THE GENERATORS OF THE 2-CLASS GROUP OF SOME FIELDS \( \mathbb{Q}(\sqrt{pq_1q_2}, i) \):
CORRECTION TO THEOREM 3 OF [5]

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Abstract: Let \( p, q_1 \) and \( q_2 \) be different primes satisfying the condition that the 2-class group of the field \( \mathbb{k} = \mathbb{Q}(\sqrt{pq_1q_2}, i) \) is of type \((2, 2, 2)\). In this paper, we are interested to give the generators of \( C_{\mathbb{k}, 2} \), the 2-class group of \( \mathbb{k} \), which corrects the Theorem 3 of A. Azizi, A. Zekhnini and M. Taous: On the generators of the 2-class group of the field \( \mathbb{k} = \mathbb{Q}(\sqrt{d}, i) \), IJPAM, Volume 81, No. 5 (2012), 773-784.

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1. Introduction

Let $k$ be an algebraic number field and let $C_{k,2}$ denote its 2-class group, that is the 2-Sylow subgroup of the ideal class group $C_k$ of $k$. The structure and the generators of $C_{k,2}$ play an important role in Number Theory, in fact they can help to determine the structure and the generators of the maximal unramified pro-2 extension of $k$, and they also help to solve the capitulation problem of the 2-ideal classes of $k$ in its unramified extensions see [6, 7, 8, 9, 10, 11, 12, 16, 17].

Let $k = \mathbb{Q}(\sqrt{d}, i)$, where $d$ is a square-free integer. In [5], we determined the generators of $C_{k,2}$ whenever it is of type (2, 2, 2), but Theorem 3 is false, the purpose of this paper is to correct this Theorem.

2. Preliminaries

Let $p$, $q_1$ and $q_2$ be different primes such that $p \equiv 1 \pmod{4}$ and $q_1 \equiv q_2 \equiv 3 \pmod{4}$. Put $d = pq_1q_2$, according to [13], $C_{k,2}$ is of type (2, 2, 2) if and only if $p$, $q_1$, $q_2$ satisfy the following two conditions:

\begin{align*}
A: & \quad p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4} \quad \text{and} \quad \left(\frac{2}{p}\right) = \left(\frac{q_1}{q_2}\right) = -\left(\frac{q_2}{q_1}\right) = 1. \\
B: & \quad \text{One of the following three conditions is satisfied:} \\
& \quad \text{(I)} \quad \left(\frac{p}{q_1}\right) \left(\frac{p}{q_2}\right) = -1 \quad \text{and} \quad \left(\frac{2}{q_1}\right) = \left(\frac{2}{q_2}\right) = -1. \\
& \quad \text{(II)} \quad \left(\frac{p}{q_1}\right) \left(\frac{p}{q_2}\right) = -1, \quad \left(\frac{2}{q_1}\right) = 1 \quad \text{and} \quad \left(\frac{2}{q_2}\right) = -1. \\
& \quad \text{(III)} \quad \left(\frac{p}{q_1}\right) = \left(\frac{p}{q_2}\right) = -1 \quad \text{and} \quad \left(\frac{2}{q_1}\right) \left(\frac{2}{q_2}\right) = -1. 
\end{align*}

**Definition 1.** Let $p$, $q_1$ and $q_2$ be different primes satisfying the condition $A$ above.

1. $p$, $q_1$ and $q_2$ are called of type $B(I)(1)$ if the following conditions hold: $\left(\frac{p}{q_1}\right) = -\left(\frac{p}{q_2}\right) = 1$ and $\left(\frac{2}{q_1}\right) = \left(\frac{2}{q_2}\right) = -1$.

2. $p$, $q_1$ and $q_2$ are called of type $B(I)(2)$ if the following conditions hold: $\left(\frac{p}{q_2}\right) = -\left(\frac{p}{q_1}\right) = 1$ and $\left(\frac{2}{q_1}\right) = \left(\frac{2}{q_2}\right) = -1$.

3. $p$, $q_1$ and $q_2$ are called of type $B(II)(1)$ if the following conditions hold: $\left(\frac{p}{q_1}\right) = -\left(\frac{p}{q_2}\right) = 1$ and $\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = 1$. 


