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ALMOST σ-CONTINUITY ON σ-STRUCTURES

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Abstract: We introduce to the notion of almost σ-continuity on σ-structures and investigate the relation among σ-continuity, weak σ-continuity and almost σ-continuity. We also investigate characterizations for such continuity by using σ-semiopen sets, σ-preopen sets, σ-β-open sets, σ-regular open sets, and σ-δ-open sets.

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1. Introduction

Császár [1] introduced the notions of generalized topology and generalized open sets as the following: Let X be a nonempty set and μ be a collection of subsets of X. Then μ is called a *generalized topology* (briefly GT) on X iff $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i\in I} G_i \in \mu$. The elements of μ are called *g-open* sets and the complements are called *g-closed* sets. In [2], Kim and Min introduced the notion of a σ-structure as: $s \subseteq 2^X$ is called a *σ-structure* on X if for $i \in I \neq \emptyset$, $U_i \in s$ implies $\cup_{i\in I} U_i \in s$. The elements of s are called *σ-open* sets and the complements are called *σ-closed* sets. The notion of σ-structures is an

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extended notion of generalized topology defined by Császár. The σ-structure s is said to be *strong* [4] if $X \in \sigma$. Then we studied the notion of σ-semiopen sets [3] in the spaces with σ-structures analogous to semi-open sets [6] introduced by Levine on a given topological space. We also introduced the notions of σ-preopen sets [4] and σ-β-open sets [5]. In particular, we showed that the family $\sigma PO(X)$ (resp., $\sigma\beta O(X)$) of all σ-preopen sets (resp., σ-β-open sets) in (X, σ) is a generalized topology in sense of Császár. In [4], we showed that the family $\sigma SO(X)$ of all σ-semiopen subsets is strong but $\sigma PO(X)$ may not be strong in (X, σ). In [5], we studied σ-β-open sets which are the generalized σ-open sets of σ-semiopen sets and σ-preopen sets, and showed that the family $\sigma\beta O(X)$ of all σ-β-open sets in (X, σ) is a strong σ-structure. In [7], we also introduced the notion of σ-regular open set and studied basic properties. In [6], we studied σ-δ^*-open sets and σ-δ-open sets which are the generalized σ-open sets of σ-regular open sets. In [2], we introduced the notion of σ-continuity and obtained characterizations of σ-continuity by using the two operators i_s and c_s. In [2], we introduced the notion of weak σ-continuity which is an extended notion of σ-continuity, and studied characterizations by using σ-semiopen sets, σ-preopen sets, σ-β-open sets and σ-regular open sets.

The purpose of this paper is to study the extended notion of σ-continuity, and so we are going to introduce the notion of almost σ-continuity. We also investigate the relation among σ-continuity, weak σ-continuity and almost σ-continuity. Then we investigate characterizations by using σ-semiopen sets, σ-preopen sets and σ-β-open sets. In particular, we study characterizations for almost σ-continuity by using σ-δ-open sets which are introduced in [6].

2. Preliminaries

We recall that the notions of the two operators i_s and c_s are defined in [2]: Let s be a σ-structure on a nonempty set X and $A \subseteq X$:

$i_s A = \cup\{S \subseteq X : S \subseteq A,\ S \text{ is } \sigma\text{-open}\}$;

$c_s A = \cap\{F \subseteq X : A \subseteq F,\ F \text{ is } \sigma\text{-closed}\}$.

Theorem 2.1 ([2]). *Let s be a σ-structure on a nonempty set X and $A, B \subseteq X$. Then:*

(1) $i_s\emptyset = \emptyset$ and $c_s X = X$.

(2) $i_s A \subseteq A$ and $A \subseteq c_s A$.

(3) If $A \subseteq B$, then $i_s A \subseteq i_s B$ and $c_s A \subseteq c_s B$.

(4) $i_s i_s A = i_s A$ *and* $c_s c_s A = c_s A$.

(5) $c_s(A) = X - i_s(X - A)$ *and* $i_s(A) = X - c_s(X - A)$.

(6) A *is* σ*-open iff* $A = i_s A$ *for* $A \neq \emptyset$; A *is* σ*-closed iff* $A = c_s A$ *for* $A \neq X$.

