alpha\beta}$ are 1-forms on $TM$. Since $\bar{\nabla}$ is torsion-free, $\nabla$ is also torsion-free, and $h_i^{\ell}$ and $h_{\alpha}^{s}$ are symmetric. From the fact $h_i^{\ell}(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$, we show that $h_i^{\ell}$ are independent of the choice of $S(TM)$. We say that

$$h(X,Y) = \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{\alpha=r+1}^{n} h_{\alpha}^{s}(X,Y)E_{\alpha}$$

is the *second fundamental tensor* of $M$.

The induced connection $\nabla$ on $TM$ is not metric and satisfies

$$(\nabla_X g)(Y,Z) = \sum_{i=1}^{r} \{h_i^{\ell}(X,Y)\,\eta_i(Z) + h_i^{\ell}(X,Z)\,\eta_i(Y)\}, \tag{9}$$

where $\eta_i$ are the 1-forms such that

$$\eta_i(X) = \bar{g}(X, N_i), \qquad i \in \{1, \cdots, r\}. \tag{10}$$

However, the connection $\nabla^*$ on $S(TM)$ is metric. The above three local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X,Y) - \sum_{k=1}^{r} h_k^\ell(X,\xi_i)\eta_k(Y), \tag{11}$$
$$\bar{g}(A^*_{\xi_i}X,N_j) = 0;$$
$$\epsilon_\alpha h_\alpha^s(X,Y) = g(A_{E_\alpha}X,Y) - \sum_{i=1}^{r} \phi_{\alpha i}(X)\eta_i(Y), \tag{12}$$
$$\bar{g}(A_{E_\alpha}X,N_i) = \epsilon_\alpha \rho_{i\alpha}(X);$$
$$h_i^*(X,PY) = g(A_{N_i}X,PY), \qquad \eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, \tag{13}$$

and $\epsilon_\beta\sigma_{\alpha\beta} = -\epsilon_\alpha\sigma_{\beta\alpha}$, where $\epsilon_\alpha = \bar{g}(E_\alpha,E_\alpha)$ is the sign of $E_\alpha$ such that $\epsilon_\alpha = \pm 1$. For an $r$-lightlike submanifold, replace $Y$ by $\xi_i$ in (12), we have

$$h_\alpha^s(X,\xi_i) \;=\; -\epsilon_\alpha\phi_{\alpha i}(X). \tag{14}$$

From (4), (8) and (14), we have

$$\bar{\nabla}_X\xi_i = -A^*_{\xi_i}X + \sum_{j=1}^{r}\tau_{ji}(X)\xi_j + \sum_{j=1}^{r} h_j^\ell(X,\xi_i)N_j - \sum_{\alpha=r+1}^{n}\epsilon_\alpha\phi_{\alpha i}(X)E_\alpha. \tag{15}$$

**Definition 1.** A lightlike submanifold $M$ of a semi-Riemannian manifold $(\bar{M},\bar{g})$ is said to be *irrotational* if $\bar{\nabla}_X\xi_i \in \Gamma(TM)$ for all $i$.

**Note 1.** For an $r$-lightlike $M$, the above definition is equivalently to

$$h_j^\ell(X,\xi_i) = 0, \quad h_\alpha^s(X,\xi_i) = \phi_{\alpha i}(X) = 0. \tag{16}$$

Denote by $\bar{R}$ and $R$ the curvature tensors of $\bar{\nabla}$ and $\nabla$ respectively. Using the local Gauss-Weingarten formulas for $M$, we obtain

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z & \\ &+ \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} \\ &+ \sum_{\alpha=r+1}^{n}\{h_\alpha^s(X,Z)A_{E_\alpha}Y - h_\alpha^s(Y,Z)A_{E_\alpha}X\} \\ &+ \sum_{i=1}^{r}\{(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z)\end{aligned} \tag{17}$$

$$
\begin{aligned}
&+ \sum_{j=1}^{r}[\tau_{ji}(X)h_j^{\ell}(Y,Z) - \tau_{ji}(Y)h_j^{\ell}(X,Z)] \\
&+ \sum_{\alpha=r+1}^{n}[\phi_{\alpha i}(X)h_\alpha^s(Y,Z) - \phi_{\alpha i}(Y)h_\alpha^s(X,Z)]\}N_i \\
&+ \sum_{\alpha=r+1}^{n}\{(\nabla_X h_\alpha^s)(Y,Z) - (\nabla_Y h_\alpha^s)(X,Z) \\
&+ \sum_{i=1}^{r}[\rho_{i\alpha}(X)h_i^{\ell}(Y,Z) - \rho_{i\alpha}(Y)h_i^{\ell}(X,Z)] \\
&+ \sum_{\beta=r+1}^{n}[\sigma_{\beta\alpha}(X)h_\beta^s(Y,Z) - \sigma_{\beta\alpha}(Y)h_\beta^s(X,Z)]\}E_\alpha.
\end{aligned}
$$

