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STRONGLY $s^{*}g^{*}$-CLOSED SETS

K. Kannan[1], D. Rajalakshmi[2 §], C.K. Shathyaashree[3]

[1,2,3]Department of Mathematics
Srinivasa Ramanujan Centre
SASTRA University
Kumbakonam, Tamil Nadu, 612 001, INDIA

Abstract: The main aim of the present paper is to define strongly $s^{*}g^{*}$-closed sets and strongly $s^{*}g^{*}$-open sets in topological spaces. Moreover we also discuss the relation between the various existing sets and these new sets.

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Key Words: $s^{*}g$-closed sets, $s^{**}g$-closed sets, strongly g^{*}-closed sets, strongly $s^{*}g^{*}$-closed sets

1. Introduction

In 1970, Levine [1] introduced the notion of generalized closed sets (g-closed sets) in topological spaces and showed that the notions compactness, countably compactness, para compactness and normality are all g-closed hereditary. According to him, a subset A of a topological space X is called g-closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X. One can see the details of g-closed sets in [2]. Since it is the natural extension of closed sets, several researchers introduced many types of generalization of closed sets.

In this sequel, Andrijevic [3], Arya and Nour [4], Bhattacharya and Lahiri [5], Dontchev [6], Dontchev and Ganster [7], Gnanambal [8], Levine [9], Maki

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§Correspondence author

et. al. [10, 11, 12], Mashhour et.al [13], Njastad [14], Chandrasekhara Rao and Palaniappan [15], Veličko [16], Veerakumar [17], Chandrasekhara Rao and Joseph [18], Kannan [19] introduced and investigated semi pre closed sets, g-semi closed sets,sg-closed sets, g-semi pre closed sets, δ-generalized closed sets, generalized pre regular closed sets, semi closed sets, $g\alpha$-closed sets, αg-closed sets, θ-generalized closed sets, α-closed sets, regular generalized closed sets, H-closed sets, g^*-closed sets, s^*g-closed sets, semi star star generalized closed sets respectively.

Parimelazhagan [20] introduced and studied the concepts of strongly g^*-closed sets. The aim of the present paper is to introduce strongly s^*g^*-closed sets in a topological space. We also discuss some properties of them.

2. Preliminaries

Throughout this paper, X or (X, τ) is a topological space. The intersection of all closed sets containing A is called closure of A and it is denoted by $cl(A)$ and the union of all open sets contained in A is called interior of A and it is denoted by $int(A)$. A subset A of a space (X, τ) is called semiopen [9] (resp. regular open [15]) if $A \subseteq cl[int(A)]$ {resp. $A = int[cl(A)]$}. The complement of semiopen (resp. regular open) set is called semiclosed (resp. regular closed).

A subset A of a topological space (X, τ) is said to be s^*g-closed [18] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X. The complement of s^*g-closed sets are called s^*g-open. The intersection of all s^*g-closed sets containing A is called s^*g-closure of A and it is denoted by $s^*g\text{-}cl(A)$ and the union of all s^*g-open sets contained in A is called s^*g-interior of A and it is denoted by $s^*g\text{-}int(A)$. The class of all s^*g-open sets containing x in X is denoted by $S^*GO(X, x)$. A subset A of a topological space (X, τ) is said to be $s^{**}g$-closed [19] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is s^*g-open in X. The complement of $s^{**}g$-closed sets are called $s^{**}g$-open.

3. Strongly s^*g^*-Closed Sets

Recall that a set A is strongly g^*-closed [20] if $cl[int(A)] \subseteq U$ whenever $A \subseteq U$ and U is g-open in X.

Definition 3.1. A set A is a strongly s^*g^*-closed set if $cl[int(A)] \subseteq U$ whenever $A \subseteq U$ and U is s^*g-open in X.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$. Then, $\phi, X, \{b\}$, $\{c\}, \{b, c\}$ are strongly $s^{*}g^{*}$-closed sets in X.

