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OPTIMAL CONVEX COMBINATION BOUNDS OF GEOMETRIC AND SECOND SEIFFERT MEANS FOR NEUMAN-SÁNDOR MEAN

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Abstract: In this paper, we present the least value α and the greatest value β such that the double inequality

$$\alpha G(a,b)+(1-\alpha)T(a,b)<M(a,b)<\beta G(a,b)+(1-\beta)T(a,b)$$

holds for all $a,b>0$ with $a\neq b$, where $G(a,b),M(a,b)$ and $T(a,b)$ are respectively the geometric, Neuman-Sándor and second Seiffert means of a and b.

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1. Introduction

For $a,b>0$ with $a\neq b$ the Neuman-Sándor mean $M(a,b)$ [1] was defined by

$$M(a,b)=\frac{a-b}{2\sinh^{-1}(\dfrac{a-b}{a+b})}, \tag{1.1}$$

where $\sinh^{-1}(x)=\log(x+\sqrt{1+x^2})$ is the inverse hyperbolic sine function.

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Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a,b)$ can be found in the literature [1,2].

Let $H(a,b) = (2ab)/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/(4\arctan\sqrt{a/b} - \pi)$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan(a-b)/(a+b)]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic mean of a and b, respectively. Then

$$\begin{aligned} &\min\{a,b\} < H(a,b) < G(a,b) < L(a,b) \\ &< P(a,b) < A(a,b) < M(a,b) < T(a,b) \\ &< Q(a,b) < C(a,b) < \max\{a,b\} \end{aligned} \tag{1.2}$$

hold for all $a, b > 0$ with $a \neq b$.

Li et al. [3] showed that the double inequality

$$L_{p_0}(a,b) < M(a,b) < L_2(a,b) \tag{1.3}$$

holds for all $a, b > 0$ with $a \neq b$, where $L_p(a,b) = [(a^{p+1} - b^{p+1})/((p+1)(a-b))]^{1/p} (p \neq -1, 0)$, $L_0(a,b) = 1/e(a^a/b^b)^{1/(a-b)}$ and $L_{-1}(a,b) = (a-b)/(\log a - \log b)$ is the p-th generalized logarithmic mean of a and b, and $p_0 = 1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$.

In [4], Neuman proved that the double inequalities

$$\alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b) \tag{1.4}$$

and

$$\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) < \mu C(a,b) + (1-\mu)A(a,b) \tag{1.5}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\cdots, \beta \geq 1/3, \lambda \leq [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2}) = 0.1345\cdots$ and $\mu \geq 1/6$.

In [5], Chu etc proved that the double inequalities

$$\alpha_1 L(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 L(a,b) + (1-\beta_1)Q(a,b) \tag{1.6}$$

and

$$\alpha_2 L(a,b) + (1-\alpha_2)C(a,b) < M(a,b) < \beta_2 L(a,b) + (1-\beta_2)C(a,b) \tag{1.7}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \geq 2/5, \beta_1 \leq 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\cdots, \alpha_2 \geq 5/8$ and $\beta_2 \leq 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$.

The main purpose of this paper is to find the least value α and the greatest value β such that the double inequality

$$\alpha G(a,b) + (1-\alpha)T(a,b) < M(a,b) < \beta G(a,b) + (1-\beta)T(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

2. Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

lemma 2.1 Let $f(x) = 1/\sqrt{1-x^2}, g(x) = \sqrt{1-x^2}, h(x) = 1/\sqrt{1+x^2}, l(x) = \arctan x$ and $k(x) = \log(x + \sqrt{1+x^2})$. Then The inequalities

$$f(x) > 1 + \frac{x^2}{2}, \tag{2.1}$$

$$g(x) < 1 - \frac{x^2}{2}, \tag{2.2}$$

$$h(x) < 1 - \frac{x^2}{2} + \frac{3}{8}x^4, \tag{2.3}$$

$$l(x) > x - \frac{x^3}{3}, \tag{2.4}$$

and

$$k(x) > x - \frac{x^3}{6} \tag{2.5}$$

hold for all $x \in (0, 1)$.

