International Journal of Pure and Applied Mathematics
Volume 102 No. 4 2015, 605-611
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v102i4.2
ijpam.eu

HAMILTON-CONNECTIVITY IN BALANCED BIPARTITE GRAPHS

Yusleidy Alcalá[1 §], Daniel Brito[2], Oscar Castro[3], Lope Marín[4]
[1,2,3,4]Departmento de Matemáticas
Escuela de Ciencias
Núcleo de Sucre
Universidad de Oriente
Cumaná 6101-A, Apartado 245, VENEZUELA

Abstract: Let G be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 4$. If for every balanced independent set S of four vertices $|N(S)| \geq n+2$, then G is Hamiltonian connected. This is an improvement of the bound given by [4].

AMS Subject Classification: 05C38, 05C45, 05C70
Key Words: hamiltonian connected, balanced bipartite graph, neighborhood union

1. Introduction

We use [2] for terminology and notation not defined here. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of a simple, finite and undirected graph G. According to the (arbitrary) orientation of a cycle C of G, the successor and predecessor of a vertex z of C are denoted by z^+ and z^-, respectively. Let $G = (A, B, E)$ be a balanced bipartite simple graph of order $2n$, i.e. a graph with a bipartition into two independent vertex sets of the same cardinality.

Received: October 3, 2014

§Correspondence author

$N(S)$ is the neighborhood union of a balanced independent set S of four vertices, i.e. an independent set containing two vertices from each side of the bipartition. G is Hamiltonian connected if for every two vertices one on each side of the bipartition there exists a Hamiltonian path connecting them.

A *2-dumbbell*, is be a graph that consists of two disjoint cycles C_1 and C_2 which are joined by two disjoint paths P_1 and P_2 with $V(C_i) \bigcap V(P_i) = \phi$ such that the neighbors of the extremes of P_1 in $C_i, (i = 1, 2)$, are consecutive with the neighbors of the extremes of P_2 in C_i.

The investigation of certain extremal problems involving neighborhood union conditions for balanced independent sets of cardinality four was initiated by Amar et al. [1]. They posed the following question: Let H_{14} denote the class of graphs obtained from the graph depicted in Fig 1, where some (or all) of the four possible edges joining the top to the bottom might be present as well.

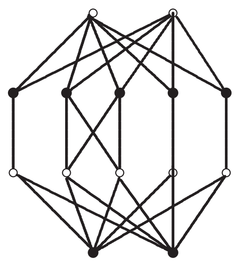

Figure 1: The minimal graph of H_{14}

Conjeture 1.1. (Amar et al. [1]). Let $G = (A, B, E)$ be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 3$. If for every balanced independent set S with $|S| = 4$, we have $|N(S)| > n$, then G is Hamiltonian or $G \in H_{14}$.

They also proved the following:

Theorem 1.2 (Amar et al. [1]). Let $G = (A, B, E)$ be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 3$. If for every balanced independent set S with $|S| = 4$, we have $|N(S)| > n + 2$, then G is Hamiltonian.

Since then, the following results have been obtained:

Theorem 1.3 (Brito y Lárez. [3]). Let $G = (A, B, E)$ be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 4$. If for every balanced independent set S with $|S| = 4$, we have $|N(S)| > n$, then G is Hamiltonian.

Theorem 1.4 (Brito et al. [4]). Let $G = (A, B, E)$ a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 3$. If for every balanced independent set S with $|S| = 4$,we have $|N(S)| > n + 2$, then G is Hamiltonian connected.

2. Previous Lemma

In the proofs the following subgraph, called $1-2$ dumbbell,the graph depicted in Fig 2, is defined as a dumbbell in which one of the extremes of each path are adjacent. Furthermore the neighbors of the ends of the paths consecutive in the cycle are adjacent to distance one-neighbors of the ends of the other path.

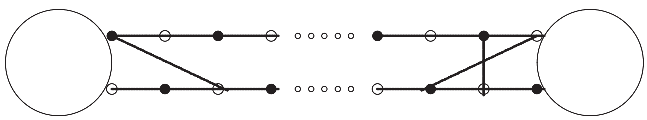

Figure 2: The 1- 2-dumbbell

Lemma 2.1 Let $G = (A, B, E)$ be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 4$, $C = a_1 b_1 a_2 b_2 \ldots a_n b_n a_1$ a hamiltonian cycle of G and D, a maximal spanning $1-2$ dumbbell of G. If G does not have a uv-hamiltonian path with $u = b_n$ and $v = a_j$, then:

1) A balanced independent set S exists such that $|S| = 4$.
2) $N^{-}(S \cap A) \cap N(S \cap B) = \emptyset$.

Proof. Let $G = (A, B, E)$ be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 4$, $C = a_1 b_1 a_2 b_2 \ldots a_n b_n a_1$, a hamiltonian cycle of

G and D, a maximal spanning $1-2$ dumbbell of G, with an orientation $a_1C_1^+b_sP_1a_tC_2^+b_jP_2a_1$.

