

**PROPERTIES OF INTEGRAL INVARIANTS OF
THE INVOLUTE-EVOLUTE OFFSETS OF RULED SURFACES**

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Abstract: In this study, we have given new characteristics results about the pitch and angle of the pitch which are the integral invariants of the involute-evolute offsets of ruled surface with geodesic Frenet frame.

AMS Subject Classification: 53A05, 53A17, 53A25

Key Words: ruled surface, involute - evolute, integral invariants, geodesic Frenet frame

1. Introduction

In differential geometry, the ruled surface is a special type of surface which can be defined by choosing a curve and a line along that curve. The ruled surfaces are one of the easiest of all surfaces to parametrize. Gaspard Monge found and investigated this surface also established the partial differential equation that satisfies all ruled surface. V. Hlavaty, and J. Hoschek also investigated ruled surfaces which are formed by one parameter set of lines [1] and [2]. In addition, H. R. Müller showed that the pitch of closed ruled surfaces are integral invariants [3]. The ruled surfaces are very important in many areas of sciences for instance kinematics, Computer-Aided Geometric Design (CAGD), Computer-Aided Manufacturing (CAM) and geometric modelling.

Received: May 12, 2015

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From past to present, many characteristic properties of the ruled surfaces and their integral invariants have been examined in Euclidean space and non-Euclidean spaces [4]-[10]. Ravani and Ku studied Bertrand offsets of ruled surface with respect to the geodesic Frenet frame [4]. Based on this work, E. Kasap and N. Kuruoğlu given new characteristic properties and integral invariants of these offsets, respectively, [5] and [6] using the book [7]. Moreover, the involute-evolute offsets of ruled surface with respect to the geodesic Frenet frame have been considered in [8]. Using line geometry, it is shown that a theory similar to that of involute-evolute curves have been developed for a ruled surface. Mannheim offsets of ruled surface with respect to the geodesic Frenet frame is defined in [9].

In this paper, based on the involute-evolute offsets of ruled surface [8] we have calculated integral invariants of these offsets with respect to the geodesic Frenet frame.

2. Preliminaries

In this section, we will present some basic concepts related to ruled surface, geodesic Frenet frame and the involute-evolute offsets of ruled surfaces.

2.1. Differential Geometry of Ruled Surface with Geodesic Frenet Frame

A ruled surface M in \mathbb{R}^3 is generated by a one-parameter family of straight lines. The straight lines are called the *rulings* . The equation of the ruled surface can be written as,

$$\varphi(s, v) = \alpha(s) + ve(s), \|e(s)\| = 1$$

where (α) is curve which is called the *base curve* of the ruled surface and the curve, which is drawn by $e(s)$ on the unit sphere S^2 is called the *spherical indicatrix curve* and e is also called the *spherical indicatrix vector* of ruled surface [4].

If the ruled surface satisfies the condition $\varphi(s + P, v) = \varphi(s, v)$ for all $s \in I$, then the ruled surface is called closed.

The unit normal of φ along a general generator $l = \varphi(s_0, v)$ of the ruled surface approaches a limiting direction as v infinitely decreases. This direction is called the *asymptotic normal direction* and it is calculated by [4]

$$g(s) = \frac{e \times e_s}{\|e_s\|}, \quad e_s = \frac{de}{ds}.$$

The point at which the unit normal of φ is perpendicular to g is called the *striction point (or central point)* on l and the curve drawn by these points are called the *striction curve* of φ [4]. For the striction curve of the ruled surface φ , we have

$$c(s) = \alpha(s) - \frac{\langle \alpha_s, e_s \rangle}{\langle e_s, e_s \rangle} e(s). \tag{1}$$

In this case, we will take the striction curve as the base curve of the ruled surface. So the ruled surface can be expressed as

$$\varphi(s, v) = c(s) + ve(s). \tag{2}$$

The direction of the unit normal at a striction point is called the *central normal* of the ruled surface φ and it is calculated by [4]

$$t = \frac{e_s}{\|e_s\|}.$$

Thus the orthonormal system $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ is called the *geodesic Frenet frame* of the ruled surface φ such that [4]

$$\begin{aligned} e &= e \\ t &= \frac{e_s}{\|e_s\|} \\ g &= \frac{e \times e_s}{\|e \times e_s\|} \end{aligned} \tag{3}$$

The derivative formulae of the geodesic Frenet frame is given as follows [4]:

$$\begin{aligned} e_q &= t \\ t_q &= \gamma g - e \\ g_q &= -\gamma t \end{aligned} \tag{4}$$

where q is called the *arc-parameter* of spherical indicatrix curve (e) and γ is called the *geodesic curvature* of (e) with respect to the unit sphere S^2 .

