SOME FUNCTIONS VIA $\Lambda_b(V_b)$-SETS

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Abstract: In this paper, we introduce and study some new types of functions by the use of $\Lambda_b$-sets and $V_b$-sets.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. The concept of $b$-open set in a topological space was introduced by Andrijevic in 1996 [1]. But one year later, this notion was also called $\gamma$-open sets due to El-Atik [7]. A subset $A$ of a topological space $(X, \tau)$ is said to be $b$-open (=$\gamma$-open [7]) if $A \subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A))$, where $\text{Cl}(A)$ denotes the closure of $A$ and $\text{Int}(A)$ denotes the interior of $A$ in $(X, \tau)$. The complement $A^c$ of a $b$-open...
set $A$ is called $b$-closed [1] ($=\gamma$-closed [7]). The family of all $b$-open (resp. $b$-closed) sets in $(X, \tau)$ is denoted by $BO(X, \tau)$ (resp. $BC(X, \tau)$). The intersection of all $b$-closed sets containing $A$ is called the $b$-closure of $A$ [1] and is denoted by $bCl(A)$. Quite recently Caldas et. al. [3] used $b$-open sets to define and investigate the $\Lambda_b$-sets (resp. $V_b$-sets) which are intersection of $b$-open (resp. union of $b$-closed) sets. The purpose of the present paper is introduce d and studied some new type of functions by using $\Lambda_b$-sets and $V_b$-sets. The family of all $b$-open sets of $(X, \tau)$ containing a point $x \in X$ is denoted by $BO(X, x)$.

2. Preliminaries

Throughout this paper $(X, \tau)$, $(Y, \sigma)$ and $(Z, \nu)$ (or simply $X$, $Y$ and $Z$) will always denote topological spaces on which no separation axioms are assumed, unless explicitly stated.

**Definition 2.1.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $b$-continuous [7] ($=\gamma$-continuous [7]) (resp. $b$-irresolute [7]($=\gamma$-irresolute [7])) if for every $A \in \sigma$ (resp. $A \in BO(Y, \sigma)$) $f^{-1}(A) \in BO(X, \tau)$, or equivalently, $f$ is $b$-continuous (resp. $b$-irresolute) if and only if for every closed set $A$ (resp. $b$-closed set $A$) of $(Y, \sigma)$, $f^{-1}(A) \in BC(X, \tau)$.

**Definition 2.2.** Let $B$ be a subset of a topological space $(X, \tau)$. $B$ is a $\Lambda_b$-set (resp. $V_b$-set) [3], if $B = B^{\Lambda_b}$ (resp. $B = B^{V_b}$), where: $B^{\Lambda_b} = \cap\{O : O \supseteq B, O \in BO(X, \tau)\}$ and $B^{V_b} = \cup\{F : F \subset B; F^c \in BO(X, \tau)\}$.

The family of all $\Lambda_b$-sets (resp. $V_b$) of $(X, \tau)$ is denoted by $\tau^{\Lambda_b}$ (resp. $\tau^{V_b}$).

**Proposition 2.3.** For a space $(X, \tau)$, the following statements hold:

(i) $\emptyset$ and $X$ are $\Lambda_b$-sets and $V_b$-sets.

(ii) Every union of $\Lambda_b$-sets (resp. $V_b$-sets) is a $\Lambda_b$-set (resp. $V_b$-set).

(iii) Every intersection of $\Lambda_b$-sets (resp. $V_b$-sets) is a $\Lambda_b$-set (resp. $V_b$-set).

**Proposition 2.4.** Let $A, B$ for some $\{B : \alpha \in \Omega\}$ be subsets of a topological space $(X, \tau)$. Then the following properties are valid [3]:

(a) $B \subset B^{\Lambda_b}$;

(b) If $A \subset B$, then $A^{\Lambda_b} \subset B^{\Lambda_b}$;

(c) $B^{\Lambda_b}A^{\Lambda_b} = B^{\Lambda_b}$;

(d) $(\bigcup_{\alpha \in \Omega} B)^{\Lambda_b} = \bigcup_{\alpha \in \Omega} B^{\Lambda_b}$;

(e) If $A \in BO(X, \tau)$, then $A = A^{\Lambda_b}$ (i.e, $A$ is an $\Lambda_b$-set);

(f) $(B^c)^{\Lambda_b} = (B^{V_b})^c$;

(g) $B^{V_b} \subset B$;
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(h) If $B \in BC(X, \tau)$, then $B = B^{V_b}$ (i.e., $A$ is a $V_b$-set);
(i) $(\bigcup_{\alpha \in \Omega} B)^{\Lambda_b} \subset \bigcup_{\alpha \in \Omega} B^{\Lambda_b}$;
(j) $(\bigcup_{\alpha \in \Omega} B)^{V_b} \supseteq \bigcup_{\alpha \in \Omega} B^{V_b}$.