(4) \( p, q_1 \) and \( q_2 \) are called of type \( B(II)(2) \) if the following conditions hold:
\[
\left( \frac{p}{q_2} \right) = -\left( \frac{p}{q_1} \right) = 1 \quad \text{and} \quad \left( \frac{2}{q_1} \right) = -\left( \frac{2}{q_2} \right) = 1.
\]

Lemma 2. Let \( d \equiv 1 \pmod{4} \) be a positive square free integer and \( \varepsilon_d = x + y\sqrt{d} \) be the fundamental unit of \( \mathbb{Q}(\sqrt{d}) \). Assume \( N(\varepsilon_d) = 1 \), then

1. \( x + 1 \) and \( x - 1 \) are not squares in \( \mathbb{N} \) i.e. \( 2\varepsilon_d \) is not a square in \( \mathbb{Q}(\sqrt{d}) \).

2. For all prime \( p \) dividing \( d \), \( p(x + 1) \) and \( p(x - 1) \) are not squares in \( \mathbb{N} \).

Proof. 1. As \( d \equiv 1 \pmod{4} \), then by [13, Corollaire 3.2] the unit index of \( \mathbb{Q}(\sqrt{d}, i) \) is equal to 1, hence by [1, Applications (ii)] we get that \( 2\varepsilon_d \) is not a square in \( \mathbb{Q}(\sqrt{d}) \), this is equivalent to \( x + 1 \) and \( x - 1 \) are not squares in \( \mathbb{N} \).

2. Assume \( p(x + 1) \) or \( p(x - 1) \) is a square in \( \mathbb{N} \), then, by the decomposition uniqueness in \( \mathbb{Z} \), there exist \( y_1, y_2 \) in \( \mathbb{Z} \) such that
\[
\begin{align*}
\left\{ \begin{array}{l}
y = y_1y_2, \\
d = pd^\prime;
\end{array} \right.
\end{align*}
\]

such that \( p(x + 1) = p^2y_1^2 \) and \( p(x - 1) = p^2y_2^2 + 2p \). This in turn yields that \( p^2(x^2 - 1) = p^2y_1^2(p^2y_2^2 + 2p) \); as \( x^2 - 1 = y^2d \), so we get \( y^2d = y_1^2(p^2y_2^2 + 2p) \), and \( y_2d = p^2y_2^2 + 2p \). Since \( d \equiv 1 \pmod{4} \) and \( p \equiv \pm 1 \pmod{4} \), we deduce that \( \mp 2 \equiv y_1^2 - y_2^2 \pmod{4} \). On the other hand, we know that for all \( a \in \mathbb{Z} \), \( a^2 \equiv 0 \) or \( 1 \pmod{4} \), thus \( \mp 2 \equiv 0, 1 \) or \(-1 \pmod{4} \). Which is absurd. \( \square \)

Let \( p, q_1 \) and \( q_2 \) be different primes satisfying the condition \( A \) above. As the norm of \( \varepsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2} \), the fundamental unit of \( \mathbb{Q}(\sqrt{pq_1q_2}) \), is 1, then by the decomposition uniqueness in \( \mathbb{Z} \), each of the numbers \( x \pm 1, 2(x \pm 1), p(x \pm 1), q_1(x \pm 1), q_2(x \pm 1), 2p(x \pm 1), 2q_1(x \pm 1), 2q_2(x \pm 1) \) and \( 2pq_1q_2(x \pm 1) \) can be a square in \( \mathbb{N} \).

Lemma 3. Let \( p, q_1 \) and \( q_2 \) be different primes satisfying the condition \( A \). Let \( \varepsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2} \) be the fundamental unit of \( \mathbb{Q}(\sqrt{pq_1q_2}) \).

1. If \( p, q_1 \) and \( q_2 \) are of type \( B(I)(1) \) or \( B(II)(1) \), then only \( 2q_1(x + 1) \) (i.e. \( 2pq_1(x - 1) \)) is a square in \( \mathbb{N} \).

2. If \( p, q_1 \) and \( q_2 \) are of type \( B(I)(2) \) or \( B(II)(2) \), then only \( 2q_2(x - 1) \) (i.e. \( 2pq_1(x + 1) \)) is a square in \( \mathbb{N} \).

