International Journal of Pure and Applied Mathematics
Volume 104 No. 4 2015, 585-597
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v104i4.10

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SOME REMARKS ON CARISTI TYPE FIXED POINT THEOREM

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Abstract: The aims of this paper is to present a new generalized versions of Caristi-type fixed point theorem, into the context of iterated mappings in complete metric spaces. Examples are given to support the usability of our results and to distinguish them from the existing ones.

AMS Subject Classification: 47H10, 54H25
Key Words: metric space, fixed point, Caristi-type mapping, coprime positive integers, lower semicontinuity

1. Introduction

A number of generalizations of the Banach principle have appeared [1] [2] [3] [7] [8] [9] [12], in the midst of fixed point theorems in complete metric spaces we consider a generalization given by J. Caristi [4] in 1976, and recall that this theorem states that any self map T of X has a fixed point, if (X, δ) is complete and there exists a lower semi continuous map φ from X in to the nonnegative

Received: July 25, 2015

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number such that for all x in X : $\delta(x,Tx) \leqslant \varphi(x) - \varphi(Tx)$. Caristi's theorem is one of the most useful results in complete metric space. Since the appearance, various proofs are been given, the reader is advised to read [11] [10] [5] [13], for example Watson in [14] established the equivalence between a completeness of the metric space and the Caristi fixed point theorem and in [17] Park and Kang proved another version as following: Let X be a metric space, then X is complete if and only if for every self map T of X with a uniformly continuous function $\varphi : X \longrightarrow [0,1)$ such that $\delta(x,Tx) \leqslant \varphi(x) - \varphi(Tx)$ has a fixed point.

Using a Bishop-Phelps's lemma [20] we can show that Caristi fixed point theorem is closely connected with a Variational Principle of Ekeland and can be derived directly, for more precision we refer the reader to [18][19][21].

In this work, applying some existing theorems in literature and new techniques we give new generalizing Versions of Caristi-type result and some theorems in the same context of iterated mappings in complete metric spaces, these results are far from being conclusive. Examples are given to support the usability of our results and distinguish them from the existing ones. We recall two theorems [Theorem 3.16-Theorem 3.17, [22]] that will be useful in the sequel.

Theorem 1. *Let X and Y be complete metric spaces and let $T : X \longrightarrow X$ be an arbitrary mapping. Suppose there exist a closed mapping $S : X \longrightarrow Y$, a lower semicontinuous mapping $\varphi : S(X) \longrightarrow \mathbb{R}_+$, and a constant $c > 0$ such that for each $x \in X$,*

$$\max\{\delta(x,Tx), c\delta(Sx,STx)\} \leqslant \varphi(Sx) - \varphi(STx).$$

Then there exists $x \in X$ such that $Tx = x$.

The following is an example of a seemingly more general formulation of Caristi's theorem which actually reduces to a simple application of the original result.

Theorem 2. *Let (X,δ) be a complete metric space and $\varphi : X \to \mathbb{R}^+$ a lower semicontinuous function. Suppose $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing, concave downward, and vanishes at 0, and suppose $T : X \to X$ satisfies :*

$$\max\{\xi(\delta(x,Tx))\} \leqslant \varphi(x) - \varphi(Tx), \qquad x \in X.$$

Then T has a fixed point.

2. Main Result

We are motivated by generalisation Caristi-Type fixed point theorem into the framework of iterated mappings, in this section we will give some new results that complete some works [1] [3] in the same sens of our.

Recall that the mapping f from X into a nonnegative numbers $\mathbb{R}_+$ is lower semicontinous if for any sequence $\{x_n\}_n \subseteq X$ with $x_n \longrightarrow \bar{x} \in X$ $(n \to \infty)$ implies $\liminf f(x_n) \geqslant f(\bar{x})$, and two integres $n, m \in \mathbb{N}$ are coprime if there exist two positive integers p and q such that $np = mq + 1$ or $mq = np + 1$.

Before stating the main result, we give an important and usufel theorem :

Theorem 3. *Let (X, δ) be a complete metric space and a mapping φ from X into a nonnegative numbers $\mathbb{R}_+$ and a positive integer p. T a self map of X, suppose that for all $x \in X$:*

$$\delta(x, Tx) \leqslant \varphi(x) - \varphi(T^p x) \tag{1}$$

and the mapping $x \mapsto \delta(x, Tx)$ is lower semicontinous.

Then T has fixed point.