Theorem 3.3. Every closed set is a strongly $s^{*}g^{*}$-closed set.

Proof. Let A be a closed set. Let $A \subseteq U$ and U is $s^{*}g$-open in X. Since A is closed, $cl[int(A)] \subseteq cl(A) = A \subseteq U$. Therefore, A is strongly $s^{*}g^{*}$-closed. □

Remark 3.4. The converse of the above theorem is not true in general. The following example supports our claim.

Example 3.5. In Example 3.2, $\{b\}$ is strongly $s^{*}g^{*}$-closed, but not closed.

Theorem 3.6. Every $s^{**}g$-closed set is strongly $s^{*}g^{*}$-closed.

Proof. Let A be $s^{**}g$-closed set in X. Since A is $s^{**}g$-closed, $cl[int(A)] \subseteq cl(A) \subseteq U$. Therefore, A is strongly $s^{*}g^{*}$-closed. □

Theorem 3.7. If a subset of a topological space X is both open and strongly $s^{*}g^{*}$-closed, then it is closed.

Proof. Since A is open, A is $s^{*}g$-open. Since A is strongly $s^{*}g^{*}$-closed, $cl[int(A)] \subseteq A$. Since A is open, $cl(A) = cl[int(A)] \subseteq A$. Therefore, A is closed. □

Corollary 3.8. If A is both open and strongly $s^{*}g^{*}$-closed in X, then it is both regular open and regular closed in X.

Proof. Since A is open and strongly $s^{*}g^{*}$-closed, A is closed by Theorem 3.7. Since A is open, $int[cl(A)] = int[A] = A$. Therefore, A is regular open. Since A is closed, $A = cl(A)$. Hence $A = cl[int(A)]$, since A is open. Thus A is regular closed. □

Corollary 3.9. If A is both open and strongly $s^{*}g^{*}$-closed in X, then it is rg-closed in X.

Theorem 3.10. A set A is strongly $s^{*}g^{*}$-closed if $cl[int(A)] - A$ contains no non empty $s^{*}g$-closed set.

Proof. Suppose that F is non empty $s^{\star}g$-closed subset of $cl[int(A)] - A$. Then, $F \subseteq cl[int(A)] \cap A^c$. Therefore, $F \subseteq A^c$ implies $A \subseteq F^c$. Here F^c is $s^{\star}g$-open and A is strongly $s^{\star}g^{\star}$-closed, we have $cl[int(A)] \subseteq F^c$. Thus $F \subseteq [clint(A)]^c$. Hence $F \subseteq cl[int(A)] \cap cl[int(A)]^c = \phi$. Therefore $cl[int(A)] - A$ no non empty $s^{\star}g$-closed set. □

Corollary 3.11. A strongly $s^{\star}g^{\star}$-closed set A is regular closed if and only if $cl[int(A)] - A$ is $s^{\star}g$-closed and $A \subseteq cl[int(A)]$.

Proof. Since A is regular closed, $cl[int(A)] = A$, $cl[int(A)] - A = \phi$ is $s^{\star}g$-closed.

Conversely suppose that $cl[int(A)] - A$ is $s^{\star}g$-closed. Since A is strongly $s^{\star}g^{\star}$-closed, $cl[int(A)] - A$ contains no non empty $s^{\star}g$-closed set by Theorem 3.10. Since $cl[int(A)] - A$ is itself $s^{\star}g$-closed, $cl[int(A)] = A$. Hence A is regular closed. □

Theorem 3.12. A set A is strongly $s^{\star}g^{\star}$-closed if $cl[int(A)] - A$ contains no non empty closed set.