Proof. The inequalities (2.1)$-$(2.3) follow immediately from the inequalities

$$[f(x)]^2 - \left(1 + \frac{x^2}{2}\right)^2 = \frac{x^4(3+x^2)}{4(1-x^2)} > 0,$$

$$[g(x)]^2 - \left(1 - \frac{x^2}{2}\right)^2 = -\frac{x^4}{4} < 0$$

and

$$[h(x)]^2 - \left(1 - \frac{x^2}{2} + \frac{3}{8}x^4\right)^2 = -\frac{x^6[(3x^2-5)^2 + 15(1+x^2)]}{64(1+x^2)} < 0$$

for all $x \in (0,1)$.

Next, we prove the inequalities (2.4) and (2.5).

Let

$$l_1(x) = l(x) - (x - \frac{x^3}{3}). \tag{2.6}$$

Then simple computations lead to

$$\lim_{x\to 0^+} l_1(x) = 0, \tag{2.7}$$

and

$$l_1'(x) = \frac{x^4}{1+x^2} > 0 \tag{2.8}$$

for $x \in (0,1)$. Therefore, the inequality (2.4) follows from (2.7), (2.8) and (2.6).

Let

$$k_1(x) = k(x) - (x - \frac{x^3}{6}). \tag{2.9}$$

Then simple computations yield

$$\lim_{x\to 0^+} k_1(x) = 0, \tag{2.10}$$

$$k_1'(x) = \frac{1}{\sqrt{1+x^2}} - 1 + \frac{x^2}{2}, \tag{2.11}$$

$$\lim_{x\to 0^+} k_1'(x) = 0, \tag{2.12}$$

and

$$k_1''(x) = x\big[1 - \frac{1}{(1+x^2)^{3/2}}\big] > 0 \tag{2.13}$$

for $x \in (0,1)$. Therefore, the inequality (2.5) follows easily from (2.10), (2.12), (2.13) and (2.9). □

lemma 2.2 Let $\varphi(x) = 1/\arctan x - x/[(1+x^2)(\arctan x)^2]$. Then the inequality

$$\varphi(x) < \frac{2}{3}x - \frac{1}{9}x^3 \tag{2.14}$$

holds for all $x \in (0,1)$.

Proof. Let

$$\varphi_1(x) = (1+x^2)\arctan x - x - (\frac{2}{3}x - \frac{1}{9}x^3)(1+x^2)(\arctan x)^2. \tag{2.15}$$

Then

$$\lim_{x\to 0^+} \varphi_1(x) = 0. \tag{2.16}$$

Differentiating $\varphi_1(x)$ yields

$$\begin{aligned} \varphi_1'(x) = & -(\frac{2}{3} + \frac{5}{3}x^2 - \frac{5}{9}x^4)(\arctan x)^2 + (\frac{2}{9}x^3 \\ & -\frac{4}{3}x) \arctan x + 2x \arctan x. \end{aligned} \tag{2.17}$$

The expression of $\varphi_1'(x)$ is represented as

$$\begin{aligned} \varphi_1'(x) = & -(\frac{2}{3} + \frac{5}{3}x^2 - \frac{5}{9}x^4) \arctan x \cdot l(x) + (\frac{2}{9}x^3 \\ & -\frac{4}{3}x) \arctan x + 2x \arctan x, \end{aligned} \tag{2.18}$$

where $l(x)$ is defined as in lemma 2.1. Owing to

$$\frac{2}{3} + \frac{5}{3}x^2 - \frac{5}{9}x^4 = \frac{1}{9}[6 + 15x^2(1 - \frac{x^2}{3})] > 0$$

for $x \in (0,1)$, making use of the inequality (2.4) with the item $-[2/3+(5/3)x^2-(5/9)x^4] \arctan x \cdot l(x)$ replaced by $-[2/3+(5/3)x^2-(5/9)x^4] \arctan x \cdot (x-x^3/3)$ in the equality (2.18) cause the conclusion that