Proof. If $p = 0$ implie $T = Id_X$. So for $p \neq 0$, let us first show that the recurrent sequence $\{x_n\}_{n \in \mathbb{N}}$ defined by $x_{n+1} = Tx_n$ with an aribtray $x \in X$ is convergent. We have for all $n \in \mathbb{N}$, $T^p x_n = x_{n+p}$ and from (1) : $\delta(x_n, x_{n+1}) \leqslant \varphi(x_n) - \varphi(x_{n+p})$, it follows that

$$\sum_{i=0}^{n} \delta(x_i, x_{i+1}) \leqslant \sum_{i=0}^{n} \varphi(x_i) - \varphi(x_{i+p}) \Rightarrow \sum_{i=0}^{n} \delta(x_i, x_{i+1}) \leqslant \sum_{i=0}^{p-1} \varphi(x_i) < \infty$$

implying that the sequence $\{T^n x\}_{n \in \mathbb{N}}$ is convergent with limit $\bar{x}$ (Cauchy sequence in complete metric space), then all the subsequences of $\{T^n x\}_{n \in \mathbb{N}}$ in particular $\{T^{pn} x\}_{n \in \mathbb{N}}$ converge to the same limit $\bar{x}$. From (1) we obtain $\varphi(T^p x) \leqslant \varphi(x)$ for all x in X, hence the sequence $\{\varphi(T^{pn} x)\}_{n \in \mathbb{N}}$ is decreasing bounded below by zero, it follows that is convergent $(\lim \varphi(T^{pn} x) < \infty)$. Since

$$\delta\left(T^{pn} x, T^{pn+1} x\right) \leqslant \varphi(T^{pn} x) - \varphi\left(T^{pn+p} x\right)$$

and $x \mapsto \delta(x, Tx)$ is lower semicontinous we have :

$$\delta(\bar{x}, T\bar{x}) \leqslant \liminf_n \delta\left(T^{pn} x, T^{pn+1} x\right) \leqslant \liminf_n \left[\varphi(T^{pn} x) - \varphi\left(T^{p(n+1)} x\right)\right] = 0$$

which implies $T\bar{x} = \bar{x}$, then T has a fixed point ∎

Here we give an example showing that this result is not a trivial case :

Example 4. Let $X = [0, \infty)$ with the usual distance $\delta(x, y) = |x - y|$, the positive integer $p = 2$ and φ a map define as follows : $\varphi(x) = \frac{2}{x}$ if $x \in]0, \infty)$ and $\varphi(0) = 0$, and T a self map of X define as follows : $Tx = x$ if $x \in \left\{0, \frac{1}{2}\right\} \cup [1, \infty)$ and $Tx = x + 1$ if $x \in \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[$. We need only consider two cases:

- First case : for all $x \in \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[$, $T^2x = x + 1$ (because in this case $Tx \in [1, \infty)$) implies that

$$\varphi(x) - \varphi\left(T^2x\right) = \frac{2}{x} - \frac{2}{x+1} = \frac{2}{x(x+1)} > 1 = |x - x - 1| = \delta(x, Tx)$$

- Second case : for all $x \in \left\{0, \frac{1}{2}\right\} \cup [1, \infty)$, $T^2x = x$ and

$$\varphi(x) - \varphi\left(T^2x\right) = \varphi(x) - \varphi(x) = 0 \geqslant 0 = |x - x| = \delta(x, Tx)$$

This shows that the inequality (1) yields, ie for all x in $[0, \infty)$

$$\delta(x, Tx) \leqslant \varphi(x) - \varphi\left(T^2x\right)$$

then all the assumptions hold, as a result T admit at least one fixed point, to be specific $T(0) = 0$.

Now we are in position to formulate our main statement.

Theorem 5. *Let (X, δ) be a complete metric space, φ a mapping from X into a nonnegative numbers and two coprime positive integers p and q. T a self map of X, suppose that for all $x \in X$:*

$$\max\left\{\delta(x, T^px), \delta(x, T^qx)\right\} \leqslant \varphi(x) - \varphi(T^{pq}x) \tag{2}$$

and the mappings $x \mapsto \delta(x, T^px)$ and $x \mapsto \delta(x, T^qx)$ are lower semicontinous.

Then T has fixed point.