Proof. Suppose that F is non empty closed subset of $cl[int(A)] - A$. Then, $F \subseteq cl[int(A)] \cap A^c$. Therefore, $F \subseteq A^c$ implies $A \subseteq F^c$. Here F^c is open and hence it is $s^{\star}g$-open and A is strongly $s^{\star}g^{\star}$-closed, we have $cl[int(A)] \subseteq F^c$. Thus, $F \subseteq [clint(A)]^c$. Hence $F \subseteq cl[int(A)] \cap cl[int(A)]^c = \phi$. Therefore, $cl[int(A)] - A$ no non empty closed set. □

Corollary 3.13. A strongly $s^{\star}g^{\star}$-closed set A is regular closed if and only if $cl[int(A)] - A$ is closed and $A \subseteq cl[int(A)]$.

Proof. Since A is regular closed, $cl[int(A)] = A$, $cl[int(A)] - A = \phi$ is closed.

Conversely suppose that $cl[int(A)] - A$ is closed. Since A is strongly $s^{\star}g^{\star}$-closed, $cl[int(A)] - A$ contains no non empty closed set by Theorem 3.12. Since $cl[int(A)] - A$ is itself closed, $cl[int(A)] = A$. Hence A is regular closed. □

Theorem 3.14. If A is strongly $s^{\star}g^{\star}$-closed and $A \subseteq B \subseteq cl[int(A)]$ then B is strongly $s^{\star}g^{\star}$-closed.

Proof. Let $B \subseteq G$, G is $s^{\star}g$-open. Since $A \subseteq B$, $B \subseteq G$, $A \subseteq G$. Since A is strongly $s^{\star}g^{\star}$-closed, $cl[int(A)] \subseteq G$. But $cl[int(B)] \subseteq cl[int(A)]$ implies

$cl[int(B)] \subseteq G$. Therefore B is strongly s^*g^*-closed. □

Theorem 3.15. Let $A \subseteq Y \subseteq X$ and suppose that A is strongly s^*g^*-closed in X then A is strongly s^*g^*-closed relative to Y.

Proof. Since A is strongly s^*g^*-closed in X, $cl[int(A)] \subseteq G$, whenever $A \subseteq G$ and G is s^*g-open in X. Since $A \subseteq Y$ and $A \subseteq G$, $A \subseteq Y \cap G$. Since G is s^*g-open in X, $Y \cap G$ is s^*g-open in Y. Let $A \subseteq Y \cap G$, $Y \cap G$ is s^*g-open in Y. Then $cl_Y[int_Y(A)] = Y \cap cl[int(A)] \subseteq Y \cap G$. Therefore, A is strongly s^*g^*-closed in Y. □

Remark 3.16. If A and B are strongly s^*g^*-closed then both $A \cap B$ and $A \cup B$ need not be strongly s^*g^*-closed. The following examples support our claim.

Example 3.17. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{b, c\}\}$. Then, $A = \{b\}$, $B = \{c\}$ are strongly s^*g^*-closed in X, but $A \cup B = \{b, c\}$ is not strongly s^*g^*-closed in X.

Example 3.18. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$. Then, $A = \{a, b\}$, $B = \{a, c\}$ are strongly s^*g^*-closed in X, but $A \cap B = \{a\}$ is not strongly s^*g^*-closed in X.

4. Strongly s^*g^*-Open Sets

Recall that a set A is strongly g^*-closed [20] if $cl[int(A)] \subseteq U$ whenever $A \subseteq U$ and U is g-open in X.

Definition 4.1. A set A is a strongly s^*g^*-open set if its complement A^C is strongly s^*g^*-closed in X.

Example 4.2. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$. Then, $\phi, X, \{a\}, \{a, b\}, \{a, c\}$ are strongly s^*g^*-open sets in X.

Theorem 4.3. A set A is a strongly s^*g^*-open set if and only if $F \subseteq int[cl(A)]$ whenever $F \subseteq A$ and F is s^*g-closed in X.

Proof. **Necessity**: Let $F \subseteq A$ and F is $s^{*}g$-closed in X. Then, $A^C \subseteq F^C$ and F^C is $s^{*}g$-open in X. Since A^C is strongly $s^{*}g^{*}$-closed, $cl\left[int(A^C)\right] \subseteq F^C$. Consequently, $F \subseteq \left\{cl\left[int(A^C)\right]\right\}^C = int\left[int(A^C)\right]^C = int\,[cl(A)]$.