Definition 2.5. A topological space $X$ is said to be:
(a) $R_0$-space [5] if for each open set $V$ of $X$ and each $x \in V$, $Cl(\{x\}) \subset V$
(b) $b$-$R_0$ [6] if every $b$-open set contains the $b$-closure of each of its singletons.
(c) $b$-$T_1$ [3] if to each pair of distinct points $x, y$ of $X$ there corresponds a $b$-open set $A$ containing $x$ but not $y$ a $b$-open set $B$ containing $y$ but not $x$, or equivalently, $(X, \tau)$ is a $b$-$T_1$-space if and only if every singleton is $b$-closed.

3. $\Lambda_b$-Continuity and $\Lambda_b$-Irresoluteness

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:
(i) $\Lambda_b$-continuous if $f^{-1}(V)$ is a $\Lambda_b$-set in $(X, \tau)$ for every open set $V$ of $(Y, \sigma)$,
(ii) $\Lambda_b$-irresolute if $f^{-1}(V)$ is a $\Lambda_b$-set in $(X, \tau)$ for every $\Lambda_b$-set $V$ of $(Y, \sigma)$,
(iii) pre-$\Lambda_b$-open (resp. pre-$b$-open) if $f(A)$ is a $\Lambda_b$-set (resp. $b$-open set) of $(Y, \sigma)$ for every $\Lambda_b$-set (resp. $b$-open set) $A$ of $(X, \tau)$.

Proposition 3.2. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:
(i) if $f$ is $\Lambda_b$-continuous,
(ii) $f : (X, \tau^{\Lambda_b}) \to (Y, \sigma)$ is continuous,
(iii) $f : (X, \tau^{V_b}) \to (Y, \sigma)$ is continuous.

Proof. Clear.

Definition 3.3. (i): A filterbase $\Lambda$ is said to be $\Lambda_b$-convergent to a point $x$ in $X$ if for any $U \in S^{\Lambda_b}$ containing $x$, there exists $B \in \Lambda$ such that $B \subset U$.
(ii): A filterbase $\Lambda$ is said to be convergent to a point $x$ in $X$ if for any open set $U$ of $X$ containing $x$, there exists $B \in \Lambda$ such that $B \subset U$.

Theorem 3.4. If a function $f : (X, \tau) \to (Y, \sigma)$ is $\Lambda_b$-continuous, then for each point $x \in X$ and each filterbase $\mathcal{F}$ in $X$ $\Lambda_b$-converging to $x$, the filter base $f(\mathcal{F})$ is convergent to $f(x)$.

Proof. Let $x \in X$ and $\mathcal{F}$ be any filterbase in $X$ $\Lambda_b$-converging to $x$. Since $f$ is $\Lambda_b$-continuous, then for any open set $V$ of $(Y, \sigma)$ containing $f(x)$, there exists
There exists a filter base \( f \) containing \( x \) such that \( f(U) \subset V \). Since \( \Lambda_b \) is \( \Lambda_b \)-converging to \( x \), there exists \( B \in F \) such that \( B \subset U \). This means that \( f(B) \subset V \) and hence the filter base \( f(F) \) is convergent to \( f(x) \). \( \square \)

**Definition 3.5.** A sequence \((x_n)\) is said to be \( \Lambda_b \)-convergent to a point \( x \) if for every \( \Lambda_b \) set \( V \) containing \( x \), there exists an index \( x_0 \) such that for \( n \geq n_0 \), \( x_n \in V \).

**Theorem 3.6.** If a function \( f : (X, \tau) \to (Y, \sigma) \) is \( \Lambda_b \)-continuous, then for each point \( x \in X \) and each net \((x_n)\) which is \( \Lambda_b \)-convergent to \( x \), the net \( f(x_n) \) is convergent to \( f(x) \).

**Proof.** The proof is similar to that of Theorem 4.15.

Recall that for a function \( f : (X, \tau) \to (Y, \sigma) \), the subset \( \{(x, f(x)) : x \in X\} \subset X \times Y \) is called the graph of \( f \) and is denoted by \( G(f) \). \( \square \)

**Definition 3.7.** A graph \( G(f) \) of a function \( f : (X, \tau) \to (Y, \sigma) \) is said to be strong \( \Lambda_b \)-set if for each \((x, y) \in (X \times Y) \setminus G(f)\), there exist \( U \in S^{\Lambda_b} \) containing \( x \) and a closed set \( V \) of \( Y \) containing \( y \) such that \((U \times V) \cap G(f) = \emptyset \).

**Lemma 3.8.** A graph \( G(f) \) of a function \( f : (X, \tau) \to (Y, \sigma) \) is strong \( \Lambda_b \)-set in \( X \times Y \) if and only if for each \((x, y) \in (X \times Y) \setminus G(f)\), there exist \( U \in S^{\Lambda_b} \) containing \( x \) and a closed set \( V \) of \( Y \) containing \( y \) such that \( f(U) \cap V = \emptyset \).

**Theorem 3.9.** If \( f : (X, \tau) \to (Y, \sigma) \) is a \( \Lambda_b \)-continuous function and \((Y, \sigma)\) is a \( T_1 \)-space, then \( G(f) \) is strong \( \Lambda_b \)-set.