**Proposition 2.** *Let $M$ be an irrotational $r$-lightlike submanifold of a semi-Riemannian manifold $(\bar{M},\bar{g})$. Then we have the following equation:*

$$\bar{g}(\bar{R}(X,Y)Z,\ \xi_i) = 0, \qquad \forall\, i. \tag{18}$$

*Proof.* Using the local Gauss-Weingarten formulas (7) and (8) for $S(TM)$, we have the following Codazzi equation for $S(TM)$:

$$
\begin{aligned}
R(X,Y)\xi_i = &-\nabla^*_X(A^*_{\xi_i}Y) + \nabla^*_Y(A^*_{\xi_i}X) + A^*_{\xi_i}[X,Y] \\
&- \sum_{j=1}^{r}\tau_{ji}(X)A^*_{\xi_j}Y + \sum_{j=1}^{r}\tau_{ji}(Y)A^*_{\xi_j}X \\
&+ \sum_{j=1}^{r}\{h_j^*(Y,A^*_{\xi_i}X) - h_j^*(X,A^*_{\xi_i}Y) - 2d\tau_{ji}(X,Y) \\
&+ \sum_{k=1}^{r}[\tau_{jk}(X)\tau_{ki}(Y) - \tau_{jk}(Y)\tau_{ki}(X)]\}\xi_j.
\end{aligned} \tag{19}
$$

Taking the scalar product with $Z$ to (19) and using (7), (9) and (11) with the facts $g(R(X,Y)Z,\xi_i) = 0$ and $S(TM)$ is non-degenerate, we have

$$
\begin{aligned}
&\nabla^*_X(A^*_{\xi_i}Y) - \nabla^*_Y(A^*_{\xi_i}X) - A^*_{\xi_i}[X,Y] \\
&+ \sum_{j=1}^{r}\{\tau_{ji}(X)A^*_{\xi_j}Y - \tau_{ji}(Y)A^*_{\xi_j}X\} = 0,
\end{aligned} \tag{20}
$$

$$(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) + \sum_{j=1}^{r}\{\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)\} = 0. \tag{21}$$

Thus (19) is reduced to

$$R(X,Y)\xi_i = \sum_{j=1}^{r}\{h_j^*(Y, A^*_{\xi_i}X) - h_j^*(X, A^*_{\xi_i}Y) - 2d\tau_{ji}(X,Y) + \sum_{k=1}^{r}[\tau_{jk}(X)\tau_{ki}(Y) - \tau_{jk}(Y)\tau_{ki}(X)]\}\xi_j. \tag{22}$$

Taking the scalar product with $\xi_i$ to (17) and using (16) and (21), we have our proposition. □

## 3. Generic Lightlike Submanifolds

Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a $2m$-dimensional indefinite Kaehler manifold ([7] ∼ [12]), where $\bar{g}$ is a semi-Riemannian metric of index $q = 2v$, $0 < v < m$, and $J$ is an almost complex structure on $\bar{M}$ satisfying, for all $X$, $Y \in \Gamma(T\bar{M})$,

$$J^2 = -I, \quad \bar{g}(JX, JY) = \bar{g}(X,Y), \quad (\bar{\nabla}_X J)Y = 0. \tag{23}$$

An indefinite complex space form, denoted by $\bar{M}(c)$, is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature $c$ such that

$$\bar{R}(X,Y)Z = \frac{c}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX - \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(T\bar{M}). \tag{24}$$

Let $M$ be a generic $r$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the screen distribution $S(TM)$ of $M$ is expressed as:

$$\begin{aligned} S(TM) =&\ J(S(TM)^{\perp}) \oplus_{orth} H_o, \\ =&\ \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o, \end{aligned}$$

where $H_o$ is a non-degenerate almost complex distribution on $M$ with respect to $J$, i.e., $J(H_o) = H_o$. Denote $H' = J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp}))$. Then $H'$

is a $r$-lightlike distribution on $S(TM)$ such that $J(H') \subset tr(TM)$. Using (25), the general decompositions of (2) and (3) reduce to

$$TM = H \oplus H', \quad T\bar{M} = H \oplus H' \oplus tr(TM),$$

where $H$ is a $2r$-lightlike almost complex distribution on $M$ such that

$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

Consider the null vector fields $U_i$ and $V_i$ such that $g(U_i, V_j) = \delta_{ij}$ and the non-null vector field $W_\alpha$ on $S(TM)$ defined by

$$U_i = -JN_i, \quad V_i = -J\xi_i, \quad W_\alpha = -JE_\alpha, \ \ \forall\, i,\, \alpha. \tag{25}$$