By definition of $1-2$ dumbbell let a_1 be one vertex in the cycle C_1 neighbor of one end of the path P_2 and b_j one vertex in the cycle C_2 neighbor of one end of the path P_2 and similarly let b_s one vertex in the cycle C_1 neighbor of one end of the path P_1 and a_t one vertex in the cycle C_2 neighbor of one end of the path P_1, such that $a_1(a_t)$ and $b_s(b_j)$ are neighbor consecutive in $C_1(C_2)$, respectively.There are also sides $a_1b_s^{++}$, $b_ja_t^{--}$, $b_j^+a_t^-$. If G does not have a uv-hamiltonian path with $u=b_n$ and $v=a_j$ then:

i) $a_1b_j \notin E(G)$; for otherwise $b_nP_2^-b_j^+b_ja_1C_1^+b_sb_s^+P_1^+a_t^-a_tC_2^+a_j$ is a uv-hamiltonian path with $u=b_n$ and $v=a_j$.

ii) $a_1a_t^- \notin E(G)$; for otherwise $b_nP_2^-b_j^+b_ja_t^{--}P_1^-b_s^+b_sC_1^-a_1a_t^-a_tC_2^+a_j$ is a uv-hamiltonian path with $u=b_n$ and $v=a_j$.

iii) $b_s^+b_j \notin E(G)$; for otherwise $b_nP_2^-b_j^+b_jb_s^+b_sC_1^-a_1b_s^{++}P_1^+a_t^-a_tC_2^+a_j$ is a uv-hamiltonian path with $u=b_n$ and $v=a_j$.

iv) $b_s^+a_t^- \notin E(G)$; for otherwise $b_nP_2^-b_j^+b_ja_t^{--}P_1^-b_s^{++}a_1C_1^+b_sb_s^+a_t^-a_tC_2^+a_j$ is a uv-hamiltonian path with $u=b_n$ and $v=a_j$.

Then there exists a balanced independent set, S, of cardinality 4, with $S=\{a_1,b_j,b_s^+,a_t^-\}$

Let's assume otherwise in 2; i.e.,$(N^-(S\cap A)\cap N(S\cap B))\neq\emptyset$. Let, without loss of generality, $S=\{a_1,b_j,b_s^+,a_t^-\}$ con $a_1,b_s^+\in A$ y $b_j,a_t^-\in B$

Let, $a_k\in(N^-(S\cap A)\cap N(S\cap B))$, such that $a_k\in(N^-(a_1,b_s^+))$ and $a_k\in(N(b_j,a_t^-))$ by definition $a_k^+a_1\in E(G)$ o $a_k^+b_s^+\in E(G)$ and $a_kb_j\in E(G)$ o $a_ka_t^-\in E(G)$.

Consider the following cases:

i) $1<k<s<t<j<n$

If $a_k^+b_s^+\in E(G)$ and $a_kb_j\in E(G)$; then

$$b_nP_2^-b_ja_kC_1^-a_k^+b_s^+P_1^+a_t^-a_tC_2^+a_j$$

is a uv-hamiltonian path with $u=b_n$ and $v=a_j$.

If $a_k^+b_s^+\in E(G)$ and $a_ka_t-\in E(G)$; then

$$b_nP_2^-b_ja_t^{--}P_1^-b_s^+a_k^+C_1^+a_ka_t^-a_tC_2^+a_j$$

is a uv-hamiltonian path with $u=b_n$ and $v=a_j$.

ii) $1 < s < k < t < j < n1$
If $a_k^+ a_1 \in E(G)$ and $a_k b_j \in E(G)$; then

$$b_n P_2^- b_j a_k P_1^- b_s^+ b_s C_1^- a_1 a_k^+ P_1^+ a_t^- a_t C_2^+ a_j$$

is a uv-hamiltonian path with $u = b_n$ and $v = a_j$.

If $a_k^+ a_1 \in E(G)$ and $a_k a_t - \in E(G)$; then

$$b_n P_2^- b_j a_t^{--} P_1^- a_k^+ a_1 C_1^+ b_s b_s^+ P_1^+ a_k a_t^- a_t C_2^+ a_j$$

is a uv-hamiltonian path with $u = b_n$ and $v = a_j$.

If $a_k^+ b_s^+ \in E(G)$ and $a_k b_j \in E(G)$; then

$$b_n P_2^- b_j a_k P_1^- b_s^{++} a_1 C_1^+ b_s b_s^+ a_k^+ P_1^+ a_t^- a_t C_2^+ a_j$$

is a uv-hamiltonian path with $u = b_n$ and $v = a_j$.

If $a_k^+ b_s^+ \in E(G)$ and $a_k a_t^- \in E(G)$; then

$$b_n P_2^- b_j a_t^{--} P_1^- a_k^+ b_s^+ b_s C_1^- a_1 b_s^{++} P_1^+ a_k a_t^- a_t C_2^+ a_j$$

is a uv-hamiltonian path with $u = b_n$ and $v = a_j$.

iii) $1 < s < t < k < j < n$.