**Proof.** Let \((x, y) \in (X \times Y) \setminus G(f)\). Then \( y \neq f(x) \). Since \( Y \) is \( T_1 \) there exits an open set \( V \) in \( Y \) such that \( f(x) \in V \) and \( y \notin V \). Since \( f \) is \( \Lambda_b \)-continuous, there exit \( U \in S^{\Lambda_b} \) containing \( x \) such that \( f(U) \subset V \). Therefore, \( f(U) \cap (Y \setminus V) = \emptyset \) and \( Y \setminus V \) is closed subset of \( Y \) containing \( y \). This show that \( G(f) \) is strong \( \Lambda_b \)-set. \( \square \)

**Definition 3.10.** A topological space \( X \) is said to be \( \Lambda_b \)-connected if there does not exist disjoint \( \Lambda_b \)-set \( A \) and \( B \) such that \( A \cup B = X \).

**Theorem 3.11.** If \( f : (X, \tau) \to (Y, \sigma) \) is a \( \Lambda_b \)-continuous surjective function and \( X \) is \( \Lambda_b \)-connected, then \( Y \) is connected.

**Proof.** Follows from the definitions. \( \square \)

**Definition 3.12.** A collection \( \{G_{\alpha} : \alpha \in \Delta\} \) is said to be \( \Lambda_b \)-cover of a subset \( A \) of a topological space \((X, \tau)\) if \( A \subset \bigcup \{G_{\alpha} : X \setminus G_{\alpha} \in S^{\Lambda_b}, \alpha \in \Delta\} \).
Definition 3.13. A topological space \( X \) is said to be
(i) \( \Lambda_b \)-compact if every \( \Lambda_b \)-open cover of \( X \) has a finite subcover;
(ii) countably \( \Lambda_b \)-compact if every \( \Lambda_b \)-open countable cover of \( X \) has a finite subcover;
(iii) \( \Lambda_b \)-Lindelöf if every cover of \( X \) by \( \Lambda_b \)-open set has a countable subcover.

Theorem 3.14. If \( f : (X, \tau) \to (Y, \sigma) \) is a \( \Lambda_b \)-continuous surjection and
\( (X, \tau) \) is \( \Lambda_b \)-compact (resp. countably \( \Lambda_b \)-compact, \( \Lambda_b \)-Lindelöf), then \( Y \) is compact (resp. countably compact, Lindelöf).

Proof. Follows from the definitions. \( \square \)

Theorem 3.15. If \( f : (X, \tau) \to (Y, \sigma) \) is a \( \Lambda_b \)-continuous injective function
and \( (Y, \sigma) \) is a \( T_2 \)-space, then \( (X, \tau) \) is \( \Lambda_b -T_2 \)-space.

Proof. For any pair of distinct points \( x \) any \( y \) in \( X \), there exist distinct open
sets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U \) and \( f(y) \in V \). Since \( f \) is \( \Lambda_b \)-continuous,
\( f^{-1}(U) \) and \( f^{-1}(V) \) are \( \Lambda_b \)-sets in \( X \) containing \( x \) any \( y \), respectively. Therefore
\( f^{-1}(U) \cap f^{-1}(V) = \emptyset \) because \( U \cap V = \emptyset \). This show that \( (X, \tau) \) is \( \Lambda_b -T_2 \) \( \square \)

Proposition 3.16. For a function \( f : (X, \tau) \to (Y, \sigma) \) the following properties are equivalent:
(i) \( f \) is \( \Lambda_b \)-irresolute,
(ii) \( f : (X, \tau^{\Lambda_b}) \to (Y, \sigma^{\Lambda_b}) \), is continuous,
(iii) \( f : (X, \tau^{V_b}) \to (Y, \sigma^{V_b}) \), is continuous.

Proof. (i) \( \implies \) (ii) This is obvious. (ii) \( \implies \) (iii): Let \( B \) be any \( V_b \)-sets of
\( (Y, \sigma) \). Than \( B^c \) is a \( \Lambda_b \)-sets of \( (Y, \sigma) \) and \( f^{-1}(B^c) \) is a \( \Lambda_b \)-sets of \( (X, \tau) \) (iii) \( \implies \)
(i): Let \( B \) be any \( \Lambda_b \)-sets of \( (Y, \sigma) \) then \( B^c \) is a \( V_b \)-sets. Also \( f^{-1}(B^c) = (f^{-1}(B))^c \)
is a \( V_b \)-set. Thus \( f^{-1}(B) \) is \( \Lambda_b \)-sets. \( \square \)

Theorem 3.17. For a function \( f : (X, \tau) \to (Y, \sigma) \), the following properties hold.
(i) If \( f \) is \( b \)-irresolute, then it is \( \Lambda_b \)-irresolute;
(ii) If \( f \) is \( b \)-open and injective, then it is pre-\( \Lambda_b \)-open