Denote by $S$ the projection morphism of $TM$ on $H$. Using $TM = H \oplus H'$, for any vector field $X$ on $M$, $JX$ is expressed as:

$$JX = FX + \sum_{i=1}^{r} u_i(X) N_i + \sum_{\alpha=r+1}^{n} w_\alpha(X) E_\alpha, \tag{26}$$

where $u_i$, $v_i$ and $w_\alpha$ are 1-forms locally defined on $\Gamma(TM)$ by

$$u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_\alpha(X) = \epsilon_\alpha g(X, W_\alpha) \tag{27}$$

and $F$ is a tensor field of type $(1,1)$ globally defined on $M$ by $F = J \circ S$. Apply $J$ to (6), (7), (15) and (5) and use (23), (25) and (26), we have

$$\nabla_X U_i = F(A_{N_i} X) + \sum_{j=1}^{r} \tau_{ij}(X) U_j + \sum_{\alpha=r+1}^{r} \rho_{i\alpha}(X) W_\alpha, \tag{28}$$

$$h_j^\ell(X, U_i) = h_i^*(X, V_j), \qquad h_i^*(X, W_\alpha) = \epsilon_\alpha h_\alpha^s(X, U_i)\,; \tag{29}$$

$$\nabla_X W_\alpha = F(A_{E_\alpha} X) + \sum_{i=1}^{r} \phi_{\alpha i}(X) U_i + \sum_{\beta=r+1}^{n} \sigma_{\alpha\beta}(X) W_\beta, \tag{30}$$

$$h_i^\ell(X, W_\alpha) = \epsilon_\alpha h_\alpha^s(X, V_i), \quad \epsilon_\beta h_\beta^s(X, W_\alpha) = \epsilon_\alpha h_\alpha^s(X, W_\beta)\,; \tag{31}$$

$$\nabla_X V_i = F(A^*_{\xi_i} X) - \sum_{j=1}^{r} \tau_{ji}(X) V_j - \sum_{\alpha=r+1}^{r} \epsilon_\alpha \phi_{\alpha i}(X) W_\alpha + \sum_{j=1}^{r} h_j^\ell(X, \xi_i) U_j, \tag{32}$$

$$h_j^\ell(X, V_i) = h_i^\ell(X, V_j), \tag{33}$$

$$(\nabla_X F)(Y) = \sum_{i=1}^{r} u_i(Y) A_{N_i} X + \sum_{\alpha=r+1}^{n} w_\alpha(Y) A_{E_\alpha} X \tag{34}$$

$$- \sum_{i=1}^{r} h_i^{\ell}(X,Y)U_i - \sum_{\alpha=r+1}^{n} h_\alpha^s(X,Y)W_\alpha,$$

$$(\nabla_X u_i)(Y) = -\sum_{j=1}^{r} u_j(Y)\tau_{ji}(X) - \sum_{\alpha=r+1}^{n} w_\alpha(Y)\phi_{\alpha i}(X) - h_i^{\ell}(X,FY), \quad (35)$$

$$(\nabla_X w_\alpha)(Y) = -\sum_{i=1}^{r} u_i(Y)\rho_{i\alpha}(X) - \sum_{\beta=r+1}^{n} w_\beta(Y)\sigma_{\beta\alpha}(X) - h_\alpha^s(X,FY). \quad (36)$$

**Definition 3.** An $r$-lightlike submanifold $M$ of $\bar{M}$ is said to be *totally umbilical* [2] if there is a smooth vector field $\mathcal{H} \in \Gamma(tr(TM))$ such that

$$h(X,Y) = \mathcal{H}\, g(X,Y).$$

In case $\mathcal{H} = 0$, we say that $M$ is *totally geodesic.*

It is easy to see that $M$ is totally umbilical if and only if, locally, there exist smooth functions $\mathcal{A}_i$ and $\mathcal{B}_\alpha$ such that

$$h_i^{\ell}(X,Y) = \mathcal{A}_i\, g(X,Y), \quad h_\alpha^s(X,Y) = \mathcal{B}_\alpha\, g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \quad (37)$$

**Theorem 4.** *Let $M$ be a totally umbilical generic $r$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $M$ is totally geodesic.*

*Proof.* From $(31)_1$ and (37), we obtain

$$\mathcal{A}_i g(X, W_\alpha) = \epsilon_\alpha \mathcal{B}_\alpha g(X, V_i), \quad \forall\, X \in \Gamma(TM).$$