If we take $p = q = 1$ an immediate consequence of the preceding statement is the following result :

Corollary 6. *(Caristi-Type Theorem)*

Let (X, δ) be a complete metric space, φ a mapping from X into a nonnegative numbers. T a self-map of X, suppose that for all x in X :

$$\delta(x, Tx) \leqslant \varphi(x) - \varphi(Tx)$$

and $x \mapsto \delta(x, Tx)$ is lower semicontinous.

Then T has a fixed point in X.

Furthermore, the following example verifies that Thoerem 3 does indeed generalize Caristi-Type theorem referred to above.

Example 7. Let $X = \left[\frac{9}{10}, 1\right]$ with the usual metric, T and φ are defined respectively as follow $Tx = \frac{2x}{1+x}$ and $\varphi(x) = \frac{10}{13x}$ $\forall x \in X$, it is clear that $x \mapsto \delta(x, Tx)$ is a continuous map so it is lower semicontinous and for all x in X :

$$|x - Tx| = \left| x - \frac{2x}{1+x} \right| \leqslant \frac{10}{13x} - \left[\frac{5}{26} \times \frac{1+3x}{x} \right] = \varphi(x) - \varphi\left(T^2 x\right) \quad (3)$$

$$|x - Tx| = \left| x - \frac{2x}{1+x} \right| \geqslant \frac{10}{13x} - \left[\frac{5}{13} \times \frac{1+x}{x} \right] = \varphi(x) - \varphi(Tx) \quad (4)$$

From (3) we have all the conditions of Thoerem 3 verified with $p = 2$, so by (4) the hypothesis of Caristi-Type Theorem is therefore not satisfied and yet T has a fixed point $T(1) = 1$.

Proof of Theorem 5. From the inequality (2) yields for all $x \in X$,

$$\delta(x, T^p x) \leqslant \varphi(x) - \varphi(T^{pq} x)$$

To shorten, let us introduce the temporary notation $f = T^p$ then $\delta(x, fx) \leqslant \varphi(x) - \varphi(f^q x)$ from (2), applying Thoerem 3 we conclude that $f\bar{x} = T^p \bar{x} = \bar{x}$.

We proceed to show that T^q admit the same fixed point $\bar{x}$, for this concider a subset $K := \left\{\bar{x}, T\bar{x}, \cdots, T^{p-1}\bar{x}\right\}$ constituted by the first $p-$elements of the orbit under T of $\bar{x}$, it follows immediately that is non-empty stable by T $(T[K] = K)$ and because X is a Hausdorff space K is closed so it is complete. By the inequality (2) we have also for all $x \in K$, $\delta(x, T^q x) \leqslant \varphi(x) - \varphi(T^{pq} x)$ and Thoerem 3 shows that T^q had a least a fixed point in K, that is, there is a positive integer $j \in \{0, 1, \ldots, p-1\}$ such that $T^q\left(T^j \bar{x}\right) = T^j \bar{x}$ and since $0 \leqslant j \leqslant p-1$ there is some positive integer $k > 0$ with $j + k = p$, thus $T^k\left(T^q\left(T^j \bar{x}\right)\right) = T^k\left(T^j \bar{x}\right)$ i.e.

$$T^{q+k+j}\bar{x} = T^{q+p}\bar{x} = T^q(T^p\bar{x}) = T^q\bar{x} = T^{k+j}\bar{x} = T^p\bar{x} = \bar{x}$$

then $T^q\bar{x} = \bar{x}$ also a fixed point for T^q. Note that for all $n \in \mathbb{N}$ we have $T^{np}\bar{x} = \bar{x}$, since p and q are coprime positive integers there exist two positive integers r and l such that $rp = lq + 1$ or $rq = lp + 1$. Assume that $rp = lq + 1$ and as T^p and T^q have a common fixed point then

$$\bar{x} = T^{rp}\bar{x} = T^{lq+1}\bar{x} = T\left(T^{lq}\bar{x}\right) = T\bar{x}$$

which completes the proof. □

Remark 8. If we replace pq by pqh in inequality (2), where h is any positive integer, we get the same result at Theorem 5.

Here is an example to show that is not always true to think that the fixed points of T^p are fixed points of T^q. But the inequality (2) is a necessary condition.