Proof. (i) Let \( B \) be a \( \Lambda_b \)-sets of \( (Y, \sigma) \). Since \( f \) is \( b \)-irresolute, we have
\( f^{-1}(B) \subset (f^{-1}(B))^{\Lambda_b} \cap \{ U | f^{-1}(B) \subset U \in BO(X, \tau) \} \subset \cap \{ f^{-1}(V) | V \subset \}
\in BO(Y, \sigma) \} = f^{-1}(\cap \{ V | B \subset V \in BO(Y, \sigma) \}) = (f^{-1}(B))^{\Lambda_b} = f^{-1}(B) \).
Therefore, we obtain \( f^{-1}(B) = (f^{-1}(B))^{\Lambda_b} \) which show that \( f^{-1}(B) \) is a \( \Lambda_b \)
set. Consequently, \( f \) is \( \Lambda_b \)-irresolute.
(ii) Let \( A \) be a \( \Lambda_b \)-set of \( (X, \tau) \). Since \( f \) is pre-\( b \)-open and injective. We have
and bijective, then
\[ f(A) = f(\cap \{ \{U | A \subset U \in BO(X, \tau) \} \}) = \cap \{ f(\{U | A \subset U \in BO(X, \tau) \}) \} = \cap \{ f(U) | A \subset U \in BO(X, \tau) \} \supset \cap \{ V | f(A) \subset V \in BO(Y, \sigma) \} = (f(A))^A_b \supset f(A). \] Therefore we obtain \( f(A) = (f(A))^A_b \). Which show that \( f(A) \) is a \( \Lambda_b \)-set. Consequently, \( f \) is pre-\( \Lambda_b \)-open. \( \square \)

**Corollary 3.18.** If \( f : (X, \tau) \to (Y, \sigma) \) is bijective, \( b \)-irresolute and pre-\( b \)-closed, then

(i) for every \( V_b \)-set \( B \) of \( (Y, \sigma) \), then \( f^{-1}(B) \) is a \( V_b \)-set of \( (X, \tau) \)

(ii) for every \( V_b \)-set \( B \) of \( (X, \tau) \), then \( f(B) \) is a \( V_b \)-set of \( (Y, \sigma) \)

**Proposition 3.19.** (i) If \( f : (X, \tau) \to (Y, \sigma) \) is a \( \Lambda_b \)-irresolute function and \( g : (Y, \sigma) \to (Z, \gamma) \) is a \( \Lambda_b \)-continuous function, then the composition \( g \circ f : (X, \tau) \to (Z, \gamma) \) is \( \Lambda_b \)-continuous.

(ii) If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \gamma) \) are both \( \Lambda_b \)-irresolute, then the composition \( g \circ f : (X, \tau) \to (Z, \gamma) \) is \( \Lambda_b \)-irresolute.

**Proof.** It follows directly from the definitions. \( \square \)

**Definition 3.20.** A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( V_b \)-closed if for each closed set \( F \) of \( X \), \( f(F) \) is a \( V_b \) set of \( (Y, \sigma) \).

**Theorem 3.21.** A function \( f : (X, \tau) \to (Y, \sigma) \) is \( V_b \)-closed if and only if for each subset \( S \) of \( Y \) and for each open set \( U \) containing \( f^{-1}(S) \), there is a \( \Lambda_b \)-set \( V \) of \( Y \) such that \( S \subset V \) and \( f^{-1}(V) \subset U \).

**Proof.** Let \( S \) be a subset of \( Y \) and \( U \) be an open subset of \( X \) such that \( f^{-1}(S) \subset U \). Then, \( Y \setminus f(X \setminus U) = V \) (say), is a \( \Lambda_b \)-set containing \( S \) such that \( f^{-1}(V) \subset U \). Conversely, let \( F \) be an arbitrary closed set of \( X \). Then \( f^{-1}(Y \setminus f(F)) \subset X \setminus F \) and \( X \setminus F \) is open in \( X \). By hypothesis, there is a \( \Lambda_b \)-set \( V \) of \( Y \) such that \( Y \setminus f(F) \subset V \) and \( f^{-1}(V) \subset X \setminus F \) hence \( Y \setminus V \subset f(F) \subset f(X \setminus f^{-1}(V)) \subset Y \setminus V \), which implies \( f(F) = Y \setminus V \). Since \( Y \setminus V \) is a \( V_b \)-set, \( f(F) \) is a \( V_b \)-set; hence \( f \) is a \( V_b \)-closed function. \( \square \)

We consider now some composition properties interms of \( V_b \)-sets.

**Theorem 3.22.** Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \gamma) \) be two functions such that \( g \circ f : (X, \tau) \to (Z, \gamma) \) is \( V_b \)-closed. Then, (i) if \( f \) is continuous and surjective, then \( g \) is \( V_b \)-closed, (ii) if \( g \) is \( b \)-irresolute, pre-\( b \)-closed and bijective, then \( f \) is \( V_b \)-closed.