3. If \( p, q_1 \) and \( q_2 \) are of type \( B(III) \), then only \( 2p(x - 1) \) (i.e. \( 2q_1q_2(x + 1) \)) is a square in \( \mathbb{N} \).
Proof. As \( pq_1q_2 \equiv 1 \, (\mod 4) \) and \( N(\varepsilon_{pq_1q_2}) = 1 \), then Lemma 2 implies that \( x \pm 1, p(x \pm 1), q_1(x \pm 1) \) and \( q_2(x \pm 1) \) are not squares in \( \mathbb{N} \). On the other hand, Lemma 5 of [2] yields that \( 2(x \pm 1) \) and \( 2pq_1q_2(x \pm 1) \) are not squares in \( \mathbb{N} \). Hence only \( 2p(x \pm 1), 2q_1(x \pm 1) \) and \( 2q_2(x \pm 1) \) can be squares in \( \mathbb{N} \).

Suppose \( p, q_1 \) and \( q_2 \) are of type \( B(1) \). If \( 2p(x \pm 1) \) is a square in \( \mathbb{N} \), then, by the decomposition uniqueness of \( x \pm 1 \) in \( \mathbb{Z} \), there exist \( y_1, y_2 \) in \( \mathbb{Z} \) such that

\[
\begin{aligned}
x \pm 1 &= 2py_1^2, \\
y &= 2y_1y_2;
\end{aligned}
\]

from which we deduce that \( \left( \frac{q_1q_2}{p} \right) = 1 \), but this contradicts the fact that \( \left( \frac{q_1q_2}{p} \right) = -1 \).

Similarly, if we assume \( 2q_2(x \pm 1) \) is a square in \( \mathbb{N} \), we get

\[
\begin{aligned}
x \pm 1 &= 2q_2y_1^2, \\
y &= 2y_1y_2;
\end{aligned}
\]

which implies that \( \left( \frac{q_2}{p} \right) = \left( \frac{2}{p} \right) \), hence \( \left( \frac{q_2}{p} \right) = 1 \), which is absurd.

Finally, if \( 2q_1(x - 1) \) is a square in \( \mathbb{N} \), then proceeding similarly we get \( \left( \frac{q_1}{q_2} \right) = -1 \), which is absurd. So the result.

The other cases are proved similarly.

We close this section by the following lemmas.

**Lemma 4** ([19]). Let \( p_1, p_2, \ldots, p_n \) be distinct primes and for each \( j \), let \( e_j = \pm 1 \). Then there exist infinitely many primes \( l \) such that \( \left( \frac{p_j}{l} \right) = e_j \), for all \( j \).

**Lemma 5** ([18], p. 205). If \( \mathcal{H} \) is an unramified ideal in some extension \( \mathbb{K}/k \) = \( k(\sqrt{x})/k \), then the quadratic residue symbol is given by the Artin symbol \( \varphi = \left( \frac{\mathbb{K}(\sqrt{x})/k}{\mathcal{H}} \right) \) as follows: \( \left( \frac{x}{\mathcal{H}} \right) = \sqrt{x}^{\varphi - 1} \).

3. Main Result

Let \( F = \mathbb{Q}(i) \) and denote by \( Am(k/F) \) the group of the ambiguous classes of \( k/F \) and by \( Am_s(k/F) \) its subgroup generated by the strongly ambiguous classes. As \( p \equiv 1 \, (\mod 4) \), so there exist \( e \) and \( f \) in \( \mathbb{N} \) such that \( p = e^2 + 4f^2 = \pi_1\pi_2 \). Put \( \pi_1 = e + 2if \) and \( \pi_2 = e - 2if \). Let \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{Q}_1 \) and \( \mathcal{Q}_2 \) be the prime ideals of \( k \) above \( \pi_1, \pi_2, q_1 \) and \( q_2 \) respectively. It is easy to see that \( \mathcal{H}_j^2 = (\pi_j) \)
Thus Formula (3) allows us to deduce that the following formula (see for example [15]):

\[ \text{If } q \in \mathbb{Q}, \text{ then } \mathcal{H}_j = (\pi_j), \text{ where } 1 \leq j \leq 2, \text{ and since also } \sqrt{e^2 + (2f)^2} = \sqrt{p} \notin \mathbb{Q}(\sqrt{pq_1q_2}), \text{ so according to [5, Proposition 1]} \mathcal{H}_j \text{ is not principal in } \mathbb{k}.