Theorem 2.2 ([2]). *Let* s *be a* σ*-structure on a nonempty set* X *and* $A \subseteq X$. *Then:*

(1) $x \in i_s A$ *iff there exists a* σ*-open set* S *containing* x *such that* $S \subseteq A$.

(2) $x \in c_s A$ *iff* $S \cap A \neq \emptyset$ *for every* σ*-open set* S *containing* x.

Let s, s' be σ-structures on X and Y, respectively. Then a function $f : X \to Y$ is said to be *σ-continuous* [2] if $f^{-1}(G) \in s$ for every $G \in s'$. Then f is σ-continuous iff $f^{-1}(i_s A) \subseteq i_s f^{-1}(A)$ for every $A \subseteq Y$.

3. Almost σ-Continuity

First, we introduce the notion of almost σ-continuous functions on σ-structures and investigate characterizations of almost σ-continuity by using the two operators i_s and c_s. From now on, every σ-structure is strong unless otherwise stated.

Definition 3.1. *Let* s, s' *be* σ*-structures on* X *and* Y*, respectively. Then* $f : X \to Y$ *is said to be almost* σ*-continuous at* $x \in X$ *if for each* σ*-open subset* V *containing* $f(x)$*, there is a* σ*-open set* U *containing* x *such that* $f(U) \subseteq i_s(c_s(V))$. *A function* $f : (X, \tau) \to (Y, \mu)$ *is said to be almost* σ*-continuous if it has the property at each point of* X.

Theorem 3.2. *Let* $f : X \to Y$ *be a function between* σ*-structures on* X *and* Y. *Then the following statements are equivalent:*

(1) f *is almost* σ*-continuous.*

(2) $f^{-1}(V) \subseteq i_s(f^{-1}(i_s(c_s(V))))$ *for every* σ*-open subset* V *of* Y.

(3) $c_s(f^{-1}(c_s(i_s(F)))) \subseteq f^{-1}(F)$ *for every* σ*-closed set* F *of* Y.

(4) $c_s(f^{-1}(c_s(i_s(c_s(B))))) \subseteq f^{-1}(c_s(B))$ *for every set* B *of* Y. *(5)* $f^{-1}(i_s(B)) \subseteq i_s(f^{-1}(i_s(c_s(i_s(B)))))$ *for every set* B *of* Y.

Proof. (1) $\Rightarrow$ (2) Let V be a σ-open set in Y and $x \in f^{-1}(V)$. There exists a σ-open set U of X containing x such that $f(U) \subseteq i_s(c_s(V))$. Since $x \in U \subseteq f^{-1}(i_s(c_s(V)))$, by definition of i_s, $x \in i_s(f^{-1}(i_s(c_s(V))))$. Hence $f^{-1}(V) \subseteq i_s(f^{-1}(i_s(c_s(V))))$.

(2) $\Rightarrow$ (3) It is obvious from Theorem 2.1.

(3) $\Rightarrow$ (4) Let B be a subset of Y. Since $c_s(B)$ is σ-closed in Y, it follows from (3).

(4) $\Rightarrow$ (5) It is obvious from Theorem 2.1.

(5) $\Rightarrow$ (1) For each $x \in X$, let V be any σ-open set of Y containing $f(x)$. Then by hypothesis, $x \in f^{-1}(V) \subseteq i_s(f^{-1}(i_s(c_s(V)))$. Since $x \in i_s(f^{-1}(i_s(c_s(V)))$, from Theorem 2.2, there exists a σ-set U containing x such that $U \subseteq f^{-1}(i_s(c_s(V)))$. It implies $f(U) \subseteq i_s(c_s(V))$. Hence f is almost σ-continuous at x.

□

Remark 3.3. First, we recall the notion of weak σ-continuous functions introduced in [7]: Let s, s' be σ-structures on X and Y, respectively. Then a function $f : X \to Y$ is said to be *weak σ-continuous* if for each $x \in X$ and each σ-open set V containing $f(x)$, there exists a σ-open set U containing x such that $f(U) \subseteq c_s(V)$. Then

$$\sigma\text{-continuous} \Rightarrow \text{almost } \sigma\text{-continuous} \Rightarrow \text{weakly } \sigma\text{-continuous}$$

The converses are not true in general as the next example.

