SOME FUNCTIONS VIA $\Lambda_b(V_b)$-SETS

G. Shanmugam$^1$, N. Karthikeyan$^2$, N. Rajesh$^3$

$^1,^2$Department of Mathematics
Jeppiaar Engineering College
Chennai 600119, Tamilnadu, INDIA

$^3$Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur, 613005, Tamilnadu, INDIA

Abstract: In this paper, we introduce and study some new types of functions by the use of $\Lambda_b$-sets and $V_b$-sets.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. The concept of $b$-open set in a topological space was introduced by Andrijevic in 1996 [1]. But one year later, this notion was also called $\gamma$-open sets due to El-Atik [7]. A subset $A$ of a topological space $(X, \tau)$ is said to be $b$-open ($=\gamma$-open [7]) if $A \subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A))$, where $\text{Cl}(A)$ denotes the closure of $A$ and $\text{Int}(A)$ denotes the interior of $A$ in $(X, \tau)$. The complement $A^c$ of a $b$-open
set $A$ is called $b$-closed [1] ($=\gamma$ -closed [7]). The family of all $b$-open (resp. $b$-closed) sets in $(X, \tau)$ is denoted by $BO(X, \tau)$ (resp. $BC(X, \tau)$). The intersection of all $b$-closed sets containing $A$ is called the $b$-closure of $A$ [1] and is denoted by $bCl(A)$. Quite recently Caldas et. al. [3] used $b$-open sets to define and investigate the $\Lambda_b$-sets (resp. $V_b$-sets) which are intersection of $b$-open (resp. union of $b$-closed) sets. The purpose of the present paper is introduced and studied some new type of functions by using $\Lambda_b$-sets and $V_b$-sets. The family of all $b$-open sets of $(X, \tau)$ containing a point $x \in X$ is denoted by $BO(X, x)$.

2. Preliminaries

Throughout this paper $(X, \tau)$, $(Y, \sigma)$ and $(Z, \nu)$ (or simply $X$, $Y$ and $Z$) will always denote topological spaces on which no separation axioms are assumed, unless explicitly stated.

**Definition 2.1.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $b$-continuous [7] ($=\gamma$ -continuous [7]) (resp. $b$-irresolute [7]($=\gamma$ -irresolute [7])) if for every $A \in \sigma$ (resp. $A \in BO(Y, \sigma)) f^{-1}(A) \in BO(X, \tau)$, or equivalently, $f$ is $b$-continuous (resp. $b$-irresolute) if and only if for every closed set $A$ (resp. $b$-closed set $A$) of $(Y, \sigma)$, $f^{-1}(A) \in BC(X, \tau)$.

**Definition 2.2.** Let $B$ be a subset of a topological space $(X, \tau)$. $B$ is a $\Lambda_b$-set (resp. $V_b$-set) [3], if $B = B^{\Lambda_b}$ (resp. $B = B^{V_b}$ ), where:$B^{\Lambda_b} = \cap\{O : O \supseteq B, O \in BO(X, \tau)\}$ and $B^{V_b} = \cup\{F : F \subset B; F^c \in BO(X, \tau)\}$.

The family of all $\Lambda_b$-sets (resp. $V_b$ of $(X, \tau)$ is denoted by $\tau^{\Lambda_b}$ (resp. $\tau^{V_b}$)

**Proposition 2.3.** For a space $(X, \tau)$, the following statements hold:
(i) $\emptyset$ and $X$ are $\Lambda_b$-sets and $V_b$-sets.
(ii) Every union of $\Lambda_b$-sets (resp.$V_b$-sets) is a $\Lambda_b$-set(resp.$V_b$-set).
(iii) Every intersection of $\Lambda_b$-sets (resp.$V_b$-sets) is a $\Lambda_b$-set (resp.$V_b$ set).

**Proposition 2.4.** Let $A, B$ for some $\{B : \alpha \in \Omega\}$ be subsets of a topological space $(X, \tau)$. Then the following properties are valid [3]:
(a) $B \subset B^{\Lambda_b}$;
(b) If $A \subset B$, then $A^{\Lambda_b} \subset B^{\Lambda_b}$;
(c) $B^{\Lambda_b} = B^{\Lambda_b}$;
(d) $(\bigcup_{\alpha \in \Omega} B)^{\Lambda_b} = \bigcup_{\alpha \in \Omega} B^{\Lambda_b}$;
(e) If $A \in BO(X, \tau)$, then $A = A^{\Lambda_b}$ (i.e, $A$ is an $\Lambda_b$-set);
(f) $(B^c)^{\Lambda_b} = (B^{V_b})^c$;
(g) $B^{V_b} \subset B$;
(h) If $B \in BC(X, \tau)$, then $B = B^{V_b}$ (i.e., $A$ is an $V_b$-set);
(i) $(\bigcup_{\alpha \in \Omega} B)^\Lambda_b \subset \bigcup_{\alpha \in \Omega} B^{\Lambda_b}$;
(j) $(\bigcup_{\alpha \in \Omega} B)^{V_b} \supseteq \bigcup_{\alpha \in \Omega} B^{V_b}$.