Similarly, if we differentiate the equation (3) with respect to the arc-parameter of α , we can give the following equations [5]:

$$\begin{aligned} e_s &= q_s t \\ t_s &= -q_s e + \gamma q_s g \\ g_s &= -\gamma q_s \end{aligned} \tag{5}$$

These equations can be considered the analogue of the equation (4). Moreover, the above equation system can be written as a matrix form:

$$\begin{bmatrix} de \\ dt \\ dg \end{bmatrix} = \begin{bmatrix} 0 & q_s & 0 \\ -q_s & 0 & \gamma q_s \\ 0 & -\gamma q_s & 0 \end{bmatrix} \begin{bmatrix} e \\ t \\ g \end{bmatrix}. \tag{6}$$

with the aim of this matrix, the Pfaffian forms (connection forms) of the geodesic Frenet frame $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ can be obtained such that $w_1 = \gamma q_s$, $w_2 = 0$ and $w_3 = q_s$ [6].

Corollary 1. *The geodesic curvature of (\mathbf{e}) with respect to the unit sphere S^2 is obtained by the equation (4), [4]*

$$\gamma = \frac{\langle \mathbf{e}, \mathbf{e}_s \times \mathbf{e}_{ss} \rangle}{\|\mathbf{e}_s\|^3}.$$

If the successive rulings intersect, the ruled surface is called *developable*.

The *distribution parameter* of the ruled surface is identified by

$$P_e = \frac{\det(\alpha_s, \mathbf{e}, \mathbf{e}_s)}{\langle \mathbf{e}_s, \mathbf{e}_s \rangle}. \quad (7)$$

Theorem 2. *The ruled surface is developable if and only if $P_e = 0$ [4].*

Theorem 3. *Let c be a striction curve of a developable surface with direction \mathbf{e} . The spherical indicatrix, \mathbf{e} , is tangent of its striction curve [4].*

Let $H = sp\{e, t, g\}$ be a moving space, H' be a constant space and H/H' be a closed motion. The instantaneous Pfaffian vector of the motion is [6]

$$\vec{w} = \gamma q_s \mathbf{e} + q_s \mathbf{g}. \quad (8)$$

The *pitch of closed ruled surface with geodesic Frenet frame* is calculated by [6]

$$L_e = - \oint_c \langle \mathbf{e}, d\mathbf{c} \rangle = - \oint_c dv \quad (9)$$

Like that the pitch of closed ruled surface with geodesic Frenet frame which is drawn by the central normal \mathbf{t}

$$L_t = - \oint_c \langle \mathbf{t}, d\mathbf{c} \rangle = - \oint_c dv \quad (10)$$

and the pitch of closed ruled surface with geodesic Frenet frame which is drawn by the asymptotic normal \mathbf{g}

$$L_g = - \oint_c \langle \mathbf{g}, d\mathbf{c} \rangle = - \oint_c dv. \quad (11)$$

The Steiner rotation vector of the motion H/H' which is defined along the striction curve of φ is $\vec{D} = \oint_c \vec{w}$ [7].