Example 9. In the complexe plane $\mathbb{C}$, let α be a primitive pth root of unity ($\alpha^p = 1$) and p,q two coprime positive integers, let $U = \{z/|z| < 1\}$ a subset of $\mathbb{C}$. T is a self map of $\mathbb{C} \setminus U$ such that $Tz = \alpha z$ and φ a map defined from $\mathbb{C} \setminus U$ into a nonnegative numbers by $\varphi(z) = |z|$, it is obvious that T is a continuous map so $x \mapsto \delta(x, T^p x)$ and $x \mapsto \delta(x, T^q x)$ are lower semicontinous and for all z in $\mathbb{C} \setminus U$:

$$\delta(z, T^p z) = |z - z| = 0 \leqslant \varphi(z) - \varphi(T^{qp} z) = 0$$

on the other side, $\delta(z, T^q z) \leqslant \varphi(z) - \varphi(T^{qp} z)$ does not hold because $\delta(z, T^q z) = |z - \alpha^q z| = |z| \times |1 - \alpha^q| > 0$ for all z in $\mathbb{C} \setminus U$. It is seen that all points of $\mathbb{C} \setminus U$ are fixed points of T^p contrariwise T^q has no fixed point in $\mathbb{C} \setminus U$.

The following corollary is immediate from Theorem 5

Corollary 10. *Under the hypothesis of Theorem 5 with the inequality* (2) *replaced by:*

$$\max\{\delta(x, T^p x), \delta(x, T^q x)\} \leqslant \varphi(x) - \varphi(Tx) \tag{5}$$

for all x in X, then T has a fixed point.

Proof. From (5) it follows that for all x in X, $0 \leqslant \varphi(Tx) \leqslant \varphi(x)$ implies $\varphi(T^{pq} x) \leqslant \varphi(x)$ i.e. $-\varphi(x) \leqslant -\varphi(T^{pq} x)$ then (5) implie (2) and all the required conditions are verified, Theorem 5 end the proof. □

It is noted that all the above results are not true if the positive integers p and q are not coprime, here is an example to show the interest of this condition:

Example 11. Let $X = \left\{x \in \mathbb{R}^2 / R_1 \leqslant \|x\| \leqslant R_2\right\}$ with $(0 < R_1 < R_2)$ and the metric δ is the Euclidean norm, T a self map of X such that $Tx = -x$ and φ a map defined from X into a nonnegative numbers by $\varphi(x) = 1$. Its obvious to see that for all even positive integer r we have $T^r x = x$, so it is convenient to choose p and q from the even positive integers set of $\mathbb{N}$, consequently the others assumptions of above results are verified

$$\max\{\delta(x, T^p x), \delta(x, T^q x)\} = \|x - x\| = 0 \leqslant 1 - 1 = 0,$$

otherwise T do not admit a fixed point in X.

An immediate consequence of Theorem 5 is the following statements.

Corollary 12. *Let (X, δ) be a complete metric space and φ a mapping from X into a nonnegative numbers and $\{p_i/1 \leqslant i \leqslant r;\ r > 1\}$, a positive integers with $(p_i \wedge p_j = 1/i \neq j)$. T a self map of X, and $x \mapsto \delta(x, T^{p_i}x)$ are lower semicontinous satisfying for all $x \in X$:*

$$\max\{\delta(x, T^{p_i}x)/1 \leqslant i \leqslant n\} \leqslant \varphi(x) - \varphi(T^m x) \tag{6}$$

where $m = \prod_{i=1}^{r} p_i$, then T has at least one fixed point.

Proof. Without loss of generality we can take $p = p_1$ and $q = \prod_{i\neq 1} p_i$, by (6) we have for all x in X,

$$\delta(x, T^p x) \leqslant \varphi(x) - \varphi(T^{pq}x)$$

then from Theorem 3, there exist an element $\bar{x}$ such that $T^p\bar{x} = \bar{x}$. Now we can choose $p = p_2$ and $q = \prod_{i\neq 2,} p_i$, a similar argument to that in the proof of Theorem 5 shows that T^{p_2} had at least a common fixed point as T^{p_1}, since $p_1 \wedge p_2 = 1$, T has a fixed point, this finishes the proof. □

Remark 13. In the hypothesis of the corollary12, it can be only assumed that two of the functions $x \mapsto \delta(x, T^{p_i}x)$, $1 \leqslant i \leqslant r$, are lower semicontinuous.