**Definition 2.5.** A topological space $X$ is said to be:

(a) $R_0$-space [5] if for each open set $V$ of $X$ and each $x \in V$, $Cl(\{x\}) \subset V$
(b) $b$-$R_0$ [6] if every $b$-open set contains the $b$-closure of each of its singletons.
(c) $b$-$T_1$ [3] if to each pair of distinct points $x, y$ of $X$ there corresponds a $b$-open set $A$ containing $x$ but not $y$ a $b$-open set $B$ containing $y$ but not $x$, or equivalently, $(X, \tau)$ is a $b$-$T_1$-space if and only if every singleton is $b$-closed.

3. $\Lambda_b$-Continuity and $\Lambda_b$-Irresoluteness

**Definition 3.1.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

(i) $\Lambda_b$-continuous if $f^{-1}(V)$ is a $\Lambda_b$-set in $(X, \tau)$ for every open set $V$ of $(Y, \sigma)$,
(ii) $\Lambda_b$-irresolute if $f^{-1}(V)$ is a $\Lambda_b$-set in $(X, \tau)$ for every $\Lambda_b$-set $V$ of $(Y, \sigma)$,
(iii) pre-$\Lambda_b$-open (resp. pre-$b$-open) if $f(A)$ is a $\Lambda_b$-set (resp. $b$-open set) of
$(Y, \sigma)$ for every $\Lambda_b$-set (resp. $b$-open set) $A$ of $(X, \tau)$.

**Proposition 3.2.** For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:

(i) $f$ is $\Lambda_b$-continuous,
(ii) $f : (X, \tau^{\Lambda_b}) \to (Y, \sigma)$ is continuous,
(iii) $f : (X, \tau^{V_b}) \to (Y, \sigma)$ is continuous.

**Proof.** Clear. □

**Definition 3.3.** (i): A filterbase $\Lambda$ is said to be $\Lambda_b$-convergent to a point $x$ in $X$ if for any $U \in S^{\Lambda_b}$ containing $x$, there exists $B \in \Lambda$ such that $B \subset U$.
(ii): A filterbase $\Lambda$ is said to be convergent to a point $x$ in $X$ if for any open set $U$ of $X$ containing $x$, there exists $B \in \Lambda$ such that $B \subset U$.

**Theorem 3.4.** If a function $f : (X, \tau) \to (Y, \sigma)$ is $\Lambda_b$-continuous, then for each point $x \in X$ and each filterbase $\mathcal{F}$ in $X$ $\Lambda_b$-converging to $x$, the filter base $f(\mathcal{F})$ is convergent to $f(x)$.

**Proof.** Let $x \in X$ and $\mathcal{F}$ be any filterbase in $X$ $\Lambda_b$-converging to $x$. Since $f$ is $\Lambda_b$-continuous, then for any open set $V$ of $(Y, \sigma)$ containing $f(x)$, there exists
there exists $B$ to be strong $\Lambda_b$-filter base $f(U)$ containing $x$ such that $f(U) \subset V$. Since $\Lambda_b$ is $\Lambda_b$-converging to $x$, there exists $B \in \mathcal{F}$ such that $B \subset U$. This means that $f(B) \subset V$ and hence the filter base $f(\mathcal{F})$ is convergent to $f(x)$. \hfill \square

**Definition 3.5.** A sequence $(x_n)$ is said to be $\Lambda_b$-convergent to a point $x$ if for every $\Lambda_b$ set $V$ containing $x$, there exists an index $x_0$ such that for $n \geq n_0$, $x_n \in V$.

**Theorem 3.6.** If a function $f : (X, \tau) \to (Y, \sigma)$ is $\Lambda_b$-continuous, then for each point $x \in X$ and each net $(x_n)$ which is $\Lambda_b$-convergent to $x$, the net $f(x_n)$ is convergent to $f(x)$.

**Proof.** The proof is similar to that of Theorem 4.15.

Recall that for a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of $f$ and is denoted by $G(f)$.

**Definition 3.7.** A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be strong $\Lambda_b$-set if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in S^{\Lambda_b}$ containing $x$ and a closed set $V$ of $Y$ containing $y$ such that $(U \times V) \cap G(f) = \emptyset$.

**Lemma 3.8.** A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is strong $\Lambda_b$-set in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in S^{\Lambda_b}$ containing $x$ and a closed set $V$ of $Y$ containing $y$ such that $f(U) \cap V = \emptyset$.