Sufficiency: Let $A^C \subseteq U$ and U is $s^{*}g$-open in X. Then, $U^C \subseteq A$ and U^C is $s^{*}g$-closed in X. By our assumption, $U^C \subseteq int[cl(A)]$. This implies that $cl\left[int(A^C)\right] \subseteq U$. Hence A^C is strongly $s^{*}g^{*}$-closed in X. Consequently, A is a strongly $s^{*}g^{*}$-open set. □

Theorem 4.4. Every open set is a strongly $s^{*}g^{*}$-open set.

Proof. Let A be a open set. Let $F \subseteq A$ and F is $s^{*}g$-closed in X. Since A is open, $F \subseteq A = int(A) \subseteq int[cl(A)]$. Therefore, A is strongly $s^{*}g^{*}$-open. □

Remark 4.5. The converse of the above theorem is not true in general. The following example supports our claim.

Example 4.6. In Example 4.2, $\{a, c\}$ is strongly $s^{*}g^{*}$-open, but not open.

Theorem 4.7. Every $s^{**}g$-open set is strongly $s^{*}g^{*}$-open.

Proof. Let A be $s^{**}g$-open set in X. Let $F \subseteq A$ and F is $s^{*}g$-closed in X. Since A is $s^{**}g$-open, $F \subseteq int(A) \subseteq int[cl(A)]$. Therefore, A is strongly $s^{*}g^{*}$-open. □

Theorem 4.8. If a subset of a topological space X is both closed and strongly $s^{*}g^{*}$-open, then it is open.

Proof. Since A is closed, A is $s^{*}g$-closed. Since A is strongly $s^{*}g^{*}$-open, $A \subseteq int[cl(A)]$. Since A is closed, $A \subseteq int[cl(A)] = int(A)$. Therefore, A is open. □

Corollary 4.9. If A is both closed and strongly $s^{*}g^{*}$-open in X, then it is both regular open and regular closed in X.

Proof. Since A is closed and strongly $s^{*}g^{*}$-open, A is open by Theorem 4.8. Since A is closed, $int[cl(A)] = int[A] = A$. Therefore, A is regular open. Since A is closed, $A = cl(A)$. Hence $A = cl[int(A)]$, since A is open. Thus A is regular closed. □

Corollary 4.10. If A is both closed and strongly $s^{\square}g^{\square}$-open in X, then it is rg-open in X.

Theorem 4.11. A set A is strongly $s^{\square}g^{\square}$-open if $A - int[cl(A)]$ contains no non empty $s^{\square}g$-closed set.

Proof. Suppose that F is non empty $s^{\square}g$-closed subset of $A - int[cl(A)]$. Then, $F \subseteq A \cap \{int[cl(A)]\}^C$. Therefore, $F \subseteq A$ and F is $s^{\square}g$-closed. Since A is strongly $s^{\square}g^{\square}$-open, we have $F \subseteq int[cl(A)]$. Also, $F \subseteq \{int[cl(A)]\}^C$. Hence $F \subseteq int[cl(A)] \cap int[cl(A)]^c = \phi$. Therefore $A - int[cl(A)]$ no non empty $s^{\square}g$-closed set. □

Corollary 4.12. A strongly $s^{\square}g^{\square}$-open set A is regular open if and only if $A - int[cl(A)]$ is $s^{\square}g$-closed and $A \subseteq int[cl(A)]$.

Proof. Since A is regular open, $int[cl(A)] = A$, $A - int[cl(A)] = \phi$ is $s^{\square}g$-closed.

Conversely suppose that $A - int[cl(A)]$ is $s^{\square}g$-closed. Since A is strongly $s^{\square}g^{\square}$-open, $A - int[cl(A)]$ contains no non empty $s^{\square}g$-closed set by Theorem 4.11. Since $A - int[cl(A)]$ is itself $s^{\square}g$-closed, $int[cl(A)] = A$. Hence A is regular open. □

Theorem 4.13. A set A is strongly $s^{\square}g^{\square}$-open if $A - int[cl(A)]$ contains no non empty closed set.