**Proof.** (i) Let \( F \) be an arbitrary closed set in \( (X, \tau) \). Then \( f \) is a \( V_b \)-set in \( (Y, \sigma) \). Since \( g \) is bijective, \( b \)-irresolute and pre-\( b \)-closed, \( (g \circ f)(F) = g(f(F)) \) is a
$V_b$-set by Corollary 3.21(ii). (ii) The proof follows immediately from definitions. Regarding the restriction $f|_A$ of a function $f : (X, \tau) \to (Y, \sigma)$ to a subset $A$ of $X$, we have the following:

\begin{align*}
\text{Theorem 3.23.} \quad & (i) \text{ If } f : (X, \tau) \to (Y, \sigma) \text{ is } V_b \text{-closed and } A \text{ is a closed set in } (X, \tau), \text{ then its restriction } f|_A : (A, \tau|_A) \to (Y, \sigma) \text{ is } V_b \text{-closed.} \\
& (ii) \text{ Let } B \text{ be a } V_b \text{-set of } (Y, \sigma) \text{ If } f : (X, \tau) \to (Y, \sigma) \text{ is } V_b \text{-closed, then } f|_A : (A, \tau|_A) \to (Y, \sigma) \text{ is } V_b \text{-closed, where } A = f^{-1}(B). \\
\end{align*}

\textbf{Proof.} (i) Let $F$ be a closed set of $(A, \tau|_A)$. Since $A$ is closed in $(X, \tau)$, $F$ is closed in $(X, \tau)$ and $f_A(F) = f(F)$ is a $V_b$-closed set of $(Y, \sigma)$. Therefore, $f|_A$ is $V_b$-closed. (ii) Let $F$ be a closed set of $A$. Then $F = A \cap H$ for some closed set $H$ of $(X, \tau)$. By Proposition 2.3, we have $f(H) \cap B$ is a $V_b$-set in $(Y, \sigma)$ since $B$ is a $V_b$-set. Using $f|_A(F) = f(A \cap H) = f(H) \cap B$, $f|_A$ is $V_b$-closed.

\textbf{Definition 3.24.} A topological space $(X, \tau)$ is said to be $T_b$-space if $\tau_{\Lambda_b} = \tau^{V_b}$.

\textbf{Theorem 3.25.} For a topological space $(X, \tau)$, the following properties are equivalent:

(i) $(X, \tau)$ is a $b$-$R_0$ space;

(ii) $(X, \tau^{V_b})$ is discrete;

(iii) $(X, \tau^{\Lambda_b})$ is discrete;

(iv) For each $x \in X$, $\{x\}$ is a $\Lambda_b$-set of $(X, \tau)$;

(v) $P = (P)^b$ for each $P \in BO(X, \tau)$;

(vi) $(X, \tau)$ is a $T_b$-space;

(vii) $(X, \tau^{\Lambda_b})$ is a $R_0$ space;

\textbf{Proof.} (i) $\Rightarrow$ (ii): It is shown in Theorem 3.11 of [4] that $(X, \tau)$ is $b$-$R_0$ if and only if it is $b$-$T_1$. In [3], it is shown that if $(X, \tau)$ is $b$-$T_1$, then $(X, \tau^{\Lambda})$ is discrete. (ii) $\Rightarrow$ (iii): This is obvious. (iii) $\Rightarrow$ (iv): For each $x \in X$, $\{x\}$ is a $\Lambda_b$-open and $\{x\}$ is a $\Lambda_b$-closed set of $(X, \tau)$. (iv) $\Rightarrow$ (v): Let $P$ be a $b$-open set of $X$. Let $y \in P^c$ then $(\{y\})^\Lambda \supset P^c$ by the assumption. By using Proposition 2.4, we have $P^c \supset \Lambda_b(\{y\}) = (P^c)^\Lambda_b$, and hence $P^c = (P^c)^\Lambda_b$. Then it follows from Proposition 4 that $P = (P^c)^b \Rightarrow (vi)$: By (v), we have $BO(X, \tau) \subset \tau^{\Lambda_b}$. First we show that $\tau^{\Lambda_b} \subset \tau^{V_b}$. Let $A$ be any $\Lambda_b$ of $(X, \tau)$. Then $A = \cap\{V \setminus A \subset V \in BO(X, x)\}$. Since $BO(X, \tau) \subset \tau^{V_b}$, By Proposition 4 we have $A \subset \tau^{V_b}$ and $\tau^{\Lambda_b} \subset \tau^{V_b}$. Next let $A \subset \tau^{V_b}$. Then $X \setminus A \subset \tau^{\Lambda_b}$. Therefore $A \subset \tau^{\Lambda_b}$ and $\tau^{V_b} \subset \tau^{\Lambda_b}$. Consequently, we obtain $\tau^{V_b} = \tau^{\Lambda_b}$ and $(X, \tau)$ is a $T_b$-space. (vi) $\Rightarrow$ (vii) Suppose that $V \in \tau^{\Lambda_b}$ and $x \in V$. Since $(X, \tau)$ is a $T_b$-space, $V \in \tau^{V_b}$ and $V^c \in \tau^{\Lambda_b}$. Since $\{x\} \cap V^c = \emptyset$, $\cl_{\tau^{\Lambda_b}}(\{x\}) \cap V^c = \emptyset$.
and $\text{Cl}_{\tau A_b}(\{x\}) \subset V$ where $\text{Cl}_{\tau A_b}(\{x\})$ denotes the closure of $\{x\}$ in $(X, \tau A_b)$.