**Theorem 3.9.** If $f : (X, \tau) \to (Y, \sigma)$ is a $\Lambda_b$-continuous function and $(Y, \sigma)$ is a $T_1$-space, then $G(f)$ is strong $\Lambda_b$-set.

**Proof.** Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since $Y$ is $T_1$ there exists an open set $V$ in $Y$ such that $f(x) \in V$ and $y \notin V$. Since $f$ is $\Lambda_b$-continuous, there exist $U \in S^{\Lambda_b}$ containing $x$ such that $f(U) \subset V$. Therefore, $f(U) \cap (Y \setminus V) = \emptyset$ and $Y \setminus V$ is closed subset of $Y$ containing $y$. This show that $G(f)$ is strong $\Lambda_b$-set. \hfill \square

**Definition 3.10.** A topological space $X$ is said to be $\Lambda_b$-connected if there does not exist disjoint $\Lambda_b$-set $A$ and $B$ such that $A \cup B = X$.

**Theorem 3.11.** If $f : (X, \tau) \to (Y, \sigma)$ is a $\Lambda_b$-continuous surjective function and $X$ is $\Lambda_b$-connected, then $Y$ is connected.

**Proof.** Follows from the definitions. \hfill \square

**Definition 3.12.** A collection $\{G_\alpha : \alpha \in \Delta\}$ is said to be $\Lambda_b$-cover of a subset $A$ of a topological space $(X, \tau)$ if $A \subset \bigcup\{G_\alpha : X \setminus G_\alpha \in S^{\Lambda_b}, \alpha \in \Delta\}$.
Definition 3.13. A topological space $X$ is said to be
(i) $\Lambda_b$-compact if every $\Lambda_b$-open cover of $X$ has a finite subcover;
(ii) countably $\Lambda_b$-compact if every $\Lambda_b$-open countable cover of $X$ has a finite subcover;
(iii) $\Lambda_b$-Lindelöf if every cover of $X$ by $\Lambda_b$-open set has a countable subcover.

Theorem 3.14. If $f : (X, \tau) \to (Y, \sigma)$ is $\Lambda_b$-continuous surjection and $(X, \tau)$ is $\Lambda_b$-compact (resp. countably $\Lambda_b$-compact, $\Lambda_b$-Lindelöf), then $Y$ is compact (resp. countably compact, Lindelöf).

Proof. Follows from the definitions.

Theorem 3.15. If $f : (X, \tau) \to (Y, \sigma)$ is a $\Lambda_b$-continuous injective function and $(Y, \sigma)$ is a $T_2$-space, then $(X, \tau)$ is $\Lambda_b$-$T_2$-space.

Proof. For any pair of distinct points $x$ any $y$ in $X$, there exist distinct open sets $U$ and $V$ in $Y$ such that $f(x) \in U$ and $f(y) \in V$. Since $f$ is $\Lambda_b$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\Lambda_b$-sets in $X$ containing $x$ any $y$, respectively. Therefore $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ because $U \cap V = \emptyset$. This show that $(X, \tau)$ is $\Lambda_b$-$T_2$.

Proposition 3.16. For a function $f : (X, \tau) \to (Y, \sigma)$ the following properties are equivalent:
(i) $f$ is $\Lambda_b$-irresolute,
(ii) $f : (X, \tau^{\Lambda_b}) \to (Y, \sigma^{\Lambda_b})$, is continuous,
(iii) $f : (X, \tau^{V_b}) \to (Y, \sigma^{V_b})$, is continuous.

Proof. (i) $\implies$ (ii) This is obvious. (ii) $\implies$ (iii): Let $B$ be any $V_b$-sets of $(Y, \sigma)$. Then $B^c$ is a $\Lambda_b$-sets of $(Y, \sigma)$ and $f^{-1}(B^c)$ is a $\Lambda_b$-sets of $(X, \tau)$ (iii) $\implies$ (i): Let $B$ be any $\Lambda_b$-sets of $(Y, \sigma)$ then $B^c$ is a $V_b$-sets. Also $f^{-1}(B^c) = (f^{-1}(B))^c$ is a $V_b$-set. Thus $f^{-1}(B)$ is $\Lambda_b$-sets.

Theorem 3.17. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties hold.
(i) If $f$ is $b$-irresolute, then it is $\Lambda_b$-irresolute;
(ii) If $f$ is $b$-open and injective, then it is pre-$\Lambda_b$-open.

Proof. (i) Let $B$ be a $\Lambda_b$-sets of $(Y, \sigma)$. Since $f$ is $b$-irrselotue, we have $f^{-1}(B) \subset (f^{-1}(B))^{\Lambda_b} = \{U | f^{-1}(B) \subset U \in BO(X, \tau)\} = \{f^{-1}(V) | B \subset V \in BO(Y, \sigma)\} = f^{-1}(\{V \subset V \in BO(Y, \sigma)\}) = (f^{-1}(B))^{\Lambda_b} = f^{-1}(B)$. Therefore, we obtain $f^{-1}(B) = (f^{-1}(B))^{\Lambda_b}$ which show that $f^{-1}(B)$ is a $\Lambda_b$-set. Consequently, $f$ is $\Lambda_b$-irresolute.