The angle of pitch of closed ruled surface with geodesic Frenet frame is calculated by [6]

$$\lambda_e = \oint_c \langle \vec{D}, \vec{e} \rangle = \oint_c w_1. \tag{12}$$

Like that the angle of pitch of closed ruled surface with geodesic Frenet frame which is drawn by the central normal \mathbf{t}

$$\lambda_t = \oint_c \langle \vec{D}, \vec{t} \rangle = \oint_c w_2 = 0 \tag{13}$$

and the angle of pitch of closed ruled surface with geodesic Frenet frame which is drawn by the asymptotic normal \mathbf{g}

$$\lambda_g = \oint_c \langle \vec{D}, \vec{g} \rangle = \oint_c w_3. \tag{14}$$

2.2. The Involute-Evolute Offsets of Ruled Surfaces

The ruled surface φ^* is said to be an involute offset of the ruled surface φ (or φ is said to be an evolute offset of the ruled surface φ^*) if there exists a one to one correspondence between their rulings such that the central normal of φ and the spherical indicatrix vector of φ^* are linearly dependent at the striction points of their corresponding rulings. The base ruled surface $\varphi(s, v)$, can be expressed as

$$\varphi(s, v) = c(s) + ve(s),$$

where (c) is its striction curve and s is the arc length of the curve c . The equation of the offset surface φ^* can be written as, in terms of its base surface φ

$$\varphi^*(s, v) = c^*(s) + ve^*(s) = c(s) + R(s)t(s) + vt(s), \tag{15}$$

where R is the distance between the corresponding striction points of φ and φ^* [8].

If $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ are the geodesic Frenet vectors of φ , then $\{\mathbf{e}^*, \mathbf{t}^*, \mathbf{g}^*\}$ are the geodesic Frenet vectors of the evolute offset φ^* of φ are given by figure 1 where θ is the angle between \mathbf{e} and \mathbf{g}^* [8].

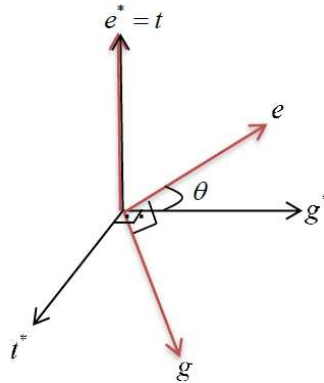


Figure 1: The relation between $\{e, t, g\}$ frame, $\{e^*, t^*, g^*\}$ frame

If $\{e, t, g\}$ are the geodesic Frenet vectors of φ , then $\{e^*, t^*, g^*\}$ are the geodesic Frenet vectors of the evolute offset φ^* of φ are given by

$$\begin{bmatrix} e^* \\ t^* \\ g^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \\ \cos \theta & 0 & \sin \theta \end{bmatrix} \begin{bmatrix} e \\ t \\ g \end{bmatrix}, \tag{16}$$

where θ is the angle between e and g^* [8].

Theorem 4. *Let φ^* be an evolute offset of the ruled surface φ . If γ is constant, then θ is constant and also for the converse, if $\theta \neq 0$ then the converse is true [8].*

Theorem 5. *Let φ^* be an evolute offset of the closed ruled surface φ with period P . φ^* is developable if and only if the spherical indicatrix curve (e) of φ is geodesic curve [8].*

3. Properties Of Integral Invariants Of The Involute-Evolute Offsets Of Ruled Surfaces

Let the ruled surface φ^* be the involute-evolute offsets of a ruled surface φ , $\{e, t, g\}$ and $\{e^*, t^*, g^*\}$ be the geodesic Frenet frame at the striction points of the ruled surface φ and φ^* , respectively, which are the involute-evolute offsets. Then, if we differentiate the equation (16) with respect to the arc-parameter of

c , we can give the following equations:

$$\begin{aligned} e^*_s &= q_s (\sin \theta + \gamma \cos \theta) t^* - q_s (\cos \theta - \gamma \sin \theta) g^* \\ t^*_s &= -q_s (\sin \theta + \gamma \cos \theta) e^* - \theta_s g^* \\ g^*_s &= q_s (\cos \theta - \gamma \sin \theta) e^* + \theta_s t^* \end{aligned} \tag{17}$$

Thus for the Pfaffian forms (connection forms) of the system $\{e^*, t^*, g^*\}$, we get

$$\begin{aligned} w_1^* &= -\theta_s \\ w_2^* &= q_s (\cos \theta - \gamma \sin \theta) \\ w_3^* &= q_s (\sin \theta + \gamma \cos \theta) \end{aligned} \tag{18}$$