Proof. Suppose that F is non empty closed subset of $A - int[cl(A)]$. Then, $F \subseteq A \cap \{int[cl(A)]\}^C$. Therefore, $F \subseteq A$ and F is $s^{\square}g$-closed since every closed set is $s^{\square}g$-closed. Since A is strongly $s^{\square}g^{\square}$-open, we have $F \subseteq int[cl(A)]$. Also, $F \subseteq \{int[cl(A)]\}^C$. Hence $F \subseteq int[cl(A)] \cap int[cl(A)]^c = \phi$. Therefore $A - int[cl(A)]$ no non empty closed set. □

Corollary 4.14. A strongly $s^{\square}g^{\square}$-open set A is regular open if and only if $A - int[cl(A)]$ is closed and $A \subseteq int[cl(A)]$.

Proof. Since A is regular open, $int[cl(A)] = A$, $A - int[cl(A)] = \phi$ is closed.

Conversely suppose that $A - int[cl(A)]$ is closed. Since A is strongly $s^{\square}g^{\square}$-open, $A - int[cl(A)]$ contains no non empty closed set by Theorem 4.13. Since $A - int[cl(A)]$ is itself closed, $int[cl(A)] = A$. Hence A is regular open. □

Theorem 4.15. If A is strongly $s^{\square}g^{\square}$-open and $int[cl(A)] \subseteq B \subseteq A$ then B is strongly $s^{\square}g^{\square}$-open.

Proof. Let $F \subseteq B$, F is $s^{\square}g$-closed. Since $B \subseteq A$, $F \subseteq B$, $F \subseteq A$. Since A is strongly $s^{\square}g^{\square}$-open, $F \subseteq int[cl(A)]$. But $int[cl(A)] \subseteq int[cl(B)]$ implies $F \subseteq int[cl(B)]$. Therefore B is strongly $s^{\square}g^{\square}$-open. □

Theorem 4.16. Let $A \subseteq Y \subseteq X$ and suppose that A is strongly $s^{\square}g^{\square}$-open in X then A is strongly $s^{\square}g^{\square}$-open relative to Y.

Proof. Since A is strongly $s^{\square}g^{\square}$-open in X, $F \subseteq int[cl(A)]$, whenever $F \subseteq A$ and F is $s^{\square}g$-closed in X. Since $A \subseteq Y$ and $F \subseteq A$, $F \subseteq Y$ and hence $F = F \cap Y \subseteq Y$. Since F is $s^{\square}g$-closed in X, $F = Y \cap F$ is $s^{\square}g$-closed in Y. Let $F \subseteq A$, F is $s^{\square}g$-closed in Y. Now, $int_Y[cl_Y(A)] = Y \cap int[cl(A)] \supseteq F$. Therefore, A is strongly $s^{\square}g^{\square}$-open in Y. □

Remark 4.17. If A and B are strongly $s^{\square}g^{\square}$-open then both $A \cap B$ and $A \cup B$ need not be strongly $s^{\square}g^{\square}$-open. The following examples support our claim.

Example 4.18. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{b, c\}\}$. Then, $A = \{a, b\}$, $B = \{a, c\}$ are strongly $s^{\square}g^{\square}$-open in X, but $A \cap B = \{a\}$ is not strongly $s^{\square}g^{\square}$-open in X.

Example 4.19. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$. Then, $A = \{b\}$, $B = \{c\}$ are strongly $s^{\square}g^{\square}$-open in X, but $A \cup B = \{b, c\}$ is not strongly $s^{\square}g^{\square}$-open in X.

Conclusion

Thus, the properties of strongly $s^{\square}g^{\square}$-closed sets and strongly $s^{\square}g^{\square}$-open sets have been studied. Further research on separation axioms and continuous mappings with the help of these sets can be undertaken.

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