From Lemma 3 we get the following assertions:

• If \( p, q_1 \) and \( q_2 \) are of type \( B(I)(1) \) or \( B(II)(1) \), then \( 2q_1(x+1) \) and \( 2pq_2(x-1) \) are squares in \( \mathbb{N} \), so Remark 1 of [5] implies that \( \mathcal{Q}_1 \) and \( \mathcal{H}_1\mathcal{H}_2\mathcal{Q}_2 \) are principal in \( \mathbb{k} \).

• If \( p, q_1 \) and \( q_2 \) are of type \( B(I)(2) \) or \( B(II)(2) \), then \( 2q_2(x-1) \) and \( 2pq_1(x+1) \) are squares in \( \mathbb{N} \), so Remark 1 of [5] implies that \( \mathcal{Q}_2 \) and \( \mathcal{H}_1\mathcal{H}_2\mathcal{Q}_1 \) are principal in \( \mathbb{k} \).

• If \( p, q_1 \) and \( q_2 \) are of type \( B(III) \), then \( 2p(x-1) \) and \( 2q_1q_2(x+1) \) are squares in \( \mathbb{N} \), so Remark 1 of [5] implies that \( \mathcal{H}_1\mathcal{H}_2 \) and \( \mathcal{Q}_1\mathcal{Q}_2 \) are squares in \( \mathbb{k} \) and \( \mathcal{Q}_1, \mathcal{Q}_2 \) are not. Moreover, as \( (\mathcal{H}_1\mathcal{Q}_1)^2 = (\pi_1q_1) \) and \( q_1\sqrt{p} \notin \mathbb{Q}(\sqrt{d}) \), so [5, Proposition 1] implies that \( \mathcal{H}_1\mathcal{Q}_1 \) is not principal in \( \mathbb{k} \).

According to the ambiguous class number formula (see [14]) we have:

\[
|\text{Am}(\mathbb{k}/F)| = \frac{h(F)2^{t-1}}{|E_F : E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^x)|},
\]

(1)

where \( h(F) \) is the class number of \( F \) and \( t \) is the number of finite and infinite primes of \( F \) ramified in \( \mathbb{k}/F \). Moreover as the class number of \( F \) is equal to 1, so the formula (1) yields that

\[
|\text{Am}(\mathbb{k}/F)| = 2^r,
\]

(2)

where \( r = \text{rank}_{\mathbb{C}}(\mathbb{k},2) = t - e - 1 \) and \( 2^e = |E_F : E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^x)| \) (see for example [20]). The relation between \( |\text{Am}(\mathbb{k}/F)| \) and \( |\text{Am}_s(\mathbb{k}/F)| \) is given by the following formula (see for example [15]):

\[
\frac{|\text{Am}(\mathbb{k}/F)|}{|\text{Am}_s(\mathbb{k}/F)|} = [E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^x) : N_{\mathbb{k}/F}(E_{\mathbb{k}})].
\]

(3)

Since \( r = \text{rank}_{\mathbb{C}}(\mathbb{k},2) = 3 \), so Formula (2) implies that \( |\text{Am}(\mathbb{k}/\mathbb{Q}(i))| = 2^3 = 8 \). Moreover, we know that \( p \equiv 1 \pmod{8} \), hence by [21] \( i \) is a norm in \( \mathbb{k}/\mathbb{Q}(i) \), thus Formula (3) allows us to deduce that

\[
\frac{|\text{Am}(\mathbb{k}/\mathbb{Q}(i))|}{|\text{Am}_f(\mathbb{k}/\mathbb{Q}(i))|} = [E_{\mathbb{Q}(i)} \cap N_{\mathbb{k}/\mathbb{Q}(i)}(\mathbb{k}^x) : N_{\mathbb{k}/\mathbb{Q}(i)}(E_{\mathbb{k}})] = [< i > : < -1 >] = 2
\]

\((E_{\mathbb{k}} = \langle i, \varepsilon_{pq_1q_2} \rangle \text{ since } x \pm 1 \text{ is not a square in } \mathbb{N} \text{ see [3]), so } |\text{Am}_s(\mathbb{k}/\mathbb{Q}(i))| = 4 \).