(ii) Let $A$ be a $\Lambda_b$-set of $(X, \tau)$. Since $f$ is pre-$b$-open and injective. We have
If \( f : (X, \tau) \to (Y, \sigma) \) is bijective, \( b \)-irresolute and pre-\( b \)-closed, then

(i) for every \( V_b \)-set \( B \) of \( (Y, \sigma) \), then \( f^{-1}(B) \) is a \( V_b \)-set of \( (X, \tau) \)

(ii) for every \( V_b \)-set \( B \) of \( (X, \tau) \), then \( f(B) \) is a \( V_b \)-set of \( (Y, \sigma) \)

**Proposition 3.19.** (i) If \( f : (X, \tau) \to (Y, \sigma) \) is a \( \Lambda_b \)-irresolute function and \( g : (Y, \sigma) \to (Z, \gamma) \) is a \( \Lambda_b \)-continuous function, then the composition \( g \circ f : (X, \tau) \to (Z, \gamma) \) is \( \Lambda_b \)-continuous.

(ii) If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \gamma) \) are both \( \Lambda_b \)-irresolute, then the composition \( g \circ f : (X, \tau) \to (Z, \gamma) \) is \( \Lambda_b \)-irresolute.

*Proof.* It follows directly from the definitions.

**Definition 3.20.** A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( V_b \)-closed if for each closed set \( F \) of \( X \), \( f(F) \) is a \( V_b \)-set of \( (Y, \sigma) \).

**Theorem 3.21.** A function \( f : (X, \tau) \to (Y, \sigma) \) is \( V_b \)-closed if and only if for each subset \( S \) of \( Y \) and for each open set \( U \) containing \( f^{-1}(S) \), there is a \( \Lambda_b \)-set \( V \) of \( Y \) such that \( S \subset V \) and \( f^{-1}(V) \subset U \).

*Proof.*. Let \( S \) be a subset of \( Y \) and \( U \) be an open subset of \( X \) such that \( f^{-1}(S) \subset U \). Then, \( Y \setminus f(X \setminus U) = V \) (say), is a \( \Lambda_b \)-set containing \( S \) such that \( f^{-1}(V) \subset U \). Conversely, let \( F \) be an arbitrary closed set of \( X \). Then \( f^{-1}(Y \setminus f(F)) \subset X \setminus F \) and \( X \setminus F \) is open in \( X \). By hypothesis, there is a \( \Lambda_b \)-set \( V \) of \( Y \) such that \( (Y \setminus f(F)) \subset V \) and \( f^{-1}(V) \subset X \setminus F \) hence \( Y \setminus V \subset f(F) \subset f(X \setminus f^{-1}(V)) \subset Y \setminus V \), which implies \( f(F) = Y \setminus V \). Since \( Y \setminus V \) is a \( V_b \)-set, \( f(F) \) is a \( V_b \)-set; hence \( f \) is a \( V_b \)-closed function.

We consider now some composition properties interms of \( V_b \)-sets.

**Theorem 3.22.**. Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \gamma) \) be two functions such that \( g \circ f : (X, \tau) \to (Z, \gamma) \) is \( V_b \)-closed. Then, (i) if \( f \) is continuous and surjective, then \( g \) is \( V_b \)-closed, (ii) if \( g \) is \( b \)-irresolute, pre-\( b \)-closed and bijective, then \( f \) is \( V_b \)-closed.

*Proof.* (i) Let \( F \) be an arbitrary closed set in \( (X, \tau) \). Then \( f \) is a \( V_b \)-set in \( (Y, \sigma) \). Since \( g \) is bijective, \( b \)-irresolute and pre-\( b \)-closed, \( (g \circ f)(F) = g(f(F)) \) is a
$V_b$-set by Corollary 3.21(ii). (ii) The proof follows immediately from definitions. Regarding the restriction $f_{|A}$ of a function $f : (X, \tau) \to (Y, \sigma)$ to a subset $A$ of $X$, we have the following:

**Theorem 3.23.** (i) If $f : (X, \tau) \to (Y, \sigma)$ is $V_b$-closed and $A$ is a closed set in $(X, \tau)$, then its restriction $f_{|A} : (A, \tau_{|A}) \to (Y, \sigma)$ is $V_b$-closed. (ii) Let $B$ be a $V_b$-set of $(Y, \sigma)$ If $f : (X, \tau) \to (Y, \sigma)$ is $V_b$-closed, then $f_{|A} : (A, \tau_{|A}) \to (Y, \sigma)$ is $V_b$-closed, where $A = f^{-1}(B)$.