$$\begin{aligned} \varphi_1'(x) < & -(\frac{2}{3} + \frac{5}{3}x^2 - \frac{5}{9}x^4)(x - \frac{x^3}{3}) \arctan x \\ & +(\frac{2}{9}x^3 - \frac{4}{3}x) \arctan x + 2x \arctan x \\ = & -\frac{x^3}{27}\left[33(1-x^2) + 3x^2 + 5x^4\right] \arctan x < 0 \end{aligned} \tag{2.19}$$

for all $x \in (0,1)$. Therefore, the inequality (2.14) follows easily from (2.15) and (2.16) together with (2.19). □

lemma 2.3 (see [5, *lemma* 2.4]). Let $\psi(x) = x/[\sqrt{1+x^2} \cdot (\sinh^{-1}(x))^2] - 1/\sinh^{-1}(x)$. Then the inequality

$$\psi(x) < -\frac{x}{3} + \frac{17x^3}{90} \tag{2.20}$$

holds for all $x \in (0,1)$.

lemma 2.4 Let $\Delta(x) = \arctan x \cdot \sinh^{-1}(x)$. Then the inequality

$$\Delta(x) > x^2 - \frac{x^4}{2} \tag{2.21}$$

holds for all $x \in (0,1)$.

Proof. By the inequalities (2.4) and (2.5), the inequality (2.21) follows easily from

$$\arctan x \cdot \sinh^{-1}(x) - (x^2 - \frac{x^4}{2}) > (x - \frac{x^3}{3})(x - \frac{x^3}{6}) - (x^2 - \frac{x^4}{2}) = \frac{x^6}{18} > 0$$

for $x \in (0, 1)$. □

lemma 2.5 Let

$$\Omega(x) = 30 - 105x^2 + 84x^4 - 24x^6. \tag{2.22}$$

Then there is the real number x_0 on $(0, 1)$ so as to $\Omega(x_0) = 0$, $\Omega(x) > 0$ for $x \in (0, x_0)$, and $\Omega(x) < 0$ for $x \in (x_0, 1)$. Further more $1/2 < x_0 < 1/\sqrt{2}$.

Proof. Simple computations lead to

$$\lim_{x \to 0^+} \Omega(x) = 30 > 0, \lim_{x \to 1^-} \Omega(x) = -15 < 0, \tag{2.23}$$

$$\Omega(\frac{1}{2}) = \frac{69}{8} > 0, \ \Omega(\frac{1}{\sqrt{2}}) = -\frac{9}{2} < 0, \tag{2.24}$$

and

$$\Omega'(x) = -6x[8(1 - x^2)(4 - 3x^2) + 3] < 0 \tag{2.25}$$

for $x \in (0, 1)$. From (2.25) we confirm that $\Omega(x)$ is strictly decreasing in (0,1). It follows from (2.23) and the monotonicity of $\Omega(x)$ that there exits $x_0 \in (0, 1)$ such that $\Omega(x_0) = 0$, $\Omega(x) > 0$ for $x \in (0, x_0)$, and $\Omega(x) < 0$ for $x \in (x_0, 1)$. From (2.24) we know that $1/2 < x_0 < 1/\sqrt{2}$. □

lemma 2.6 Let $\lambda = 1 - \pi/[4 \log(1 + \sqrt{2})] = 0.108893 \cdots$ and

$$\begin{aligned} F(x) = \ & 2(11\lambda - 3)x^{16} + (35 - 59\lambda)x^{14} + (17\lambda - 91)x^{12} \\ & +2(97 - 13\lambda)x^{10} + 4(58\lambda - 99)x^8 + 5(115 \\ & -19\lambda)x^6 - (367\lambda + 491)x^4 + 20(5\lambda \\ & +11)x^2 + 8(2\lambda - 5). \end{aligned} \tag{2.26}$$

Then the inequality

$$F(x) < 0 \tag{2.27}$$

holds for all $x \in (0, 1/\sqrt{2})$.