We conclude that:
• If \( p, q_1 \) and \( q_2 \) are of type \( B(III) \), then \( \text{Am}_s(\mathbb{K}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{Q}_1] \rangle \).

• Else, \( \text{Am}_s(\mathbb{K}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle \).

Therefore there exists in \( \mathbb{K} \) an unambiguous ideal \( \mathcal{I} \) of order 2 such that

\[
\begin{cases}
\mathcal{C}l_2(\mathbb{K}) = \langle [\mathcal{H}_1], [\mathcal{Q}_1], [\mathcal{I}] \rangle & \text{if } p, q_1 \text{ and } q_2 \text{ are of type } B(III), \\
\mathcal{C}l_2(\mathbb{K}) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{I}] \rangle & \text{otherwise}.
\end{cases}
\]

By Chebotarev theorem, \( \mathcal{I} \) can always be chosen as a prime ideal of \( \mathbb{K} \) above a prime \( l \) in \( \mathbb{Q} \), which splits completely in \( \mathbb{K} \). So we can determine \( \mathcal{I} \) by using Lemma 4.

(a) Suppose \( p, q_1 \) and \( q_2 \) are of type \( B(III) \). Let \( l \equiv 1 \pmod{4} \) be a prime integer such that \( \left( \frac{pq_1q_2}{l} \right) = 1 \) and \( \left( \frac{l}{2} \right) = -1 \). Thus \( l \) splits completely in \( \mathbb{K} \), let \( \mathcal{I} \) be a prime ideal of \( \mathbb{K} \) lies above \( l \), which is an unambiguous ideal. Let us prove that \( \mathcal{I}, \mathcal{Q}_1\mathcal{I}, \mathcal{H}_1\mathcal{I} \) and \( \mathcal{Q}_1\mathcal{H}_1\mathcal{I} \) are not principal in \( \mathbb{K} \).

• \( \mathcal{I} \) is not principal in \( \mathbb{K} \), otherwise we will have \( N_{\mathbb{K}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{I}) \) principal in \( \mathbb{Q}(\sqrt{pq_1q_2}) \), so there exist \( \alpha_1, \alpha_2 \) in \( \mathbb{Q} \) such that \( l = \pm(\alpha_1^2 - \alpha_2^2pq_1q_2) \). This implies that \( \left( \frac{l}{2} \right) = 1 \); which is absurd.

• If \( \mathcal{Q}_1\mathcal{I} \) is principal in \( \mathbb{K} \), then \( N_{\mathbb{K}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{Q}_1\mathcal{I}) = \mathbb{Q}^2N_{\mathbb{K}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{I}) \) is also principal in \( \mathbb{Q}(\sqrt{pq_1q_2}) \) (note that the ideal of \( \mathbb{Q}(\sqrt{pq_1q_2}) \) above \( q_1 \) remains inert in \( \mathbb{K} \)). On the other hand, under our conditions \( \mathbb{Q}(\sqrt{pq_1q_2}) \) is cyclic of order 2 (see [13]), thus \( N_{\mathbb{K}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{I}) \) is principal in \( \mathbb{Q}(\sqrt{pq_1q_2}) \), this implies the contradiction \( \left( \frac{l}{2} \right) = 1 \).

• If \( \mathcal{H}_1\mathcal{I} \) is principal in \( \mathbb{K} \), then \( N_{\mathbb{K}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{H}_1\mathcal{I}) \) is principal in \( \mathbb{Q}(\sqrt{pq_1q_2}) \). So there exist \( \alpha_1, \alpha_2 \) in \( \mathbb{Q} \) such that \( pl = \pm(\alpha_1^2 - \alpha_2^2pq_1q_2) \); which yields that \( p \) divides \( \alpha_1 \), hence there exists \( \beta \) in \( \mathbb{Q} \) such that \( l = \pm(\beta^2p - \alpha_2^2q_1q_2) \); this in turn implies that \( \left( \frac{l}{p} \right) = \left( \frac{pq_2}{p} \right) = -1 \); but this contradicts the condition \( B(III) \).