Replace $X$ by $W_\alpha$ and $X$ by $U_i$ to this equations by turns, we have $\mathcal{A}_i = 0$ for all $i$ and $\mathcal{B}_\alpha = 0$ for all $\alpha$. Thus $M$ is totally geodesic. □

**Theorem 5.** *Let $M$ be a generic $r$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $H$ is an integrable distribution on $M$ if and only if we have*

$$h(X, FY) = h(FX, Y), \quad \forall\, X,\, Y \in \Gamma(H).$$

*Moreover, if $M$ is totally umbilical, then $H$ is a parallel distribution on $M$.*

*Proof.* Taking $Y \in \Gamma(H)$, we have $FY = JY \in \Gamma(H)$ and $u_i(Y) = w_\alpha(Y) = 0$ for all $i$ and $\alpha$. Applying $J$ to (4) with $Y \in \Gamma(H)$ and using (34), we have

$$h_i^{\ell}(X, FY) = g(\nabla_X Y, V_i), \quad h_\alpha^s(X, FY) = \epsilon_\alpha g(\nabla_X Y, W_\alpha), \quad (38)$$

$$(\nabla_X F)(Y) = -\sum_{i=1}^{r} h_i^\ell(X,Y)U_i - \sum_{\alpha=r+1}^{n} h_\alpha^s(X,Y)W_\alpha. \quad (39)$$

By straightforward calculations from two equations of (38), we have

$$h(X,FY) - h(FX,Y) = \sum_{i=1}^{r} g([X,Y],V_i)N_i + \sum_{\alpha=r+1}^{n} \epsilon_\alpha g([X,Y],W_\alpha)E_\alpha.$$

If $H$ is an integrable distribution on $M$, then $[X,Y] \in \Gamma(H)$ for all $X$, $Y \in \Gamma(H)$. This implies $g([X,Y],V_i) = g([X,Y],W_\alpha) = 0$. Thus we get $h(X,FY) = h(FX,Y)$. Conversely if $h(X,FY) = h(FX,Y)$ for all $X$, $Y \in \Gamma(H)$, then we have $g([X,Y],V_i) = g([X,Y],W_\alpha) = 0$. Thus $[X,Y] \in \Gamma(H)$ for all $X$, $Y \in \Gamma(H)$ and $H$ is an integrable distribution on $M$.

If $M$ is totally umbilical, then, from Theorem 3.1 and (38), we have

$$g(\nabla_X Y, V_i) = g(\nabla_X Y, W_\alpha) = 0.$$

This implies $\nabla_X Y \in \Gamma(H)$ for all $X$, $Y \in \Gamma(H)$, and hence $H$ is a parallel distribution on $M$. □

**Theorem 6.** *Let $M$ be a generic $r$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $F$ is parallel on $H$ with respect to $\nabla$ if and only if $H$ is a parallel distribution on $M$.*

*Proof.* Assume that $F$ is parallel on $H$ with respect to $\nabla$. For any $X$, $Y \in \Gamma(H)$, we have $(\nabla_X F)Y = 0$. Taking the scalar product with $V_j$ and $E_\beta$ to (39) with $(\nabla_X F)Y = 0$, we have $h_i^\ell(X,Y) = 0$ for all $i$ and $h_\alpha^s(X,Y) = 0$ for all $\alpha$ respectively. From (38), we have $g(\nabla_X Y, V_i) = 0$ for all $i$ and $g(\nabla_X Y, W_\alpha) = 0$ for all $\alpha$. This imply $\nabla_X Y \in \Gamma(H)$ for all $X$, $Y \in \Gamma(H)$. Thus $H$ is a parallel distribution on $M$.

Conversely if $H$ is a parallel distribution on $M$, then, from (38), we have

$$h_i^\ell(X,FY) = 0, \quad h_\alpha^s(X,FY) = 0, \quad \forall\, X,\, Y \in \Gamma(H),\ i,\ \alpha.$$

For any $Y \in \Gamma(H)$, we show that $F^2 Y = -Y$ by (26). Replace $Y$ by $FY$ to the equations, we have $h_i^\ell(X,Y) = 0$ for all $i$ and $h_\alpha^s(X,Y) = 0$ for all $\alpha$. From these results and (39), the proof is complete. □

**Theorem 7.** *Let $M$ be a generic $r$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If $F$ is parallel with respect to $\nabla$ with respect to $\nabla$, then $H, J(ltr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on $M$ and $M$ is locally a product manifold $M^r \times M^{n-r} \times M^{m-n}$, where $M^r$, $M^{n-r}$ and $M^{m-n}$ are leaves of $J(ltr(TM))$, $J(S(TM^\perp))$ and $H$ respectively.*