**Proof.** (i) Let $F$ be a closed set of $(A, \tau_{|A})$. Since $A$ is closed in $(X, \tau)$, $F$ is closed in $(X, \tau)$ and $f_{|A}(F) = f(F)$ is a $V_b$-closed set of $(Y, \sigma)$. Therefore, $f_{|A}$ is $V_b$-closed. (ii) Let $F$ be a closed set of $A$. Then $F = A \cap H$ for some closed set $H$ of $(X, \tau)$. By Proposition 2.3, we have $f(H) \cap B$ is a $V_b$-set in $(Y, \sigma)$ since $B$ is a $V_b$-set. Using $f_{|A}(F) = f(A \cap H) = f(H) \cap B$, $f_{|A}$ is $V_b$-closed.

**Definition 3.24.** A topological space $(X, \tau)$ is said to be $T_b$-space if $\tau_{\Lambda_b} = \tau_{V_b}$.

**Theorem 3.25.** For a topological space $(X, \tau)$, the following properties are equivalent:

(i) $(X, \tau)$ is a $b$-$R_0$ space;
(ii) $(X, \tau_{V_b})$ is discrete;
(iii) $(X, \tau_{\Lambda_b})$ is discrete;
(iv) For each $x \in X$, $\{x\}$ is a $\Lambda_b$-set of $(X, \tau)$;
(v) $P = (P)^V_b$ for each $P \in BO(X, \tau)$;
(vi) $(X, \tau)$ is a $T_b$-space;
(vii) $(X, \tau_{\Lambda_b})$ is a $R_0$ space;

**Proof.** (i) $\Rightarrow$ (ii): It is shown in Theorem 3.11 of [4] that $(X, \tau)$ is $b$-$R_0$ if and only if it is $b$-$T_1$. In [3], it is shown that if $(X, \tau)$ is $b$-$T_1$, then $(X, \tau_{\Lambda_b})$ is discrete. (ii) $\Rightarrow$ (iii): This is obvious. (iii) $\Rightarrow$ (iv): For each $x \in X$, $\{x\}$ is $\tau_{\Lambda_b}$-open and $\{x\}$ is a $\Lambda_b$-set of $(X, \tau)$. (iv) $\Rightarrow$ (v): Let $P$ be a $b$-open set of $X$. Let $y \in P^c$ then $(\{y\})_{\Lambda_b} \supset P^c$ by the assumption. By using Proposition 2.4, we have $P^c \supset \cup\{A_b(\{y\}) : y \in P^c\} = (P^c)_{\Lambda_b}^\Lambda$ and hence $P^c = (P^c)_{\Lambda_b}^\Lambda$. Then it follows from Proposition 4 that $P = (P^c)^V_b(v) \Rightarrow (vi)$: By (v), we have $BO(X, \tau) \subset \tau_{\Lambda_b}$. First we show that $\tau_{\Lambda_b} \subset \tau_{V_b}$. Let $A$ be any $\Lambda_b$ of $(X, \tau)$. Then $A = \cap\{V \mid A \subset V \in BO(X, x)\}$. Since $BO(X, \tau) \subset \tau_{V_b}$, By Proposition 4 we have $A \in \tau_{V_b}$ and $\tau_{\Lambda_b} \subset \tau_{V_b}$. Next let $A \in \tau_{V_b}$. Then $X \setminus A \in \tau_{\Lambda_b}$. Therefore $A \in \tau_{\Lambda_b}$ and $\tau_{V_b} \subset \tau_{\Lambda_b}$. Consequently, we obtain $\tau_{V_b} = \tau_{\Lambda_b}$ and $(X, \tau)$ is a $T_b$-space. (vi) $\Rightarrow$ (vii) Suppose that $V \in \tau_{\Lambda_b}$ and $x \in V$. Since $(X, \tau)$ is a $T_b$-space, $V \in \tau_{V_b}$ and $V^c \in \tau_{\Lambda_b}$. Since $\{x\} \cap V^c = \emptyset$. Cl_{\tau_{\Lambda_b}}(\{x\}) \cap V^c = \emptyset$
and $\text{Cl}_{\tau_\Lambda_b}(\{x\}) \subset V$ where $\text{Cl}_{\tau_\Lambda_b}(\{x\})$ denotes the closure of $\{x\}$ in $(X, \tau_\Lambda_b)$.