• If \( \mathcal{Q}_1\mathcal{H}_1\mathcal{I} \) is principal in \( \mathbb{K} \), then

\[
N_{\mathbb{K}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{Q}_1\mathcal{H}_1\mathcal{I}) = \mathbb{Q}^2N_{\mathbb{K}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{Q}_1\mathcal{H}_1\mathcal{I})
\]

is principal in \( \mathbb{Q}(\sqrt{pq_1q_2}) \). So \( N_{\mathbb{K}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{Q}_1\mathcal{H}_1\mathcal{I}) \) is principal in \( \mathbb{Q}(\sqrt{pq_1q_2}) \); which implies that \( \left( \frac{l}{p} \right) = \left( \frac{pq_2}{p} \right) = -1 \), but this contradicts the condition \( B(III) \).

(b) Suppose \( p, q_1 \) and \( q_2 \) are of type \( B(I) \) or \( B(II) \). As in the assertion (a), we choose a prime \( l \equiv 1 \pmod{4} \) satisfying the conditions \( \left( \frac{pq_1q_2}{l} \right) = 1 \), \( \left( \frac{l}{2} \right) = -1 \) and \( \left( \frac{pq_2}{p} \right) \left( \frac{q_2}{p} \right) = -1 \). Thus \( l \) splits completely in \( \mathbb{K} \). Let \( \mathcal{I} \) be a prime ideal of \( \mathbb{K} \) above \( l \), so \( \mathcal{I} \) is unambiguous ideal and the ideals \( \mathcal{I}, \mathcal{H}_1\mathcal{I}, \mathcal{H}_2\mathcal{I} \) and \( \mathcal{H}_1\mathcal{H}_2\mathcal{I} \) are not principal in \( \mathbb{K} \). Indeed:

• \( \mathcal{I} \) is not principal in \( \mathbb{K} \), the same proof as in (a).
Note that $K_2 = \mathbb{Q}(\sqrt{q_1}, \sqrt{pq_2}, i)$ is an unramified quadratic extension of $k$ (see [4]), hence Lemma 5 implies that $\varphi_{K_2/k}(\mathcal{H}_j I) \neq 1$, where $j \in \{1, 2\}$ and $\varphi_{K_2/k}$ is the d’Artin map of $K_2/k$. Consequently $\mathcal{H}_j I$ is not principal in $k$.

If $\mathcal{H}_1 \mathcal{H}_2 I$ is principal in $k$, then $N_{k/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{H}_1 \mathcal{H}_2 I) = P^2 N_{k/\mathbb{Q}(\sqrt{pq_1q_2})}(I)$ is principal in $\mathbb{Q}(\sqrt{pq_1q_2})$, where $P$ is the ideal of $\mathbb{Q}(\sqrt{pq_1q_2})$ above $p$; consequently $N_{k/\mathbb{Q}(\sqrt{pq_1q_2})}(I)$ is principal in $\mathbb{Q}(\sqrt{pq_1q_2})$, which is absurd. This completes the proof of the main theorem.

**Theorem 6.** Let $k = \mathbb{Q}(\sqrt{d}, i)$, where $d = pq_1q_2$ with $p$, $q_1$ and $q_2$ are primes satisfying the conditions A et B. Let $C_{k,2}$ be the 2-class group of $k$. Put $p = \pi_1 \pi_2$, where $\pi_1$ and $\pi_2$ are in $\mathbb{Z}[i]$, denote by $\mathcal{H}_1$ (resp. $\mathcal{H}_2$ and $\mathcal{Q}_1$) the prime ideal of $k$ lies above $\pi_1$ (resp. $\pi_2$ and $q_1$). Then there exists an unambiguous ideal $I$ of $k$ of order 2 such that

1. If $p$, $q_1$ and $q_2$ are of type B(III), then $C_{k,2} = \langle [\mathcal{H}_1], [\mathcal{Q}_1], [I] \rangle$.

2. Else, $C_{k,2} = \langle [\mathcal{H}_1], [\mathcal{H}_2], [I] \rangle$.

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