Example 3.4. Let $X = \{a, b, c, d\}$:

(1) Consider a σ-structure $\sigma = \{\{a, b\}, \{a, b, c\}, X\}$ on X and a function $f : (X, \sigma) \to (X, \sigma)$ defined as follows: $f(a) = f(c) = f(d) = d$ and $f(b) = b$. Then f is almost γ-continuous but not γ-continuous.

(2) Consider a σ-structure $\sigma = \{\{b, d\}, \{a, b\}, \{a, b, d\}\}$ on X and a function $f : (X, \sigma) \to (X, \sigma)$ defined as follows: $f(c) = f(b) = c$ and $f(a) = f(d) = d$. Then since $c_s(\{b, d\} = c_s(\{a, b\} = c_s(\{a, b, d\}) = X$, obviously f is weakly σ-continuous. Note $f^{-1}(\{b, d\}) = \{a, d\}$ and $i_s(f^{-1}(i_s(c_s(\{b, d\})))) = i_s(f^{-1}(\{a, b, d\})) = i_s(\{a, d\}) = \emptyset$. Hence f is not almost σ-continuous.

Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then A is said to be *σ-regular open* if $A = i_s(c_s(A))$. The complement of σ-regular open set is called a *σ-regular closed set*.

Lemma 3.5 ([7]). *Let s be a σ-structure on a nonempty set X and $A \subseteq X$. If A is σ-closed set, then $i_s(A)$ is σ-regular open.*

Theorem 3.6. *Let $f : X \to Y$ be a function between σ-structures on X and Y. Then the following statements are equivalent:*

(1) f is almost σ-continuous.

(2) $f^{-1}(V) = i_s(f^{-1}(V))$ for every σ-regular open subset V of Y.

(3) $f^{-1}(F) = c_s(f^{-1}(F))$ for every σ-regular closed set F of Y.

Proof. (1) $\Rightarrow$ (2) Let V be any σ-regular open subset of Y. By (5) of Theorem 3.2, $f^{-1}(V) \subseteq i_s(f^{-1}(V))$. Hence $f^{-1}(V) = i_s(f^{-1}(V))$.

(2) $\Rightarrow$ (1) For $B \subseteq Y$, $c_s(i_s(B))$ is σ-closed and by Lemma 3.5, $i_s(c_s(i_s(B)))$ is σ-regular open. So from $i_s(B) \subseteq i_s(c_s(i_s(B)))$ and the condition (2), it follows $f^{-1}(i_s(B)) \subseteq f^{-1}(i_s(c_s(i_s(B)))) = i_s(f^{-1}(i_s(c_s(i_s(B)))))$. By (5) of Theorem 3.2, f is almost σ-continuous.

(2) $\Leftrightarrow$ (3) Obvious.

□

Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then A is said to be

(1) *σ-semiopen* [3] if $A \subseteq c_s(i_s(A))$,

(2) *σ-preopen* [4] if $A \subseteq i_s(c_s(A))$,

(3) *σ-β-open* [5] if $A \subseteq c_s(i_s(c_s(A)))$.

Theorem 3.7. *Let $f : X \to Y$ be a function between σ-structures on X and Y. Then the following statements are equivalent:*

(1) f is almost σ-continuous.

(2) $c_s(f^{-1}(G)) \subseteq f^{-1}(c_s(G))$ for every σ-β-open set G of Y.

(3) $c_s(f^{-1}(G)) \subseteq f^{-1}(c_s(G))$ for every σ-semiopen set G of Y.

Proof. (1) $\Rightarrow$ (2) Let G be any σ-β-open set. Then $c_s(G) = c_s(i_s(c_s(G)))$, and so $c_s(G)$ is σ-regular closed. From (3) of Theorem 3.6, it follows $c_s(f^{-1}(G)) \subseteq c_s(f^{-1}(c_s(G))) = f^{-1}(c_s(G))$. Hence $c_s(f^{-1}(G)) \subseteq f^{-1}(c_s(G))$.

(2) $\Rightarrow$ (3) Since any σ-semiopen set is σ-β-open, it is obvious.

(3) $\Rightarrow$ (1) Let F be any σ-regular closed set of Y. Since F is also σ-semiopen, by (3), $c_s(f^{-1}(F)) \subseteq f^{-1}(c_s(F)) = f^{-1}(F)$. This implies that $c_s(f^{-1}(F)) = f^{-1}(F)$ and hence by (3) of Theorem 3.6, f is almost σ-continuous.