\((\text{vii}) \Rightarrow (i)\): Let $V \in BO(X, \tau)$ and $x \in V$. Since $BO(X, \tau) \subset \tau_\Lambda_b$, by (vii), $\text{Cl}_{\tau_\Lambda_b}(\{x\}) \subset V$. Since $\text{Cl}_{\tau_\Lambda_b}(\{x\}) \subset \tau V_b$, we have $\text{Cl}_{\tau_\Lambda_b}(\{x\}) = \bigcup \{F : F \in BC(X, \tau), F \subset \text{Cl}_{\tau_\Lambda_b}(\{x\})\}$ and $x \in \text{Cl}_{\tau_\Lambda_b}(\{x\})$. There exist $F \in BC(X, \tau)$ such that $x \in X$ and hence we have $b\text{Cl}(\{x\}) \subset F \subset \text{Cl}_{\tau_\Lambda_b}(\{x\}) \subset V$. This show that $(X, \tau)$ is a $b$-$R_0$-space.

**Corollary 3.26.** If $(X, \tau)$ is a $b$-$R_0$-space, then $(X, \tau^\Lambda_b)$ is a $R_0$-space.

**Definition 3.27.** A function $f : (X, \tau) \to (Y, \sigma)$ is called a $\Lambda_b$-homeomorphism if it is $\Lambda_b$-irresolute, pre-$\Lambda_b$-open and bijective.

**Theorem 3.28.** For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties hold:

(i) If $f$ is $\Lambda_b$- irresolute injection and $(Y, \sigma)$ is a $T_b$-space, then $(X, \tau)$ is a $T_b$-space.

(ii) If $f$ is pre-$\Lambda_b$- open surjection and $(X, \tau)$ is a $T_b$-space, then $(Y, \sigma)$ is a $T_b$-space.

(iii) Let $f$ be $\Lambda_b$- homeomorphism. Then $(X, \tau)$ is a $T_b$-space, if and only if $(Y, \sigma)$ is a $T_b$-space.

**Proof.** (i) This follows from Theorem 4.17. (ii) This is analogous to the proof of (i). (iii) This is an immediate consequence of (i) and (ii).

### 4. Weakly Pre-$\Lambda_b$-Open Functions

**Definition 4.1.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly pre-$\Lambda_b$-open if the image of every $\Lambda_b$-set set in $X$ is open in $Y$.

Clearly, every pre-$\Lambda_b$-open function is weakly pre-$\Lambda_b$-open. But the converse is not true in general.

**Example 4.2.** Let $X = \{(a, b, c)\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then the identity function $f$ on $X$ is weakly pre-$\Lambda_b$-open but not pre-$\Lambda_b$-open.

**Remark 4.3.** It is evident that, the concepts weakly pre-$\Lambda_b$-openness and $\Lambda_b$-continuity are coincide if the function is a bijective.

**Theorem 4.4.** A function $f : (X, \tau) \to (Y, \sigma)$ is weakly pre-$\Lambda_b$-open if and only if for every subset $U$ of $X$, $f(\text{Int}_{\tau_\Lambda_b}(U)) \subset \text{Int}(f(U))$. 
Proof. Let $f$ be weakly pre-$\Lambda_b$-open map. Now we have $\text{Int}(f(U)) \subset U$ and $\text{Int}_{\tau_{\Lambda_b}}(U)$ is a $\Lambda_b$-set. Hence we obtain that $f(\text{Int}_{\tau_{\Lambda_b}}(U)) \subset f(U)$. As $f(\text{Int}_{\tau_{\Lambda_b}}(U))$ is open, then $f(\text{Int}_{\tau_{\Lambda_b}}(U)) \subset \text{Int}(f(U))$. Conversely, assume that $U$ be a $\Lambda_b$-set in $X$. Then $f(U) = f(\text{Int}_{\tau_{\Lambda_b}}(U)) \subset \text{Int}(f(U))$. But usually $\text{Int}(f(U)) \subset f(U)$. Consequently $f(U) = \text{Int}(f(U))$ and hence $f$ is weakly pre-$\Lambda_b$-open. 

Lemma 4.5. A function $f : (X, \tau) \to (Y, \sigma)$ is weakly pre-$\Lambda_b$-open then $\text{Int}_{\tau_{\Lambda_b}}(f^{-1}(G)) \subset f^{-1}(\text{Int}(G))$ for every subset $G \subset Y$.

Proof. Let $G$ be any arbitrary subset of $Y$. Then $\text{Int}_{\tau_{\Lambda_b}}(f^{-1}(G))$ is a $\Lambda_b$-set in $X$ and $f$ is weakly pre-$\Lambda_b$-open, then $f(\text{Int}_{\tau_{\Lambda_b}}(f^{-1}(G))) \subset \text{Int}(f^{-1}(G)) \subset \text{Int}(G)$. Thus $\text{Int}_{\tau_{\Lambda_b}}(f^{-1}(G)) \subset f^{-1}(\text{Int}(G))$. 

Definition 4.6. A subset $S$ is called a $\Lambda_b$-neighbourhood of a point of $x$ of $X$ if there exist a $\Lambda_b$-set $U$ such that $x \in U \subset S$.