Proof. Let $x \in (0, 1/\sqrt{2})$. Making use of the transform $x^2 = 1/t$ $(t \in (2, +\infty))$ for $F(x)$ yields

$$F(x) = \frac{1}{t^8} F_1(t), \tag{2.28}$$

where

$$\begin{aligned} F_1(t) = \ & 8(2\lambda - 5)t^8 + 20(5\lambda + 11)t^7 - (367\lambda + 491)t^6 \\ & +5(115 - 19\lambda)t^5 + 4(58\lambda - 99)t^4 + 2(97 \\ & -13\lambda)t^3 + (17\lambda - 91)t^2 + (35 - 59\lambda)t \\ & +2(11\lambda - 3). \end{aligned} \tag{2.29}$$

Simple computations lead to

$$\lim_{t \to 2^+} F_1(t) = -4(1539\lambda + 47) < 0, \tag{2.30}$$

$$\begin{aligned} F_1'(t) = \ & 64(2\lambda - 5)t^7 + 140(5\lambda + 11)t^6 - 6(367\lambda \\ & +491)t^5 + 25(115 - 19\lambda)t^4 + 16(58\lambda - 99)t^3 \\ & +6(97 - 13\lambda)t^2 + 2(17\lambda - 91)t + (35 - 59\lambda), \end{aligned} \tag{2.31}$$

$$\lim_{t \to 2^+} F_1'(t) = -(9759\lambda + 1345) < 0, \tag{2.32}$$

$$\begin{aligned} F_1''(t) = \ & 2[224(2\lambda - 5)t^6 + 420(5\lambda + 11)t^5 - 15(367\lambda \\ & +491)t^4 + 50(115 - 19\lambda)t^3 + 24(58\lambda - 99)t^2 \\ & +6(97 - 13\lambda)t + (17\lambda - 91)], \end{aligned} \tag{2.33}$$

$$\lim_{t \to 2^+} F_1''(t) = -2(4111 - 5621\lambda) < 0, \tag{2.34}$$

$$\begin{aligned} F_1'''(t) = \ & 12[224(2\lambda - 5)t^5 + 350(5\lambda + 11)t^4 - 10(367\lambda \\ & +491)t^3 + 25(115 - 19\lambda)t^2 + 8(58\lambda - 99)t \\ & +(97 - 13\lambda)], \end{aligned} \tag{2.35}$$

$$\lim_{t \to 2^+} F_1'''(t) = -252(167 - 571\lambda) < 0, \tag{2.36}$$

$$\begin{aligned} F_1^{(4)}(t) = \ & 24[560(2\lambda - 5)t^4 + 700(5\lambda + 11)t^3 - 15(367\lambda \\ & +491)t^2 + 25(115 - 19\lambda)t + 4(58\lambda - 99)], \end{aligned} \tag{2.37}$$

$$\lim_{t \to 2^+} F_1^{(4)}(t) = -48(3653 - 11591\lambda) < 0, \tag{2.38}$$

$$\begin{aligned} F_1^{(5)}(t) = \ & 120[448(2\lambda - 5)t^3 + 420(5\lambda + 11)t^2 \\ & -6(367\lambda + 491)t + 5(115 - 19\lambda)], \end{aligned} \tag{2.39}$$

$$\lim_{t \to 2^+} F_1^{(5)}(t) = -120(4757 - 11069\lambda) < 0, \tag{2.40}$$

$$F_1^{(6)}(t) = 720[224(2\lambda-5)t^2+140(5\lambda+11)t -(367\lambda+491), \tag{2.41}$$

$$\lim_{t\to 2^+} F_1^{(6)}(t) = -720(1891-2825\lambda) < 0, \tag{2.42}$$

and

$$F_1^{(7)}(t) = -5040\Big[\frac{\pi}{\ln(1+\sqrt{2})}(32t+25)+64(3x-5)\Big] < 0 \tag{2.43}$$

for $t \in (2, +\infty)$. Therefore, the inequality (2.27) follows from (2.42), (2.40), (2.38), (2.36), (2.34), (2.32), (2.30) and (2.28) together with (2.43). □

lemma 2.7 Let $\lambda = 1 - \pi/[4\log(1+\sqrt{2})] = 0.108893\cdots$ and

$$H(x) = -16\lambda x^{16}+2(3-23\lambda)x^{14}+(71\lambda-29)x^{12} +2(33\lambda+31)x^{10}-4(10\lambda+33)x^8+8(33 -47\lambda)x^6-(271\lambda+311)x^4+4(29\lambda +45)x^2+8(2\lambda-5), \tag{2.44}$$