*Proof.* Assume that $F$ is parallel with respect to $\nabla$. Then $F$ is parallel on $H$ with respect to $\nabla$. Theorem 3.3 implies $H$ is a parallel distribution on $M$. Apply the operator $F$ to (34) with $(\nabla_X F)Y = 0$ and use the facts $FU_i = FW_\alpha = 0$ for all $i$ and $\alpha$, we have

$$\sum_{i=1}^{r} u_i(Y)F(A_{N_i}X) + \sum_{\alpha=r+1}^{n} w_\alpha(Y)F(A_{E_\alpha}X) = 0.$$

Replacing $Y$ by $U_k$ and $W_\beta$ to this by turns and using (27), we have

$$F(A_{N_i}X) = 0, \qquad F(A_{E_\alpha}X) = 0.$$

Taking the scalar product with $W_\beta$ and $N_k$ to (34) by turns, we have

$$h^s_\alpha(X,Y) = \sum_{i=1}^{r} u_i(Y)w_\alpha(A_{N_i}X) + \sum_{\beta=r+1}^{n} w_\beta(Y)w_\alpha(A_{E_\beta}X), \tag{40}$$

$$\sum_{i=1}^{r} u_i(Y)g(A_{N_i}X, N_k) + \sum_{\alpha=r+1}^{n} w_\alpha(Y)g(A_{E_\alpha}X, N_k) = 0. \tag{41}$$

Replacing $Y$ by $\xi_j$ to (40), we get $\phi_{\alpha i}(X) = 0$ due to (14). Also replacing $Y$ by $W_\beta$ to (41), we have $\rho_{i\alpha}(X) = 0$ due to (12). From these results, (28) and (30), we get

$$\nabla_X U_i = \sum_{j=1}^{r} \tau_{ij}(X)U_j, \quad \nabla_X W_\alpha = \sum_{\beta=r+1}^{n} \sigma_{\alpha\beta}(X)W_\beta.$$

Thus $J(ltr(TM))$, $J(S(TM^\perp))$ and $H$ are parallel distributions on $M$ such that

$$TM = J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)) \oplus_{orth} H.$$

By the decomposition theorem of de Rham [1], we show that $M = M^r \times M^{n-r} \times M^{m-n}$, where $M^r$, $M^{n-r}$ and $M^{m-n}$ are some leaves of $J(ltr(TM))$, $J(S(TM^\perp))$ and $H$ respectively. □

**Definition 8.** A screen distribution $S(TM)$ is called *totally umbilical* [2] if there exist smooth functions $\mathcal{C}_i$ on any coordinate neighborhood $\mathcal{U}$ in $M$ such that

$$h^*_i(X, PY) = \mathcal{C}_i\, g(X,Y). \tag{42}$$

In case $\mathcal{C}_i = 0$ on $\mathcal{U}$ for all $i$, we say that $S(TM)$ is *totally geodesic*.

Due to (13) and (42), we know that $S(TM)$ is totally umbilical in $M$ if and only if each shape operator $A_{N_i}$ of $S(TM)$ satisfies

$$g(A_{N_i}X, PY) = \mathcal{C}_i\, g(X, PY), \tag{43}$$

for some smooth function $\mathcal{C}_i$ on $\mathcal{U} \subseteq M$.

**Theorem 9.** *Let $M$ be a generic $r$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. We have the following assertions.*

*(1) If each $V_i$ is parallel with respect to $\nabla$, then $\tau_{ij} = \phi_{\alpha i} = h^\ell(X, \xi_i) = 0$. In this case $M$ is irrotational. Moreover, we have*

$$A^*_{\xi_i}X = \sum_{j=1}^{r} u_j(A^*_{\xi_i}X)U_j + \sum_{\alpha=r+1}^{n} w_\alpha(A^*_{\xi_i}X)W_\alpha, \quad \forall\, X \in \Gamma(TM).$$

*(ii) If each $U_i$ is parallel with respect to $\nabla$, then $\tau_{ij} = \rho_{i\alpha} = 0$ and*

$$A_{N_i}X = \sum_{j=1}^{r} u_j(A_{N_i}X)U_j + \sum_{\alpha=r+1}^{n} w_\alpha(A_{N_i}X)W_\alpha, \quad \forall\, X \in \Gamma(TM).$$

*(iii) If each $W_\alpha$ is parallel with respect to $\nabla$, then $\phi_{\alpha i} = 0$ and*

$$A_{E_\alpha}X = \sum_{i=1}^{r} u_i(A_{E_\alpha}X)U_i + \sum_{\beta=r+1}^{n} w_\beta(A_{E_\alpha}X)W_\beta, \quad \forall\, X \in \Gamma(TM).$$