(viii) $\Rightarrow$ (i): Let $V \in BO(X, \tau)$ and $x \in V$. Since $BO(X, \tau) \subset \tau A_b$, by (vii), $\text{Cl}_{\tau A_b}(\{x\}) \subset V$. Since $\text{Cl}_{\tau A_b}(\{x\}) \subset \tau V b$, we have $\text{Cl}_{\tau A_b}(\{x\}) = \bigcup\{F : F \in BC(X, \tau), F \subset \text{Cl}_{\tau A_b}(\{x\})\}$ and $x \in \text{Cl}_{\tau A_b}(\{x\})$. There exist $F \in BC(X, \tau)$ such that $x \in X$ and hence we have $b \text{Cl}(\{x\}) \subset F \subset \text{Cl}_{\tau A_b}(\{x\}) \subset V$. This shows that $(X, \tau)$ is a $b$-$R_0$-space.

Corollary 3.26. If $(X, \tau)$ is a $b$-$R_0$-space, then $(X, \tau A_b)$ is a $R_0$-space.

Definition 3.27. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a $\Lambda_b$-homeomorphism if it is $\Lambda_b$-irresolute, pre-$\Lambda_b$-open and bijective.

Theorem 3.28. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following properties hold:

(i) If $f$ is $\Lambda_b$-irresolute injection and $(Y, \sigma)$ is a $T_b$-space, then $(X, \tau)$ is a $T_b$-space.

(ii) If $f$ is pre-$\Lambda_b$-open surjection and $(X, \tau)$ is a $T_b$-space, then $(Y, \sigma)$ is a $T_b$-space.

(iii) Let $f$ be $\Lambda_b$-homeomorphism. Then $(X, \tau)$ is a $T_b$-space, if and only if $(Y, \sigma)$ is a $T_b$-space.

Proof. (i) This follows from Theorem 4.17. (ii) This is analogous to the proof of (i). (iii) This is an immediate consequence of (i) and (ii).

4. Weakly Pre-$\Lambda_b$-Open Functions

Definition 4.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be weakly pre-$\Lambda_b$-open if the image of every $\Lambda_b$-set set in $X$ is open in $Y$.

Clearly, every pre-$\Lambda_b$-open function is weakly pre-$\Lambda_b$-open. But the converse is not true in general.

Example 4.2. Let $X = \{(a, b, c)\}$ and $\tau = \emptyset, \{a\}, X\}$. Then the identity function $f$ on $X$ is weakly pre-$\Lambda_b$-open but not pre-$\Lambda_b$-open.

Remark 4.3. It is evident that, the concepts weakly pre-$\Lambda_b$-openness and $\Lambda_b$-continuity are coincide if the function is a bijective.

Theorem 4.4. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is weakly pre-$\Lambda_b$-open if and only if for every subset $U$ of $X$, $f(\text{Int}_{\tau A_b}(U)) \subset \text{Int}(f(U))$. 
Proof. Let \( f \) be weakly pre-\( \Lambda_b \)-open map. Now we have \( \text{Int} \, f(U) \subset U \) and \( \text{Int}_{\tau_{\Lambda_b}}(U) \) is a \( \Lambda_b \)-set. Hence we obtain that \( f(\text{Int}_{\tau_{\Lambda_b}}(U)) \subset f(U) \). As \( f(\text{Int}_{\tau_{\Lambda_b}}(U)) \) is open, then \( f(\text{Int}_{\tau_{\Lambda_b}}(U)) \subset \text{Int}(f(U)) \). Conversely, assume that \( U \) be a \( \Lambda_b \)-set in \( X \). Then \( f(U) = f(\text{Int}_{\tau_{\Lambda_b}}(U)) \subset \text{Int}(f(U)) \). But usually \( \text{Int}(f(U)) \subset f(U) \). Consequently \( f(U) = \text{Int}(f(U)) \) and hence \( f \) is weakly pre-\( \Lambda_b \)-open.

**Lemma 4.5.** A function \( f : (X, \tau) \to (Y, \sigma) \) is weakly pre-\( \Lambda_b \)-open then \( \text{Int}_{\tau_{\Lambda_b}}(f^{-1}(G)) \subset f^{-1}(\text{Int}(G)) \) for every subset \( G \) of \( Y \).

Proof. Let \( G \) be an arbitrary subset of \( Y \). Then \( \text{Int}_{\tau_{\Lambda_b}}(f^{-1}(G)) \) is a \( \Lambda_b \)-set in \( X \) and \( f \) is weakly pre-\( \Lambda_b \)-open, then \( f(\text{Int}_{\tau_{\Lambda_b}}(f^{-1}(G))) \subset \text{Int}(f^{-1}(G)) \subset \text{Int}(G) \). Thus \( \text{Int}_{\tau_{\Lambda_b}}(f^{-1}(G)) \subset f^{-1}(\text{Int}(G)) \).

**Definition 4.6.** A subset \( S \) is called a \( \Lambda_b \)-neighbourhood of a point of \( x \) of \( X \) if there exist a \( \Lambda_b \)-set \( U \) such that \( x \in U \subset S \).