Then the inequality

$$H(x) < 0 \tag{2.45}$$

holds for all $x \in (1/2, 1)$.

Proof. Let $x \in (1/2, 1)$. Making use of the transform $x^2 = 1/t$ ($t \in (1, 4)$) for $H(x)$ leads to

$$H(x) = \frac{1}{t^8}H_1(t), \tag{2.46}$$

where

$$H_1(t) = 8(2\lambda-5)t^8+4(29\lambda+45)t^7-(271\lambda+311)t^6 +8(33-47\lambda)t^5-4(10\lambda+33)t^4+2(33\lambda +31)t^3+(71\lambda-29)t^2+2(3-23\lambda)t-16\lambda. \tag{2.47}$$

Simple computations yield

$$\lim_{t\to 1^+} H_1(t) = -480\lambda < 0, \tag{2.48}$$

$$H_1'(t) = 2\big[32(2\lambda-5)t^7+14(29\lambda+45)t^6-3(271\lambda +311)t^5+20(33-47\lambda)t^4-8(10\lambda+33)t^3 +3(33\lambda+31)t^2+(71\lambda-29)t+(3-23\lambda)\big], \tag{2.49}$$

$$\lim_{t\to 1^+} H_1'(t) = -2432\lambda < 0, \tag{2.50}$$

$$\begin{aligned} H_1''(t) = \;& 2\big[224(2\lambda-5)t^6+84(29\lambda+45)t^5-15(271\lambda \\ & +311)t^4+80(33-47\lambda)t^3-24(10\lambda+33)t^2 \\ & +6(33\lambda+31)t+(71\lambda-29)\big], \end{aligned} \tag{2.51}$$

$$\lim_{t\to 1^+} H_1''(t) = -9824\lambda < 0, \tag{2.52}$$

$$\begin{aligned} H_1'''(t) = \;& 12\big[224(2\lambda-5)t^5+70(29\lambda+45)t^4-10(271\lambda \\ & +311)t^3+40(33-47\lambda)t^2-8(10\lambda+33)t \\ & +(33\lambda+31)\big], \end{aligned} \tag{2.53}$$

$$\lim_{t\to 1^+} H_1'''(t) = -12(2159\lambda-7) < 0, \tag{2.54}$$

and

$$\begin{aligned} H_1^{(4)}(t) = \;& 24\big[560(2\lambda-5)t^4+140(29\lambda+45)t^3 \\ & -15(271\lambda+311)t^2+40(33-47\lambda)t \\ & -4(10\lambda+33)\big]. \end{aligned} \tag{2.55}$$

Completing the square for $H_1^{(4)}(t)$ educes that

$$\begin{aligned} H_1^{(4)}(t) &= -24\Big\{140(t-1)^3[4(5-2\lambda)(t-1)+(35 \\ &\quad -61\lambda)]+15(171-989\lambda)(t \\ &\quad +\frac{31-665\lambda}{3(171-989\lambda)}-1)^2 \\ &\quad +\frac{4[171839\lambda-7(164270\lambda^2+593)]}{3(171-989\lambda)}\Big\} \\ &< 0 \end{aligned} \tag{2.56}$$

for $t\in(1,4)$. Therefore, the inequality (2.45) follows from (2.54), (2.52), (2.50), (2.48) and (2.46) together with (2.56). □