If $a_k^+ a_1 \in E(G)$ and $a_k a_t^- \in E(G)$; then

$$b_n P_2^- b_j a_t C_2^+ a_k a_t^- P_1^- b_s^+ b_s C_1^- a_1 a_k^+ C_2^+ a_j$$

is a uv-hamiltonian path with $u = b_n$ and $v = a_j$.

If $a_k^+ b_s^+ \in E(G)$ and $a_k a_t^- \in E(G)$; then

$$b_n P_2^- b_j a_t C_2^+ a_k a_t^- P_1^- b_s^{++} a_1 C_1^+ b_s b_s^+ a_k^+ C_2^+ a_j$$

is a uv-hamiltonian path with $u = b_n$ and $v = a_j$.

iv) $1 < s < t < j < k < n$.

If $a_k^+a_1 \in E(G)$ and $a_k^-b_j \in E(G)$; then

$$b_nP_2^-a_k^+a_1C_1^+b_sb_s^+P_1^+a_t^{--}b_ja_k^-P_2^-b_j^+a_t^-a_tC_2^+a_j$$

is a uv-hamiltonian path with $u = b_n$ and $v = a_j$.

If $a_1a_k^+ \in E(G)$ and $a_ka_t^- \in E(G)$; then

$$b_nP_2^-a_k^+a_1C_1^+b_sb_s^+P_1^+a_t^-a_kP_2^-b_ja_tC_2^+a_j$$

is a uv-hamiltonian path with $u = b_n$ and $v = a_j$.

If $a_k^+b_s^+ \in E(G)$ and $a_kb_j \in E(G)$; then

$$b_nP_2^-a_k^+b_s^+b_sC_1^-a_1b_s^{++}P_1^+a_t^-b_j^+P_2^+a_kb_ja_tC_2^+a_j$$

is a uv-hamiltonian path with $u = b_n$ and $v = a_j$.

If $a_k^+b_s^+ \in E(G)$ and $a_ka_t^- \in E(G)$; then

$$b_nP_2^-a_k^+b_s^+b_sC_1^-a_1b_s^{++}P_1^+a_t^-a_kP_2^-b_ja_tC_2^+a_j$$

is a uv-hamiltonian path with $u = b_n$ and $v = a_j$.

3. Main Result

In this paper we show the following theorem:

Theorem 3.1 Let $G = (A, B, E)$ a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 4$. If for every balanced independent set S with $|S| = 4$, we have $|N(S)| \geq n + 2$, then G is Hamiltonian connected.

Proof. Let $G = (A, B, E)$ be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 4$, $C = a_1b_1a_2b_2 \ldots a_nb_na_1$ a hamiltonian cycle of G and D, a maximal spanning $1-2$ dumbbell of G, with an orientation $a_1C_1^+b_sP_2a_tC_2^+b_jP_1a_1$ By definition of $1-2$ dumbbell let a_1 be one vertex in the cycle C_1 neighbor of one end of the path P_1 and b_j one vertex in the cycle C_2 neighbor of one end of the path P_1 and similarly let b_s be one vertex in the cycle C_1 neighbors of one end of the path P_2 and a_t one vertex in the cycle

C_2 neighbor of one end of the path P_2, such that $a_1(a_t)$ and $b_s(b_j)$ are neighbor consecutive in $C_1(C_2)$, respectively.There are also sides $a_1 b_s^{++}$, $b_j a_t^{--}$, $b_j^+ a_t^-$.

Suppose that G does not have a uv-Hamiltonian path with $u = b_n$ and $v = a_j$.

Let, without loss of generality, a_1, b_s^+ be in A and b_j, a_t^- in B.

Let $S = \{a_1, b_j, b_s^+, a_t^-\}$ be the independent set with $N(a_1, b_s^+) \cap N^+(b_j, a_t^-) = \emptyset$, given by Lemma 2.1.

Then,

$$\begin{aligned} n+2 \leq |N(S)| &= |N(a_1, b_s^+)| + |N(b_j, a_t^-)| \\ &= |N^-(a_1, b_s^+)| + |b_n| + |N(b_j, a_t^-)| \\ &\leq ||N^-(a_1, b_s^+)| \cup |N(b_j, a_t^-)|| + 1 \\ &\leq |A| + 1 \leq n+1, \end{aligned}$$

which contradicts the hypothesis of the theorem.

This proves the theorem.

As an immediate consequence we have

Corollary 3.1. Let G be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 4$. If for every four independent balanced vertices the sum of their degrees is at least $n+2$, then G is hamiltonian connected.

References

[1] D. Amar, S. Brandt, D. Brito and O. Ordaz, Neighborhood conditions for balanced independent sets in bipartite graphs, *Discrete Mathematics*, **181** (1998) 31-36.

[2] J. Bondy and U. Murty, *Graphs Theory with Applications*, North Holland New York, USA (1976).

[3] D. Brito, G. Lárez, Neighborhood conditions for balanced bipartite graphs to be Hamiltonian, *International Journal of Pure and Applied Mathematics*, **34** (2007), 509-512.

[4] D. Brito, P. Mago and L. Marín, Neighborhood conditions for balanced bipartite graphs to be Hamiltonian connected. *International Mathematical Forum*, **26** (2010), 1291-1295.