*Moreover, if all of $V_i$, $U_i$ and $W_\alpha$ are parallel on $TM$ with respect to $\nabla$, then $S(TM)$ is totally geodesic in $M$ and $\tau_{ij} = \phi_{\alpha i} = \rho_{i\alpha} = 0$ on $\Gamma(TM)$. In this case, each null transversal vector fields $N_i$ of $M$ is a constant on $M$.*

*Proof.* If $V_i$ is parallel with respect to $\nabla$, then, taking the scalar product with $W_\beta$ and $V_k$ to (32) by turns, we have $\phi_{\alpha i}(X) = 0$ and $h^\ell_j(X, \xi_i) = 0$ respectively. Thus $M$ is irrotational. Also, taking the scalar product with $U_k$ to (32), we have $\tau_{ji}(X) = 0$. Thus we have $F(A^*_{\xi_i}X) = 0$. From this result and (26), we obtain

$$J(A^*_{\xi_i}X) = \sum_{j=1}^{r} u_j(A^*_{\xi_i}X)N_j + \sum_{\alpha=r+1}^{n} w_\alpha(A^*_{\xi_i}X)E_\alpha.$$

Apply $J$ to this equation and use (23) and (25), we obtain (i). In a similar way, by using (28) and (30) and by virtue of (25) and (26), we have (ii) and (iii).

Assume that all of $V_i$, $U_i$ and $W_\alpha$ are parallel on $TM$ with respect to $\nabla$. Substituting the equation of (i) into $(29)_1$, we have

$$u_j(A_{N_i}X) = v_i(A^*_{\xi_j}X) = g(A^*_{\xi_j}X,\, U_i) = 0.$$

Also, substituting the equation of (iii) into $(29)_2$, we have

$$w_\alpha(A_{N_i}X) = \epsilon_\alpha v_i(A_{E_\alpha}X) = g(A_{E_\alpha}X,\, U_i) = 0.$$

From the last two equations and the equation of (ii), we see that $A_{N_i}X = 0$ for all $X \in \Gamma(TM)$. From this and (43), we have $\mathcal{C}_i = 0$. Thus $S(TM)$ is totally geodesic in $M$ and all 1-forms $\tau_{ij}$, $\phi_{\alpha i}$ and $\rho_{i\alpha}$, defined by (5) and (6), vanish identically. Using this results and (5), we show that $N$ is a constant on $M$. □

**Theorem 10.** *Let $M$ be a totally umbilical generic $r$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that $S(TM)$ is totally umbilical in $M$. Then $S(TM)$ is totally geodesic in $M$.*

*Proof.* Assume that $M$ is totally umbilical. By Theorem 3.1, we have $h^\ell_i = h^s_\alpha = 0$ for all $i$ and $\alpha$. If $S(TM)$ is totally umbilical in $M$, we have $\mathcal{C}_i g(X, V_j) = 0$ and $\mathcal{C}_i g(X, W_\alpha) = 0$ due to (29). Replacing $X$ by $U_j$ to the first equation or replacing $X$ by $W_\beta$ to the second equation, we get $\mathcal{C}_i = 0$. Thus $S(TM)$ is totally geodesic in $M$. □

**Theorem 11.** *Let $M$ be a totally umbilical generic $r$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that $S(TM)$ is totally umbilical. Then $M$ is locally a product manifold $M^r \times M^s \times M^t$, where $M^r$, $M^s$ and $M^t$ are some leafs of $Rad(TM)$, $H_o^\perp = Span\{V_i, U_i, W_\alpha\}$ and $H_o$ respectively and $s = n + r$, $t = m - n - 2r$.*

*Proof.* As $M$ is totally umbilical. By Theorem 3.2, $H$ is a parallel distribution $M$. Thus, for all $X,\, Y \in \Gamma(H_o)$, we have $\nabla_X Y \in \Gamma(H)$. From (7) and (39), we have

$$\begin{aligned} h^*_i(X, FY) = {} & g(\nabla_X FY,\, N_i) = g((\nabla_X F)Y + F(\nabla_X Y),\; N_i) \\ = {} & g(F(\nabla_X Y),\, N_i) = -g(\nabla_X Y, JN_i) = g(\nabla_X Y,\, U_i), \end{aligned} \tag{44}$$

due to $FY \in \Gamma(H_o)$. If $S(TM)$ is totally umbilical in $M$, then we have $h^*_i = 0$ due to Theorem 2.6. By (7) and (44), we get

$$g(\nabla_X Y, N_i) = 0, \quad g(\nabla_X Y, U_i) = 0, \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H_o).$$