Theorem 4.7. For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:

(i) $f$ is weakly pre-$\Lambda_b$-open;
(ii) For each subset $U$ of $X$, $f(\text{Int}_{\tau_{\Lambda_b}}(U)) \subset \text{Int}(f(U))$;
(iii) For each $x \in X$ and each $\Lambda_b$-neighbourhood $U$ of $x$ in $X$, there exists a neighbourhood $V$ of $f(x)$ in $Y$ such that $V \subset f(U)$.

Proof. $(i) \Rightarrow (ii)$: It follows from Theorem 4.4. $(ii) \Rightarrow (iii)$: Let $x \in X$ and $U$ be an arbitrary $\Lambda_b$-neighbourhood of $x$ in $X$. Then there exists a $\Lambda_b$-set $V$ in $X$ such that $x \in V \subset U$. Then by $(ii)$, we have $f(V) = f(\text{Int}_{\tau_{\Lambda_b}}(V)) \subset \text{Int}(f(V))$ and hence $f(V) = \text{Int}(f(V))$. Therefore, it follows that $f(V)$ is open in $Y$ such that $f(x) \in f(V) \subset f(U)$. $(iii) \Rightarrow (i)$: Let $U$ be an arbitrary $\Lambda_b$-set in $X$, Then for each $y \in f(U)$, by $(iii)$ there exists a neighbourhood $V_y$ of $y$ in $Y$ such that $V_y \subset f(U)$. As $V_y$ is a neighbourhood of $y$, there exists an open set $W_y$ in $Y$ such that $y \in W_y \subset V_y$. Thus, $f(U) = \cup\{W_y : y \in f(U)\}$ which is an open set in $Y$. This implies that $f$ is weakly pre-$\Lambda_b$-open function. 

Theorem 4.8. A function $f : (X, \tau) \to (Y, \sigma)$ is weakly pre-$\Lambda_b$-open if and only if for any subset $B$ of $Y$ and for any $V_b$-set $F$ of $X$ containing $f^{-1}(B)$, there exists a closed set $G$ of $Y$ containing $B$ such that $f^{-1}(G) \subset F$.

Proof. Similar to the proof of Theorem 3.24.

Theorem 4.9. A function $f : (X, \tau) \to (Y, \sigma)$ is weakly pre-$\Lambda_b$-open if and only if $f(\text{Cl}(B)) \subset \text{Cl}_{\tau_{\Lambda_b}}(f(B))$ for every subset $B$ of $Y$. 
Proof. Suppose that $f$ is weakly pre-$\Lambda_b$-open. For any subset $B$ of $Y$, $f^{-1}(B) \subseteq \text{Cl}_{\Lambda_b}(f^{-1}(B))$. Therefore by Theorem 4.8, there exists a closed set $F$ in $Y$ such that $B \subseteq F$ and $f^{-1}(F) \subseteq \text{Cl}_{\Lambda_b}(f^{-1}(B))$. Therefore, we obtain $f^{-1}(\text{Cl}(B)) \subseteq f^{-1}(F) \subseteq \text{Cl}_{\Lambda_b}(f^{-1}(B))$. Conversely, let $B \subseteq Y$ and $F$ be a $\Lambda_b$-set of $X$ containing $f^{-1}(B)$. Put $W = \text{Cl}(B)$, then we have $B \subseteq W$ and $W$ is closed and $f^{-1}(W) \subseteq \text{Cl}_{\Lambda_b}(f^{-1}(B)) \subseteq F$. Then by Theorem 4.11, $f$ is weakly pre-$\Lambda_b$-open. \hfill $\square$

Lemma 4.10. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two functions and $g \circ f : X \to Z$ is weakly pre-$\Lambda_b$-open. If $g$ is continuous injective, then $f$ is weakly pre-$\Lambda_b$-open.

Proof. Let $U$ be a $\Lambda_b$-set in $X$, then $(g \circ f)(U)$ is open in $Z$, since $g \circ f$ is weakly pre-$\Lambda_b$-open. Again $g$ is an injective continuous function, $f(U) = g^{-1}(g \circ f(U))$ is open in $Y$. This show that $f$ is weakly pre-$\Lambda_b$-open. \hfill $\square$

Theorem 4.11. If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are two weakly pre-$V_b$-closed functions, then $g \circ f : X \to Z$ is a weakly pre-$V_b$-closed function.

Proof. Obvious. \hfill $\square$

Furthermore, we have the following.

Theorem 4.12. If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two functions. Then

(i) If $f$ is $V_b$-closed and $g$ is weakly pre-$V_b$-closed, then $g \circ f$ is closed;

(ii) If $f$ is weakly pre-$V_b$-closed and $g$ is $V_b$-closed, then $g \circ f$ is pre-$V_b$-closed;

(iii) If $f$ is $V_b$-closed and $g$ is weakly pre-$V_b$-closed, then $g \circ f$ is weakly pre-$V_b$-closed.