We change the assumption in the last corollary $\{p_i/1 \leqslant i \leqslant r;\ r > 1\}$ a positive integers pairwise coprime by $\{p_i/1 \leqslant i \leqslant r;\ r > 1\}$ are setwise coprime, then we have the following result:

Corollary 14. *Under the assumptions of Corollary 12, rather than $(p_i \wedge p_j = 1/i \neq j)$ we have $p_1 \wedge p_2 \wedge \cdots \wedge p_r = 1$, then T has a fixed point.*

Proof. By (6) we have for all x in X, $\delta(x, T^{p_1}x) \leqslant \varphi(x) - \varphi(T^m x) = \varphi(x) - \varphi\left(T^{p_1 \times \prod_{i\neq 1} p_i} x\right)$, then by Theorem 3 there exist $\bar{x}$ in X such that $T^{p_1}\bar{x} = \bar{x}$, as in the proof of Theorem 5 we take $K = \left\{\bar{x}, T\bar{x}, \ldots, T^{p_1 - 1}\bar{x}\right\}$ and we show for all $i \in \{2, 3, \cdots, r\}$ that $T^{p_i}\bar{x} = \bar{x}$. Since $\{p_i/1 \leqslant i \leqslant r;\ r > 1\}$ are positive integers setwise coprime, they exist $\alpha_i \geqslant 0$, $i \in \{1, 2, \cdots, r\}$ such that

$$\sum_{j\in I_1} \alpha_j p_j = \sum_{k\in I_2} \alpha_k p_k + 1$$

with $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{1, 2, \cdots, r\}$, note that for all $i \in \{1, 2, \cdots, r\}$: $T^{\alpha_i p_i}\bar{x} = \bar{x}$, then

$$\bar{x} = T^{\sum_{j\in I_1} \alpha_j p_j}\bar{x} = T\left(T^{\sum_{k\in I_2} \alpha_k p_k}\bar{x}\right) = T\bar{x}$$

this complete the proof. □

The following result will now be derived directly from Theorem 5 and generalizes the corollary 12 but the proof is similar.

Corollary 15. *Under the assumptions of Corollary 12, rather than* $m = \prod_{i=1}^{r} p_i$ *we take* $m = \prod_{i=1}^{r} p_i^{\alpha_i}$ *where* $\alpha_i = \vartheta_{p_i}(m)$ *is the* p_i*-adic valuation of* m*, then* T *has a fixed point in* X*.*

Proof. Without loss of generality we can assume that all $p_i > 1$ and so $\alpha_i > 1$, from (6) we obtain for all $x \in X$

$$\delta\left(x, T^{p_i}x\right) \leqslant \varphi\left(x\right) - \varphi\left(T^{p_i q}x\right)$$

where $q = p_i^{\alpha_i - 1} \times \prod_{j \neq i} p_j^{\alpha_j}$, since $x \mapsto \delta\left(x, T^{p_i}x\right)$ is lower semicontinous there exist $\bar{x} \in X$ such that $T^{p_i}\bar{x} = \bar{x}$, by Theorem 3. A similar argument to that in the proof of Theorem 5 shows that T^{p_j} with $i \neq j$ had at least a common fixed point as T^{p_i}, since $p_i \wedge p_j = 1$ for all $i \neq j$, T has a fixed point $\bar{x} \in X$, this finishes the proof.

□

Next corollary follows directly from Theorem 5 and Caristi-Type Theorem.

Corollary 16. *Under the assumptions of Theorem 5, and for all* $x \in X$ *the inequality* (2) *replaced by:*

$$\max\left\{\delta\left(x, T^{p}x\right), \delta\left(x, T^{q}x\right)\right\} \leqslant \min\left\{\varphi\left(T^{p}x\right), \varphi\left(T^{q}x\right)\right\} - \varphi\left(T^{p+q}x\right) \quad (7)$$

then T *has a fixed point in* X*.*

Proof. Let put $\psi_1 = \varphi \circ T^q$ and $\psi_2 = \varphi \circ T^p$ from (7) we have:

$$\begin{cases} \delta\left(x, T^{p}x\right) & \leqslant & \psi_1\left(x\right) - \psi_1\left(T^{p}x\right) \\ \delta\left(x, T^{q}x\right) & \leqslant & \psi_2\left(x\right) - \psi_2\left(T^{q}x\right) \end{cases}$$

for all x in X, and

$$\begin{cases} \psi_1\left(T^{p}x\right) & \leqslant & \psi_1\left(x\right) \\ \psi_2\left(T^{q}x\right) & \leqslant & \psi_2\left(x\right) \end{cases} \Rightarrow \begin{matrix} \psi_1\left(T^{pq}x\right) & \leqslant & \psi_1\left(T^{p}x\right) \\ \psi_2\left(T^{pq}x\right) & \leqslant & \psi_2\left(T^{q}x\right) \end{matrix}$$