This results and (38) imply $\nabla_X Y \in \Gamma(H_o)$ for all $X, Y \in \Gamma(H_o)$. Thus $H_o$ is a parallel distribution on $M$. By Theorem 3.1 and 3.6, we have $h_i^\ell = h_\alpha^s = A_{N_i} = \phi_{\alpha i} = 0$ and $A_{E_\alpha} X = \sum_{i=1}^r \rho_{i\alpha}(X)\xi_i$. Thus (8) and (28), (30) and (32) deduce respectively to

$$\nabla_X \xi = -\sum_{j=1}^r \tau_{ji}(X)\xi_j,$$
$$\nabla_X U_i = \sum_{j=1}^r \tau_{ij}(X)U_j + \sum_{\alpha=r+1}^n \rho_{i\alpha}(X)W_\alpha,$$
$$\nabla_X W_\alpha = -\sum_{i=1}^r \rho_{i\alpha}(X)V_j + \sum_{\beta=r+1}^n \sigma_{\alpha\beta}(X)W_\beta,$$
$$\nabla_X V_i = -\sum_{j=1}^r \tau_{ji}(X)V_j + \sum_{j=1}^r h_j^\ell(X,\xi_i)U_j.$$

Thus $Rad(TM)$, $H_o^\perp$ and $H_o$ are parallel distributions on $M$ such that

$$TM = Rad(TM) \oplus_{orth} H_o^\perp \oplus_{orth} H_o,$$

where $H_o^\perp = Span\{V_i, U_i, W_\alpha\}$. Thus we have $M = M^r \times M^s \times M^t$, where $M^r$, $M^s$ and $M^t$ are some leaves of $Rad(TM)$, $H_o^\perp$ and $H_o$ respectively and $s = n + r$, $t = m - n - 2r$. □

**Theorem 12.** *Let $M$ be an irrotational generic $r$-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. Then we have $c = 0$.*

*Proof.* Taking the scalar product with $\xi_i$ to (24) and using (18) and (27), we have

$$\frac{c}{4}\{u_i(X)\bar{g}(JY,Z) - u_i(Y)\bar{g}(JX,Z) + 2u_i(Z)\bar{g}(X,JY)\} = 0.$$

Taking $X = Z = U_i$ and $Y = \xi_i$ and using (25) and (27), we have $c = 0$. □

**Corollary 13.** *There exist no irrotational generic $r$-lightlike submanifolds $M$ of an indefinite complex space form $\bar{M}(c)$ with $c \neq 0$.*

**Theorem 14.** *Let $M$ be a generic $r$-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. If $S(TM)$ is totally umbilical in $M$, then we have $c = 0$ and $\mathcal{C}_i = 0$, on any coordinate neighborhood $\mathcal{U} \subset M$. Moreover,*

*(i) $c = 0$ implies the ambient space $\bar{M}(c)$ is a semi-Euclidean space,*

*(ii)* $\mathcal{C}_i = 0$, *on any* $\mathcal{U} \subset M$, *implies* $S(TM)$ *is totally geodesic in* $M$.

*Proof.* As $S(TM)$ is totally umbilical, using (29), (33) and (42), we have

$$h^\ell_j(X, U_i) = \mathcal{C}_i\, g(X, V_j), \quad h^s_\alpha(X, U_i) = \epsilon_\alpha \mathcal{C}_i\, g(X, W_\alpha), \quad \forall\, X \in \Gamma(TM). \tag{45}$$

Taking the scalar product with $N_i$ to (17) and use (12), we have

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)Z,\, N_i) &= \bar{g}(R(X, Y)Z,\, N_i) \\ &+ \sum_{j=1}^{r} \{h^\ell_j(X, Z)\eta_i(A_{N_j}Y) - h^\ell_j(Y, Z)\eta_i(A_{N_j}X)\}, \\ &+ \sum_{\alpha=r+1}^{n} \epsilon_\alpha \{h^s_\alpha(X, Z)\rho_{i\alpha}(Y) - h^s_\alpha(Y, Z)\rho_{i\alpha}(X)\}.\end{aligned} \tag{46}$$

Using the Gauss-Weingarten equations (7) and (8) for $S(TM)$, we obtain the Codazzi equation for $S(TM)$:

$$\begin{aligned}g(R(X, Y)PZ,\, N_i) &= (\nabla_X h^*_i)(Y,\, PZ) - (\nabla_Y h^*_i)(X,\, PZ) \\ &+ \sum_{j=1}^{r} \{h^*_j(X,\, PZ)\tau_{ij}(Y) - h^*_j(Y,\, PZ)\tau_{ij}(X)\}.\end{aligned} \tag{47}$$