Proof. Obvious. \hfill $\square$

Theorem 4.13. If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two functions such that $g \circ f : X \to Z$ is weakly pre-$\Lambda_b$-closed.

(i) If $f$ is $V_b$-irresolute surjective, then $g$ is closed.

(ii) If $g$ is $V_b$-continuous injective, then $f$ is pre-$V_b$-closed.

Proof. (i) Suppose $F$ is an arbitrary $V_b$-closed set in $Y$. As $f$ is $V_b$-irresolute, $f^{-1}(F)$ is $V_b$-set in $X$. Since $g \circ f$ is weakly pre-$V_b$-closed and $f$ is surjective, $(g \circ f(f^{-1}(F))) = g(F)$, which is closed in $Z$. This implies that $g$ is a closed function. (ii) Suppose $F$ is any $V_b$-closed set in $X$. Since $g \circ f$ closed.
is weakly pre-$V_b$-closed, $(g \circ f)(F)$ is closed in $Z$. Again $g$ is a $V_b$-continuous injective function, $g^{-1}(g \circ f(F)) = f(F)$, which is $V_b$-closed in $Y$. This shows that $f$ is pre-$V_b$-closed.

**Theorem 4.14.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. Then the function $g : (X, \tau) \rightarrow (Y, \sigma)$ is a weakly pre-$V_b$-closed if and only $g(X)$ is closed in $Y$ and $g(V) - g(X - V)$ is open in $g(X)$ whenever $V$ is $\Lambda_b$-set in $X$.

**Proof.** Necessity: Suppose $g : (X, \tau) \rightarrow (Y, \sigma)$ is a weakly pre-$V_b$-closed function. Since $X$ is $\Lambda_b$-set, $g(X)$ is closed in $Y$ and $g(V) - g(X - V) = g(X) - g(X - V)$ is open in $g(X)$ when $V$ is $\Lambda_b$-set in $X$. Sufficiency: Suppose $g(X)$ is closed in $Y$, $g(V) - g(X - V)$ is open in $g(X)$ when $V$ is $\Lambda_b$-set in $X$, and let $C$ be closed in $X$. Then $g(C) = g(X) - (g(X - C) - g(C))$ is closed in $g(X)$ and hence, closed in $Y$.

**Corollary 4.15.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. Then a surjective $g : (X, \tau) \rightarrow (Y, \sigma)$ is a weakly pre-$\Lambda_b$-closed if and only if $g(V) - g(X - V)$ is open in $Y$ whenever $U$ is $\Lambda_b$-set in $X$.

**Corollary 4.16.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces and let $g : (X, \tau) \rightarrow (Y, \sigma)$ be a $\Lambda_b$-continuous weakly pre-$\Lambda_b$-closed surjective function. Then the topology on $Y$ is $\{g(V) - g(X - V) : V$ is $V_b$-set in $X\}$.

**Proof.** Let $W$ be open in $Y$. Then $g^{-1}(W)$ is $\Lambda_b$-set in $X$, and $g(g^{-1}(W)) - g(X - g^{-1}(W)) = W$. Hence, all open sets in $Y$ are of the form $g(V) - g(X - V)$, $V$ is $\Lambda_b$-set in $X$. On the other hand, all sets of the form $g(V) - g(X - V)$, $V$ is $\Lambda_b$-set in $X$, are open in $Y$ from Corollary 4.21.

**Definition 4.17.** A topological space $(X; \tau)$ is said to be $V_b$-normal if for any pair of disjoint $V_b$-sets $F_1$ and $F_2$ of $X$, there exist disjoint open sets $U$ and $V$ such that $F_1 \subset U$ and $F_2 \subset V$.

**Theorem 4.18.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces with $X$ is $V_b$ normal and let $g : (X, \tau) \rightarrow (Y, \sigma)$ be a $V_b$-continuous weakly pre-$V_b$-closed surjective function. Then $Y$ is normal.

**Proof.** Let $K$ and $M$ be disjoint closed subsets of $Y$. Then $g^{-1}(K), g^{-1}(M)$ are disjoint $V_b$-sets of $X$. Since $X$ is $V_b$-normal, there exist disjoint open sets $V$ and $W$ such that $g^{-1}(K) \subset V$ and $g^{-1}(M) \subset W$. Then $K \subset g(V) - g(X - V)$ and $M \subset g(W) - g(X - W)$. Further by Corollary 4.21, $g(V) - g(X - V)$ and $g(W) - g(X - W)$ are open sets in $Y$ and clearly $(g(V) - g(X - V)) \cap (g(W)g(X - W)) = \emptyset$. This shows that $Y$ is normal.
References