Substituting $w_1 = \gamma q_s, w_2 = 0$ and $w_3 = q_s$ the Pfaffian forms (connection forms) of the system $\{e, t, g\}$ in the equation (18), we have

$$\begin{aligned} w_1^* &= -\theta_s \\ w_2^* &= w_3 \cos \theta - w_1 \sin \theta \\ w_3^* &= w_3 \sin \theta + w_1 \cos \theta \end{aligned} \tag{19}$$

Let $H^* = sp\{e^*, t^*, g^*\}$ be the moving frame along the striction curve of φ^* and $H^{*'} be the fixed space. For the instantaneous Pfaffian vector of the motion $H^*/H^{*'}$, we get$

$$\vec{w}^* = -\theta_s e^* + (w_3 \cos \theta - w_1 \sin \theta) t^* + (w_3 \sin \theta + w_1 \cos \theta) g^* \tag{20}$$

and when using the equation (16) we have

$$\vec{w}^* = \vec{w} - \theta_s e^* \tag{21}$$

From the relations the Pfaffian forms (connection forms) of the system $\{e^*, t^*, g^*\}$ and the instantaneous Pfaffian vector of the motion $H^*/H^{*'}$, we obtain that

$$\begin{aligned} \vec{D}^* &= \mathfrak{f}_{c^*} \vec{w}^* = \mathfrak{f}_{c+Rt} \vec{w} - \mathfrak{f}_{c+Rt} \theta_s \vec{e}^* \\ &= \vec{D} + \mathfrak{f}_{Rt} \vec{w} - e^* \mathfrak{f}_{c+Rt} \theta_s \\ &= \vec{D} + e \mathfrak{f}_{Rt} w_1 + g \mathfrak{f}_{Rt} w_3 - t \mathfrak{f}_{c+Rt} \theta_s \end{aligned} \tag{22}$$

Theorem 6. *The angle of the pitch of closed ruled surfaces with geodesic Frenet frame which are drawn by the spherical indicatrix vector, the asymptotic*

normal vector and the central normal vector, respectively, we can write

$$\begin{aligned}\lambda_{e^*} &= \lambda_t - \mathcal{f}_{c+Rt} \theta_s = -\mathcal{f}_{c+Rt} \theta_s \\ \lambda_{t^*} &= -\sin \theta \lambda_e + \cos \theta \lambda_g - \sin \theta \mathcal{f}_{Rt} w_1 + \cos \theta \mathcal{f}_{Rt} w_3 \\ \lambda_{g^*} &= \cos \theta \lambda_e + \sin \theta \lambda_g + \cos \theta \mathcal{f}_{Rt} w_1 + \sin \theta \mathcal{f}_{Rt} w_3\end{aligned}\quad (23)$$

Proof. For the angle of the pitch of closed ruled surface φ^* with geodesic Frenet frame, we can write

$$\begin{aligned}\lambda_{e^*} &= \langle \vec{D}^*, e^* \rangle = \langle \vec{D} + e \mathcal{f}_{Rt} w_1 + g \mathcal{f}_{Rt} w_3 - t \mathcal{f}_{c+Rt} \theta_s, t \rangle \\ &= \lambda_t - \mathcal{f}_{c+Rt} \theta_s = -\mathcal{f}_{c+Rt} \theta_s\end{aligned}\quad (24)$$

Similarly from the equations (16) the angle of the pitch of closed ruled surfaces with geodesic Frenet frame which are drawn by the asymptotic normal and the central normal, respectively, we can write

$$\begin{aligned}\lambda_{t^*} &= \langle \vec{D}^*, t^* \rangle = \langle \vec{D} + e \mathcal{f}_{Rt} w_1 + g \mathcal{f}_{Rt} w_3 - t \mathcal{f}_{c+Rt} \theta_s, -\sin \theta e + \cos \theta g \rangle \\ &= -\sin \theta \langle \vec{D}, e \rangle - \sin \theta \mathcal{f}_{Rt} w_1 + \cos \theta \langle \vec{D}, g \rangle + \cos \theta \mathcal{f}_{Rt} w_3 \\ &= -\sin \theta \lambda_e + \cos \theta \lambda_g - \sin \theta \mathcal{f}_{Rt} w_1 + \cos \theta \mathcal{f}_{Rt} w_3\end{aligned}\quad (25)$$