□

Theorem 3.8. *Let $f : X \to Y$ be a function between σ-structures on X and Y. Then the following statements are equivalent:*

(1) f is almost σ-continuous.

(2) $f^{-1}(G) \subseteq i_s(f^{-1}(i_s(c_s(G))))$ for every σ-preopen set G of Y.

Proof. (1) $\Rightarrow$ (2) Let G be any σ-preopen set of Y. Then since $i_s(c_s(G))$ is σ-regular open in Y, from (2) of Theorem 3.6, it follows $f^{-1}(G) \subseteq f^{-1}(i_s(c_s(G))) = i_s(f^{-1}(i_s(c_s(G))))$. Hence $f^{-1}(G) \subseteq i_s(f^{-1}(i_s(c_s(G))))$.

(2) $\Rightarrow$ (1) Let V be any σ-regular open set in Y. Then since V is σ-preopen, from (2), it follows that $f^{-1}(V) \subseteq i_s(f^{-1}(i_s(c_s(V))) = i_s(f^{-1}(V))$. Thus $f^{-1}(V) = i_s(f^{-1}(V))$ and by (2) of Theorem 3.6, f is almost σ-continuous.

□

Definition 3.9 ([6]). *Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then a subset A in a σ-structure σ is said to be δ-σ-open if for each $x \in A$, there exists a σ-regular open set G such that $x \in G \subseteq A$. The collection of all σ-δ-open sets is denoted by δ_σ*

Theorem 3.10 ([6]). *Let s be a σ-structure on a nonempty set X. Then the non-empty elements of δ_σ coincide with the unions of σ-regular open sets.*

Theorem 3.11 ([6]). *Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then*

(1) $x \in i_\delta A$ iff there exists a σ-regular open set S containing x such that $S \subseteq A$. (2) $x \in c_\delta A$ iff $S \cap A \neq \emptyset$ for every σ-regular open set S containing x.

Theorem 3.12 ([6]). *Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then*

(1) $i_\delta \emptyset = \emptyset$ and $c_\delta X = X$.

(2) $i_\delta A \subseteq A$ and $A \subseteq c_\delta A$.

(3) If $A \subseteq B$, then $i_\delta A \subseteq i_\delta B$ and $c_\delta A \subseteq c_\delta B$.

(4) $i_\delta i_\delta A = i_\delta A$ and $c_\delta c_\delta A = c_\delta A$.

(5) $c_\delta(A) = X - i_\delta(X - A)$ and $i_\delta(A) = X - c_\delta(X - A)$.

(7) A is σ-δ-open iff $A = i_\delta(A)$.

(8) A is σ-δ-closed iff $A = c_\delta(A)$.

Theorem 3.13. *Let $f : X \to Y$ be a function between σ-structures on X and Y. Then the following statements are equivalent:*

(1) f is almost σ-continuous.

(2) $c_s(f^{-1}(c_s(i_s(c_\delta(B))))) \subseteq f^{-1}(c_\delta(B))$ for every set B of Y.

(3) $c_s(f^{-1}(c_s(i_s(c_s(B))))) \subseteq f^{-1}(c_\delta(B))$ for every set B of Y.

(4) $c_s(f^{-1}(c_s(i_s(c_s(G))))) \subseteq f^{-1}(c_s(G))$ for every σ-open set G of Y.

(5) $c_s(f^{-1}(c_s(i_s(c_s(G))))) \subseteq f^{-1}(c_s(G))$ for every σ-preopen set G of Y.

Proof. (1) $\Rightarrow$ (2) Let B be any subset in Y; then $c_\delta(B)$ is σ-closed, by (3) of Theorem 3.2, it is obtained.

(2) $\Rightarrow$ (3) For any subset B of Y, $c_s(B) \subseteq c_\delta(B)$ and by (2), it is obtained.

(3) $\Rightarrow$ (4) For every σ-open subset G of Y, $c_s(G)$ is σ-regular closed. By Theorem 3.10, $c_s(G)$ is σ-δ-closed and so $c_s(G) = c_\delta(c_s(G))$ by (8) of Theorem 3.12. Hence (4) is obtained.