Using (9), (23), (24), (27), (42), (46) and (47), we get

$$\begin{aligned}&\mathcal{C}_i \sum_{j=1}^{r} \{h^\ell_j(X, PZ)\eta_j(Y) - h^\ell_j(Y, PZ)\eta_j(X)\} \\ &+ \sum_{\alpha=r+1}^{n} \epsilon_\alpha \{h^s_\alpha(X, PZ)\rho_{i\alpha}(Y) - h^s_\alpha(Y, PZ)\rho_{i\alpha}(X)\} \\ &+ \sum_{j=1}^{r} \{h^\ell_j(X, PZ)\eta_i(A_{N_j}Y) - h^\ell_j(Y, PZ)\eta_i(A_{N_j}X)\} \\ =\ & \{Y[\mathcal{C}_i] - \sum_{j=1}^{r} \mathcal{C}_j \tau_{ij}(Y) - \frac{c}{4}\eta_i(Y)\} g(X, PZ) \\ &- \{X[\mathcal{C}_i] - \sum_{j=1}^{r} \mathcal{C}_j \tau_{ij}(X) - \frac{c}{4}\eta_i(X)\} g(Y, PZ) \\ &- \frac{c}{4}\{\bar{g}(JX, PZ)v_i(Y) - \bar{g}(JY, PZ)v_i(X) - 2\bar{g}(X, JY)v_i(PZ)\}.\end{aligned} \tag{48}$$

Replacing $X$ by $\xi_k$ in this equation, we have

$$\begin{aligned}
&\mathcal{C}_i h_j^\ell(Y, PZ) + \sum_{\alpha=r+1}^{n} \epsilon_\alpha h_\alpha^s(Y, PZ)\rho_{i\alpha}(\xi_j) \\
&+ \sum_{k=1}^{r} h_k^\ell(Y, PZ)\eta_i(A_{N_k}\xi_j) + \mathcal{C}_i \sum_{k=1}^{r} h_k^\ell(\xi_j, PZ)\eta_k(Y) \\
&- \sum_{\alpha=r+1}^{n} \epsilon_\alpha h_\alpha^s(\xi_j, PZ)\rho_{i\alpha}(Y) - \sum_{k=1}^{r} h_k^\ell(\xi_j, PZ)\eta_i(A_{N_k}Y) \\
&= \{\xi_j[\mathcal{C}_i] - \sum_{k=1}^{r} \mathcal{C}_k \tau_{ik}(\xi_j) - \frac{c}{4}\delta_{ij}\} g(Y, PZ) \\
&- \frac{c}{4}\{u_j(PZ)v_i(Y) + 2u_j(Y)v_i(PZ)\},
\end{aligned} \tag{49}$$

for all $Y$, $Z \in \Gamma(TM)$. Replacing $PZ$ by $U_k$ to (49) and using (26) and (45), we have

$$\begin{aligned}
&\mathcal{C}_i \mathcal{C}_k g(Y, V_j) + \sum_{\alpha=r+1}^{n} \epsilon_\alpha \mathcal{C}_k g(Y, W_\alpha)\rho_{i\alpha}(\xi_j) \\
&+ \sum_{h=1}^{r} \mathcal{C}_k g(Y, V_h)\eta_i(A_{N_h}\xi_j) \\
&= \{\xi_j[\mathcal{C}_i] - \sum_{k=1}^{r} \mathcal{C}_k \tau_{ik}(\xi_j) - \frac{c}{4}\delta_{ij}\} g(Y, U_k) - \frac{c}{4}\delta_{jk} v_i(Y).
\end{aligned} \tag{50}$$

Replacing $Y$ with $V_k$, $U_j$ and $W_\alpha$ to (50) by turns, we have

$$\xi_j[\mathcal{C}_i] - \sum_{k=1}^{r} \mathcal{C}_k \tau_{ik}(\xi_j) - \frac{c}{2}\delta_{ij} = 0, \quad \mathcal{C}_k\{\mathcal{C}_i + \eta_i(A_{N_j}\xi_j)\} = 0, \quad \mathcal{C}_k \rho_{i\alpha}(\xi_j) = 0,$$

respectively. This shows that $\mathcal{C}_i = 0$ and $c = 0$. Thus we have our theorem. □

**Corollary 15.** *There exist no generic $r$-lightlike submanifolds $M$ of an indefinite complex space form $\bar{M}(c)$ with $c \neq 0$ such that $S(TM)$ is totally umbilcal in $M$*

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