and

$$\begin{aligned}\lambda_{g^*} &= \langle \vec{D}^*, g^* \rangle = \langle \vec{D} + e \mathcal{f}_{Rt} w_1 + g \mathcal{f}_{Rt} w_3 - t \mathcal{f}_{c+Rt} \theta_s, \cos \theta e + \sin \theta g \rangle \\ &= \cos \theta \langle \vec{D}, e \rangle + \cos \theta \mathcal{f}_{Rt} w_1 + \sin \theta \langle \vec{D}, g \rangle + \sin \theta \mathcal{f}_{Rt} w_3 \\ &= \cos \theta \lambda_e + \sin \theta \lambda_g + \cos \theta \mathcal{f}_{Rt} w_1 + \sin \theta \mathcal{f}_{Rt} w_3\end{aligned}\quad (26)$$

□

Corollary 7. *If we take θ is constant, then we have the following equalities*

from the equations (23),

$$\begin{aligned} \lambda_{e^*} &= 0 \\ \lambda_{t^*} &= -\sin \theta \lambda_e + \cos \theta \lambda_g + \oint_{Rt} -w_1 \sin \theta + w_3 \cos \theta \\ &= -\sin \theta \lambda_e + \cos \theta \lambda_g + \oint_{Rt} w_2^* \\ \lambda_{g^*} &= \cos \theta \lambda_e + \sin \theta \lambda_g + \oint_{Rt} w_1 \cos \theta + w_3 \sin \theta \\ &= \cos \theta \lambda_e + \sin \theta \lambda_g + \oint_{Rt} w_3^* \end{aligned} \tag{27}$$

Corollary 8. *If we take $\theta = 0$, then we have the following equalities from the equations (23),*

$$\begin{aligned} \lambda_{e^*} &= 0 \\ \lambda_{t^*} &= \lambda_g + \oint_{Rt} w_3 \\ \lambda_{g^*} &= \lambda_e + \oint_{Rt} w_1 \end{aligned} \tag{28}$$

Corollary 9. *If we take $\theta = \pi/2$, then we have the following equalities from the equations (23),*

$$\begin{aligned} \lambda_{e^*} &= 0 \\ \lambda_{t^*} &= -\lambda_e - \oint_{Rt} w_1 \\ \lambda_{g^*} &= \lambda_g + \oint_{Rt} w_3 \end{aligned} \tag{29}$$

For the Steiner translation vector of the motion $H^*/H^{*'}$, we can write that

$$\vec{V}^* = \oint_{(c^*)} \vec{dc}^*$$

and from the equation (15)

$$\vec{V}^* = \oint_{(c+Rt)} \overrightarrow{d(c+Rt)} = \oint_{(c+Rt)} \vec{dc} + R \oint_{(c+Rt)} \vec{dt} + t \oint_{(c+Rt)} \vec{dR} \tag{30}$$

and if we substitute the equation (6) in the equation (30), we obtain that,

$$\vec{V}^* = \vec{V} - R\lambda_g e + R\lambda_e g + \oint_{(Rt)} \vec{dc} + t \oint_{(c+Rt)} \vec{dR} - \text{Re} \oint_{(Rt)} w_3 + \text{Rg} \oint_{(Rt)} w_1 \tag{31}$$

Theorem 10. *The pitch of closed ruled surfaces with geodesic Frenet frame which are drawn by the spherical indicatrix vector, the asymptotic normal vector and the central normal vector, respectively, we can write*

$$\begin{aligned}
 L_{e^*} &= L_t + \left\langle \underset{(Rt)}{\oint} \vec{dc}, t \right\rangle + \underset{(c+Rt)}{\oint} \vec{dR} \\
 L_{t^*} &= -\sin \theta L_e + \cos \theta L_g + R\lambda_{g^*} + \left\langle \underset{(Rt)}{\oint} \vec{dc}, t^* \right\rangle \\
 L_{g^*} &= \cos \theta L_e + \sin \theta L_g - R\lambda_t^* + \left\langle \underset{(Rt)}{\oint} \vec{dc}, g^* \right\rangle
 \end{aligned} \tag{32}$$