(4) $\Rightarrow$ (5) Let G be any σ-preopen subset of Y. Then $c_s(G) = c_s(i_s(c_s(G)))$. Set $A = i_s(c_s(G))$. Since A is σ-open, by (4), $c_s(f^{-1}(c_s(i_s(c_s(A))))) \subseteq f^{-1}(c_s(A))$. From $c_s(A) = c_s(G)$, it follows $c_s(f^{-1}(c_s(i_s(c_s(G))))) \subseteq f^{-1}(c_s(G))$.

(5) $\Rightarrow$ (1) Let F be any σ-regular closed subset of Y. Set $G = i_s(F)$. Then $G = i_s(F)$ is σ-preopen and from (5), $c_s(f^{-1}(c_s(i_s(c_s(G))))) \subseteq f^{-1}(c_s(G))$. Since $c_s(G) = F$ and $c_s(i_s(c_s(G))) = A$, $c_s(f^{-1}(F)) \subseteq f^{-1}(F)$, and by Theorem 3.6, f is almost σ-continuous.

□

Theorem 3.14. *Let $f : X \to Y$ be a function between σ-structures on X and Y. Then the following statements are equivalent:*

(1) f is almost σ-continuous.

(2) $f(c_s(B)) \subseteq c_\delta(f(B))$ for every set B of X.

(3) $f^{-1}(F) = c_s(f^{-1}(F))$ *for every* σ*-*δ*-closed set* F *of* Y*.*

(4) $f^{-1}(G) = i_s(f^{-1}(G))$ *for every* σ*-*δ*-open set* G *of* Y*.*

(5) $f^{-1}(i_\delta(B)) \subseteq i_s(f^{-1}(B))$ *for every set* B *of* Y*.*

(6) $c_s(f^{-1}(B)) \subseteq f^{-1}(c_\delta(B))$ *for every set* B *of* Y*.*

Proof. (1) $\Rightarrow$ (2) Let B be any subset in Y. Let $x \in c_s(B)$ and V any σ-open set of Y containing $f(x)$. By almost σ-continuity, there exists a σ-set U containing x such that $f(U) \subseteq i_s(c_s(V))$. Since $x \in c_s(B)$, $B \cap U \neq \emptyset$ and so $\emptyset \neq f(U) \cap f(B) \subseteq i_s(c_s(V)) \cap f(B)$. Since $i_s(c_s(V))$ is any σ-regular open set, by Theorem 3.11, $f(x) \in c_\delta(f(B))$, and so we have $f(c_s(B)) \subseteq c_\delta(f(B))$.

(2) $\Rightarrow$ (3) Let F be any σ-c_δ-closed set of Y. Then from (2) and Theorem 3.12, it follows $f(c_s(f^{-1}(F))) \subseteq c_\delta(f(f^{-1}(F))) \subseteq c_\delta(F) = F$. Hence $f^{-1}(F) = c_s(f^{-1}(F))$.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (5) Let B be any subset of Y. Then since $i_\delta(B)$ is a σ-δ-open set of Y, from (4), $f^{-1}(i_\delta(B)) = i_s(f^{-1}(i_\delta(B))) \subseteq i_s(f^{-1}(B))$. Hence $f^{-1}(i_\delta(B)) \subseteq i_s(f^{-1}(B))$.

(5) $\Rightarrow$ (6) It follows from (5) of Theorem 3.12.

(6) $\Rightarrow$ (1) Let B be any subset of Y. Then by (6), $c_s(f^{-1}(c_s(i_s(c_\delta(B))))) \subseteq f^{-1}(c_\delta(c_s(i_s(c_\delta(B)))))$. Since $c_s(i_s(c_\delta(B)))$ is a σ-regular closed set, it is σ-δ-closed and $c_\delta(c_s(i_s(c_\delta(B)))) = c_s(i_s(c_\delta(B)))$. Finally, from $c_\delta(B)$ is σ-closed, it follows $c_s(f^{-1}(c_s(i_s(c_\delta(B))))) \subseteq f^{-1}(c_\delta(c_s(i_s(c_\delta(B))))) = f^{-1}(c_s(i_s(c_\delta(B)))) \subseteq f^{-1}(c_\delta(B))$. Hence by (2) of Theorem 3.13, f is almost σ-continuous. □

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