3. Main Results

Theorem 1. *The double inequality*

$$\alpha G(a,b) + (1-\alpha)T(a,b) < M(a,b) < \beta G(a,b) + (1-\beta)T(a,b)$$

holds true for $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha \geq 1/5$ *and* $\beta \leq 1-\pi/[4\log(1+\sqrt{2})] = 0.108893\cdots$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $x = (a-b)/(a+b) \in (0,1)$ and $\lambda = 1-\pi/[4\log(1+\sqrt{2})] = 0.108893\cdots$. Then

$$\begin{aligned} &\frac{G(a,b)}{A(a,b)} = \sqrt{1-x^2}, \frac{M(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1}(x)}, \\ &\frac{T(a,b)}{A(a,b)} = \frac{x}{\arctan x}. \end{aligned} \tag{3.1}$$

Firstly we prove that

$$\frac{1}{5}G(a,b) + \frac{4}{5}T(a,b) < M(a,b), \tag{3.2}$$

and

$$\lambda G(a,b) + (1-\lambda)T(a,b) > M(a,b). \tag{3.3}$$

From (3.1) we have

$$\begin{aligned} & \frac{\frac{1}{5}G(a,b) + \frac{4}{5}T(a,b) - M(a,b)}{A(a,b)} \\ = \; & \frac{1}{5}\sqrt{1-x^2} + \frac{4x}{5\arctan x} - \frac{x}{\sinh^{-1}(x)} \\ = \; & d(x). \end{aligned} \tag{3.4}$$

Equation (3.4) leads to

$$\lim_{x\to 0^+} d(x) = 0, \tag{3.5}$$

and

$$\begin{aligned} d'(x) \; = \; & \frac{4}{5}\Big[\frac{1}{\arctan x} - \frac{x}{(1+x^2)(\arctan x)^2}\Big] - \frac{x}{5\sqrt{1-x^2}} \\ & + \frac{x}{\sqrt{1+x^2}[\sinh^{-1}(x)]^2} - \frac{1}{\sinh^{-1}(x)} \\ = \; & \frac{4}{5}\varphi(x) - \frac{x}{5}f(x) + \psi(x), \end{aligned} \tag{3.6}$$

where $f(x)$, $\varphi(x)$ and $\psi(x)$ are defined as in Lemma 2.1, 2.2 and 2.3, respectively. From the inequalities (2.1), (2.14), (2.20) and the equation (3.6) we educe

$$\begin{aligned} d'(x) &< \frac{4}{5}(\frac{2x}{3}-\frac{x^3}{9})-\frac{x}{5}(1+\frac{x^2}{2})+(-\frac{x}{3}+\frac{17x^3}{90}) \\ &= 0 \end{aligned} \tag{3.7}$$

for all $x \in (0,1)$. Therefore, the inequality (3.2) follows easily from (3.4) and (3.5) together with (3.7).

From (3,1) one has

$$\begin{aligned} &\frac{\lambda G(a,b)+(1-\lambda)T(a,b)-M(a,b)}{A(a,b)} \\ =\ & \lambda\sqrt{1-x^2}+(1-\lambda)\frac{x}{\arctan x}-\frac{x}{\sinh^{-1}(x)} \\ =\ & \frac{x}{2\arctan x\cdot\sinh^{-1}(x)}D(x), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} D(x) = \ & \frac{2\lambda\sqrt{1-x^2}}{x}\Delta(x)+2(1-\lambda)\sinh^{-1}(x) \\ & -2\arctan x, \end{aligned} \tag{3.9}$$

thereinto $\Delta(x)$ is defined as in lemma 2.4. Using the inequality (2.21) for $D(x)$ leads to

$$\begin{aligned} & D(x) > \lambda\sqrt{1-x^2}(2x-x^3)+2(1-\lambda)\sinh^{-1}(x) \\ & -2\arctan x = D_1(x). \end{aligned} \tag{3.10}$$