Proof. For the pitch of the closed ruled surface φ^* with geodesic Frenet frame, we can write

$$L_{e^*} = \left\langle \vec{V}^*, e^* \right\rangle$$

and according to the equations (31) and (15) the last statement reduces to

$$\begin{aligned}
 L_{e^*} &= \left\langle \vec{V} - R\lambda_g e + R\lambda_e g + \underset{(Rt)}{\oint} \vec{dc} + t \underset{(c+Rt)}{\oint} \vec{dR} - \text{Re} \underset{(Rt)}{\oint} w_3 + \text{Rg} \underset{(Rt)}{\oint} w_1, t \right\rangle \\
 L_{e^*} &= L_t + \left\langle \underset{(Rt)}{\oint} \vec{dc}, t \right\rangle + \underset{(c+Rt)}{\oint} \vec{dR}
 \end{aligned} \tag{33}$$

□

Similarly from the equations (10) and (11) the angle of the pitch of closed ruled surfaces with geodesic Frenet frame which are drawn by the asymptotic normal and the central normal, respectively, we can write

$$\begin{aligned}
 L_{t^*} &= \left\langle \vec{V}^*, t^* \right\rangle \\
 L_{t^*} &= \left\langle \vec{V} + R(\lambda_e g - \lambda_g e) + \underset{(Rt)}{\oint} \vec{dc} + t \underset{(c^*)}{\oint} \vec{dR} \right. \\
 &\quad \left. + \text{R} \underset{(Rt)}{\oint} (w_1 g - w_3 e), \cos \theta g - \sin \theta e \right\rangle \\
 L_{t^*} &= -\sin \theta L_e + \cos \theta L_g + R\lambda_{g^*} + \left\langle \underset{(Rt)}{\oint} \vec{dc}, t^* \right\rangle
 \end{aligned} \tag{34}$$

and

$$\begin{aligned}
 L_{g^*} &= \left\langle \vec{V}^*, \vec{g}^* \right\rangle \\
 L_{g^*} &= \left\langle \vec{V} + R(\lambda_e g - \lambda_g e) + \oint_{(Rt)} \vec{dc} + t \oint_{(c^*)} \vec{dR} \right. \\
 &\quad \left. + R \oint_{(Rt)} (w_1 g - w_3 e), \cos \theta e + \sin \theta g \right\rangle \\
 L_{g^*} &= \cos \theta L_e + \sin \theta L_g - R \lambda_t^* + \left\langle \oint_{(Rt)} \vec{dc}, g^* \right\rangle
 \end{aligned} \tag{35}$$

Corollary 11. *If we take $\theta = 0$, then we have the following equalities from the equations (32),*

$$\begin{aligned}
 L_{e^*} &= L_t + \left\langle \oint_{(Rt)} \vec{dc}, t \right\rangle + \oint_{(c+Rt)} \vec{dR} \\
 L_{t^*} &= L_g + R(\lambda_e + \oint_{Rt} w_1) + \left\langle \oint_{(Rt)} \vec{dc}, g \right\rangle \\
 L_{g^*} &= L_e - R(\lambda_g + \oint_{Rt} w_3) + \left\langle \oint_{(Rt)} \vec{dc}, e \right\rangle
 \end{aligned} \tag{36}$$

Corollary 12. *If we take $\theta = \pi/2$, then we have the following equalities from the equations (32),*

$$\begin{aligned}
 L_{e^*} &= L_t + \left\langle \oint_{(Rt)} \vec{dc}, t \right\rangle + \oint_{(c+Rt)} \vec{dR} \\
 L_{t^*} &= -L_e + R(\lambda_g + \oint_{Rt} w_3) + \left\langle \oint_{(Rt)} \vec{dc}, -t \right\rangle \\
 L_{g^*} &= L_g - R(-\lambda_e - \oint_{Rt} w_1) + \left\langle \oint_{(Rt)} \vec{dc}, g \right\rangle.
 \end{aligned} \tag{37}$$

Acknowledgments

The authors thank the referees for their careful reading of the manuscript and helpful comments. The first author was supported TÜBİTAK - The Scientific and Technological Research Council of Turkey.

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