**Theorem 4.7.** For a function \( f : (X, \tau) \to (Y, \sigma) \), the following are equivalent:

1. \( f \) is weakly pre-\( \Lambda_b \)-open;
2. For each subset \( U \) of \( X \), \( f(\text{Int}_{\tau_{\Lambda_b}}(U)) \subset \text{Int}(f(U)) \);
3. For each \( x \in X \) and each \( \Lambda_b \)-neighbourhood \( U \) of \( x \) in \( X \), there exists a neighbourhood \( V \) of \( f(x) \) in \( Y \) such that \( V \subset f(U) \).

Proof. \((i) \Rightarrow (ii)\): It follows from Theorem 4.4. \((ii) \Rightarrow (iii)\): Let \( x \in X \) and \( U \) be an arbitrary \( \Lambda_b \)-neighbourhood of \( x \) in \( X \). Then there exists a \( \Lambda_b \)-set \( V \) in \( X \) such that \( x \in V \subset U \). Then by \((ii)\), we have \( f(V) = f(\text{Int}_{\tau_{\Lambda_b}}(V)) \subset \text{Int}(f(U)) \) and hence \( f(V) = \text{Int}(f(V)) \). Therefore, it follows that \( f(V) \) is open in \( Y \) such that \( f(x) \in f(V) \subset f(U) \). \((iii) \Rightarrow (i)\): Let \( U \) be an arbitrary \( \Lambda_b \)-set in \( X \), Then for each \( y \in f(U) \), by \((iii)\) there exists a neighbourhood \( V_y \) of \( y \) in \( Y \) such that \( V_y \subset f(U) \). As \( V_y \) is a neighbourhood of \( y \), there exists an open set \( W_y \) in \( Y \) such that \( y \in W_y \subset V_y \). Thus, \( f(U) = \bigcup(W_y : y \in f(U)) \) which is an open set in \( Y \). This implies that \( f \) is weakly pre-\( \Lambda_b \)-open function.

**Theorem 4.8.** A function \( f : (X, \tau) \to (Y, \sigma) \) is weakly pre-\( \Lambda_b \)-open if and only if for any subset \( B \) of \( Y \) and for any \( \Lambda_b \)-set \( F \) of \( X \) containing \( f^{-1}(B) \), there exists a closed set \( G \) of \( Y \) containing \( B \) such that \( f^{-1}(G) \subset F \).

Proof. Similar to the proof of Theorem 3.24.

**Theorem 4.9.** A function \( f : (X, \tau) \to (Y, \sigma) \) is weakly pre-\( \Lambda_b \)-open if and only if \( f(\text{Cl}(B)) \subset \text{Cl}_{\tau_{\Lambda_b}}(f(B)) \) for every subset \( B \) of \( Y \).
Proof. Suppose that \( f \) is weakly pre-\( \Lambda_b \)-open. For any subset \( B \) of \( Y \), \( f^{-1}(B) \subset \text{Cl}_{\tau_B}((f^{-1}(B)) \). Therefore by Theorem 4.8, there exists a closed set \( F \) in \( Y \) such that \( B \subset F \) and \( f^{-1}(F) \subset \text{Cl}_{\tau_B}(f^{-1}(B)) \). Therefore, we obtain \( f^{-1}(\text{Cl}(B)) \subset f^{-1}(F) \subset \text{Cl}_{\tau_B}(f^{-1}(B)) \). Conversely, let \( B \subset Y \) and \( F \) be a \( \Lambda_b \)-set of \( X \) containing \( f^{-1}(B) \). Put \( W = \text{Cl}(B) \), then we have \( B \subset W \) and \( W \) is closed and \( f^{-1}(W) \subset \text{Cl}_{\tau_B}(f^{-1}(B)) \subset F \). Then by Theorem 44, \( f \) is weakly pre-\( \Lambda_b \)-open. \( \square \)

**Lemma 4.10.** Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be two functions and \( g \circ f : X \to Z \) is weakly pre-\( \Lambda_b \)-open. If \( g \) is continuous injective, then \( f \) is weakly pre-\( \Lambda_b \)-open.

Proof. Let \( U \) be a \( \Lambda_b \)-set in \( X \), then \( (g \circ f)(U) \) is open in \( Z \), since \( g \circ f \) is weakly pre-\( \Lambda_b \)-open. Again \( g \) is an injective continuous function, \( f(U) = g^{-1}(g \circ f(U)) \) is open in \( Y \). This show that \( f \) is weakly pre-\( \Lambda_b \)-open. \( \square \)

**Theorem 4.11.** If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) are two weakly pre-\( V_b \)-closed functions, then \( g \circ f : X \to Z \) is a weakly pre-\( V_b \)-closed function.

Proof. Obvious. \( \square \)

Furthermore, we have the following.