Some tedious, but not difficult, calculations yield

$$\lim_{x\to 0^+} D_1(x)=0, \tag{3.11}$$

$$\lim_{x\to 1^-} D_1(x)=0, \tag{3.12}$$

$$D_1'(x)=\frac{\lambda(2-7x^2+4x^4)}{\sqrt{1-x^2}}+\frac{2(1-\lambda)}{\sqrt{1+x^2}}-\frac{2}{1+x^2}, \tag{3.13}$$

$$\lim_{x\to 0^+} D_1'(x)=0, \tag{3.14}$$

$$\lim_{x\to 1^-} D_1'(x)=-\infty, \tag{3.15}$$

$$\begin{aligned} D_1''(x) = \ & -\frac{\lambda(12x-23x^3+12x^5)}{(1-x^2)^{3/2}}-\frac{2(1-\lambda)x}{(1+x^2)^{3/2}} \\ & +\frac{4x}{(1+x^2)^2}, \end{aligned} \tag{3.16}$$

$$\lim_{x\to 0^+} D''(x) = 0, \tag{3.17}$$

$$D_1''(\frac{1}{4}) = \frac{256}{289} - \frac{32\sqrt{17}}{289}(1-\lambda) - \frac{679\sqrt{15}}{900}\lambda > 0, \tag{3.18}$$

$$\lim_{x\to 1^-} D_1''(x) = -\infty, \tag{3.19}$$

$$\begin{aligned} D_1'''(x) = & \frac{\lambda(24x^6 - 60x^4 + 45x^2 - 12)}{(1-x^2)^{5/2}} \\ & - \frac{2(1-\lambda)(1-2x^2)}{(1+x^2)^{5/2}} + \frac{4(1-3x^2)}{(1+x^2)^3}, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} D_1^{(4)}(x) &= \frac{\lambda x(30 - 105x^2 + 84x^4 - 24x^6)}{(1-x^2)^{7/2}} \\ &\quad + \frac{6(1-\lambda)x(3-2x^2)}{(1+x^2)^{7/2}} - \frac{48x(1-x^2)}{(1+x^2)^4} \\ &= \frac{\lambda x \Omega(x)}{(1-x^2)^{7/2}} + \frac{6(1-\lambda)x(3-2x^2)}{(1+x^2)^{7/2}} \\ &\quad - \frac{48x(1-x^2)}{(1+x^2)^4}, \end{aligned} \tag{3.21}$$

where $\Omega(x)$ is defined as in lemma 2.5.

Next, we distinguish between two cases. In the two cases, the real number x_0 satisfy $\Omega(x_0) = 0$ and $1/2 < x_0 < 1/\sqrt{2}$, which had been proved in lemma 2.5.

Case 1. Let $x \in (0, x_0) \subset (0, 1/\sqrt{2})$. (3.21) is rewritten into

$$\begin{aligned} D_1^{(4)}(x) = & \frac{\lambda x \Omega(x)}{(1-x^2)^4} g(x) + \frac{6(1-\lambda)x(3-2x^2)}{(1+x^2)^3} h(x) \\ & - \frac{48x(1-x^2)}{(1+x^2)^4}, \end{aligned} \tag{3.22}$$

where $g(x)$ and $h(x)$ are defined as in lemma 2.1. Based on lemma 2.5, making use of the inequalities (2.2) and (2.3) leads to

$$\begin{aligned} D_1^{(4)}(x) &< \frac{\lambda x(30 - 105x^2 + 84x^4 - 24x^6)}{(1-x^2)^4}(1 - \frac{x^2}{2}) \\ &\quad + \frac{6(1-\lambda)x(3-2x^2)}{(1+x^2)^3}(1 - \frac{x^2}{2} + \frac{3}{8}x^4) \\ &\quad - \frac{48x(1-x^2)}{(1+x^2)^4} \\ &= \frac{3x}{4(1-x^4)^4} F(x), \end{aligned} \tag{3.23}$$

where $F(x)$ had been denoted by (2.26). So the inequality

$$D_1^{(4)}(x) < 0 \tag{3.24}$$

follows from (3.23) and lemma 2.6.