**Theorem 4.12.** If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be any two functions. Then

(i). If \( f \) is \( V_b \)-closed and \( g \) is weakly pre-\( V_b \)-closed, then \( g \circ f \) is closed;

(ii) If \( f \) is weakly pre-\( V_b \)-closed and \( g \) is \( V_b \)-closed, then \( g \circ f \) is pre-\( V_b \)-closed;

(iii) If \( f \) is pre-\( V_b \)-closed and \( g \) is weakly pre-\( V_b \)-closed, then \( g \circ f \) is weakly pre-\( V_b \)-closed;

Proof. Obvious. \( \square \)

**Theorem 4.13.** If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be any two functions such that \( g \circ f : X \to Z \) is weakly pre-\( \Lambda_b \)-closed.

(i) If \( f \) is \( V_b \)-irresolute surjective, then \( g \) is closed.

(ii) If \( g \) is \( V_b \)-continuous injective, then \( f \) is pre-\( V_b \)-closed.

Proof. (i) Suppose \( F \) is an arbitrary \( V_b \)-closed set in \( Y \). As \( f \) is \( V_b \)-irresolute, \( f^{-1}(F) \) is \( V_b \)-set in \( X \). Since \( g \circ f \) is weakly pre-\( V_b \)-closed and \( f \) is surjective, \( (g \circ f(f^{-1}(F))) = g(F) \), which is closed in \( Z \). This implies that \( g \) is a closed function. (ii) Suppose \( F \) is any \( V_b \)-closed set in \( X \). Since \( g \circ f \)
is weakly pre-$V_b$-closed, $(g \circ f)(F)$ is closed in $Z$. Again $g$ is a $V_b$-continuous injective function, $g^{-1}(g \circ f(F)) = f(F)$, which is $V_b$-closed in $Y$. This shows that $f$ is pre-$V_b$-closed. \qed

**Theorem 4.14.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. Then the function $g : (X, \tau) \to (Y, \sigma)$ is a weakly pre-$V_b$-closed if and only $g(X)$ is closed in $Y$ and $g(V) - g(X - V)$ is open in $g(X)$ whenever $V$ is $\Lambda_b$-closed in $X$.

**Proof.** Necessity: Suppose $g : (X, \tau) \to (Y, \sigma)$ is a weakly pre-$V_b$-closed function. Since $X$ is $\Lambda_b$-set, $g(X)$ is closed in $Y$ and $g(V) - g(X - V) = g(X) - g(X - V)$ is open in $g(X)$ when $V$ is $\Lambda_b$-closed in $X$. Sufficiency: Suppose $g(X)$ is closed in $Y$, $g(V) - g(X - V)$ is open in $g(X)$ when $V$ is $\Lambda_b$-closed in $X$, and let $C$ be closed in $X$. Then $g(C) = g(X) - (g(X - C) - g(C))$ is closed in $g(X)$ and hence, closed in $Y$. \qed

**Corollary 4.15.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. Then a surjective $g : (X, \tau) \to (Y, \sigma)$ is a weakly pre-$\Lambda_b$-closed if and only if $g(V) - g(X - V)$ is open in $Y$ whenever $U$ is $\Lambda_b$-closed in $X$.

**Corollary 4.16.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces and let $g : (X, \tau) \to (Y, \sigma)$ be a $\Lambda_b$-continuous weakly pre-$\Lambda_b$-closed surjective function. Then the topology on $Y$ is $\{g(V) - g(X - V) : V \text{ is } V_b\text{-set in } X\}$.

**Proof.** Let $W$ be open in $Y$. Then $g^{-1}(W)$ is $\Lambda_b$-set in $X$, and $g(g^{-1}(W)) - g(X - g^{-1}(W)) = W$. Hence, all open sets in $Y$ are of the form $g(V) - g(X - V)$, $V$ is $\Lambda_b$-set in $X$. On the other hand, all sets of the form $g(V) - g(X - V)$, $V$ is $\Lambda_b$-set in $X$, are open in $Y$ from Corollary 4.21. \qed

**Definition 4.17.** A topological space $(X; \tau)$ is said to be $V_b$-normal if for any pair of disjoint $V_b$-sets $F1$ and $F2$ of $X$, there exist disjoint open sets $U$ and $V$ such that $F1 \subset U$ and $F2 \subset V$.

**Theorem 4.18.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces with $X$ is $V_b$ normal and let $g : (X, \tau) \to (Y, \sigma)$ be a $V_b$-continuous weakly pre-$V_b$-closed surjective function. Then $Y$ is normal.

**Proof.** Let $K$ and $M$ be disjoint closed subsets of $Y$. Then $g^{-1}(K)$, $g^{-1}(M)$ are disjoint $V_b$-sets of $X$. Since $X$ is $V_b$-normal, there exist disjoint open sets $V$ and $W$ such that $g^{-1}(K) \subset V$ and $g^{-1}(M) \subset W$. Then $K \subset g(V) - g(X - V)$ and $M \subset g(W) - g(X - W)$. Further by Corollary 4.21, $g(V) - g(X - V)$ and $g(W) - g(X - W)$ are open sets in $Y$ and clearly $(g(V) - g(X - V)) \cap (g(W)g(X - W)) = \emptyset$. This shows that $Y$ is normal. \qed
References