Case 2. Let $x \in [x_0, 1) \subset (1/2, 1)$. (3.21) is rewritten into

$$\begin{aligned} D_1^{(4)}(x) = \; & \frac{\lambda x \Omega(x)}{(1-x^2)^3} f(x) + \frac{6(1-\lambda)x(3-2x^2)}{(1+x^2)^3} h(x) \\ & - \frac{48x(1-x^2)}{(1+x^2)^4}, \end{aligned} \tag{3.25}$$

where $f(x)$ and $h(x)$ are defined as in lemma 2.1. Based on lemma 2.5, making use of the inequalities (2.1) and (2.3) yields

$$\begin{aligned} D_1^{(4)}(x) \; < \; & \frac{\lambda x(30 - 105x^2 + 84x^4 - 24x^6)}{(1-x^2)^3}(1 + \frac{x^2}{2}) \\ & + \frac{6(1-\lambda)x(3-2x^2)}{(1+x^2)^3}(1 - \frac{x^2}{2} + \frac{3x^4}{8}) \\ & - \frac{48x(1-x^2)}{(1+x^2)^4} \\ = \; & \frac{3x}{4(1-x^2)^3(1+x^2)^4} H(x), \end{aligned} \tag{3.26}$$

where $H(x)$ had been denoted by (2.44). Thus the inequality

$$D_1^{(4)}(x) < 0 \tag{3.27}$$

follows from (3.26) and lemma 2.7.

Synthesizing the above two cases we affirm that

$$D_1^{(4)}(x) < 0 \tag{3.28}$$

for all $x \in (0, 1)$. (3.28) show that $D_1''(x)$ is strictly concave function in the interval (0,1). It follows from (3.17), (3.18) and (3.19) together with the concavity of $D_1''(x)$ that there exists $\mu_1 \in (0, 1)$ such that $D_1''(x) > 0$ for $x \in (0, \mu_1)$ and $D_1''(x) < 0$ for $x \in (\mu_1, 1)$, hence $D_1'(x)$ is strictly increasing in $(0, \mu_1)$ and strictly decreasing in $(\mu_1, 1)$. From (3.14) and (3.15) together with the monotonicity of $D_1'(x)$ we know that there exits $\mu_2 \in (\mu_1, 1) \subset (0, 1)$ such that $D_1'(x) > 0$ for $x \in (0, \mu_2)$ and $D_1'(x) < 0$ for $x \in (\mu_2, 1)$, thus $D_1(x)$ is strictly increasing in $(0, \mu_2)$ and strictly decreasing in $(\mu_2, 1)$. It follows from (3.11) and (3.12) together with the monotonicity of $D_1(x)$ that

$$D_1(x) > 0 \tag{3.29}$$

for all $x \in (0,1)$. Therefore, the inequality (3.3) follows easily from (3.8), (3.10) and (3.29).

Finally, we prove that $1/5G(a,b) + 4/5T(a,b)$ is the best possible lower convex combination bound and $\lambda G(a,b) + (1-\lambda)T(a,b)$ is the best possible upper convex combination bound of the geometric and the second Seiffert mean for the Neuman-Sándor mean.

(3.1) leads to

$$\frac{T(a,b) - M(a,b)}{T(a,b) - G(a,b)} = \frac{x/\arctan x - x/\sinh^{-1}(x)}{x/\arctan x - \sqrt{1-x^2}} = B(x). \tag{3.30}$$

From (3.30) one has

$$\lim_{x \to 0^+} B(x) = \frac{1}{5}, \tag{3.31}$$

and

$$\lim_{x \to 1^-} B(x) = 1 - \frac{\pi}{4\log(1+\sqrt{2})} = \lambda. \tag{3.32}$$

If $\alpha < 1/5$, then (3.30) and (3.31) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that $M(a,b) < \alpha G(a,b) + (1-\alpha)T(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (0, \delta_1)$.

If $\beta > \lambda$, then (3.30) and (3.32) lead to the conclusion that there exists $0 < \delta_2 < 1$ such that $M(a,b) > \lambda G(a,b) + (1-\lambda)T(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$. □

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