therefore it follows that

$$\begin{cases} \delta\left(x, T^{p}x\right) & \leqslant & \psi_1\left(x\right) - \psi_1\left(T^{pq}x\right) \\ \delta\left(x, T^{q}x\right) & \leqslant & \psi_2\left(x\right) - \psi_2\left(T^{pq}x\right) \end{cases}$$

and, in consequence,

$$\max\left\{\delta\left(x, T^{p}x\right), \delta\left(x, T^{q}x\right)\right\} \leqslant \delta\left(x, T^{p}x\right) + \delta\left(x, T^{q}x\right) \leqslant \psi\left(x\right) - \psi\left(T^{pq}x\right)$$

with $\psi = \psi_1 + \psi_2$, so the Theorem 5 imlpies that there exists an element $\bar{x}$ such that $T\bar{x} = \bar{x}$. □

Next, we derive three theorems from the above results.

Theorem 17. *Let (X,δ) be a complete metric space, p, q and h non-zero positive integers such that $2p<q$ and $q-p$ is a multiple of h, and $T:X\longrightarrow X$ satisfying:*

1. $T^p(X)$ is closed,
2. $\forall x\in X$,

$$\max\left\{\delta\left(T^px,T^{p+1}x\right),\delta\left(T^qx,T^{q+1}x\right)\right\}\leqslant\varphi\left(T^{h+q}x\right)-\varphi\left(T^{h+q+1}x\right) \quad (8)$$

3. $\varphi:T^p(X)\longrightarrow\mathbb{R}_+$ is lower semicontinous
4. T^{q-p} is continuous mapping.

Then T has a fixed point.

Proof. We first observe that we can define a partial order on complete metric subspace $X_1=T^p(X)$, to do this, consider $x,y\in X_1$ and define the order $\preccurlyeq$ as follows:

$$x\preccurlyeq y\Longleftrightarrow\max\left\{\delta(x,y),\delta\left(T^{q-p}x,T^{q-p}y\right)\right\}\leqslant\varphi\left(T^{h+q-p}x\right)-\varphi\left(T^{h+q-p}y\right)$$

The procedure is to show that every chain has an upper bound. Let $\{x_\alpha\}_{\alpha\in I}$ be any chain in $(X_1,\preccurlyeq)$, and for $\alpha,\beta\in I$ set $\alpha\leqslant\beta\Leftrightarrow x_\alpha\preccurlyeq x_\beta$.

Then $\left\{\varphi\left(T^{h+q-p}x_\alpha\right)\right\}_\alpha$ is nonincreasing net so there exists $t\in\mathbb{R}_+$ such that

$$\lim\varphi\left(T^{h+q-p}x_\alpha\right)=t$$

Let $\epsilon>0$. Then there exists α_0 such that $\alpha\geqslant\alpha_0$ implies

$$t\leqslant\varphi\left(T^{h+q-p}x_\alpha\right)\leqslant t+\epsilon$$

and so for $\beta\geqslant\alpha\geqslant\alpha_0$,

$$\max\left\{\delta(x_\alpha,x_\beta),\delta\left(T^{q-p}x_\alpha,T^{q-p}x_\beta\right)\right\}\leqslant\varphi\left(T^{h+q-p}x_\alpha\right)-\varphi\left(T^{h+q-p}x_\beta\right)\leqslant\epsilon$$

then $\{x_\alpha\}_\alpha$ is a Cauchy net in X_1 which it is convergent to some $\bar{x}\in X_1$. In the same manner we can see that the net $\{T^{q-p}x_\alpha\}_\alpha$ converge to limit $T^{q-p}\bar{x}\in X$ by assumption T^{q-p} is continous, hence T^h is also continous because h is a multiple of $q-p$ implies that $T^{h+q-p}x_\alpha \longrightarrow T^{h+q-p}\bar{x}$, since φ is lower semicontinous it is clear to see that

$$\varphi\left(T^{h+q-p}\bar{x}\right) \leqslant t$$

Therefore $\{x_\alpha\}_\alpha$ is an increasing net so for all $\alpha, \beta \in I$ with $\beta \geqslant \alpha$ implies

$$\begin{aligned} \max\left\{\delta\left(x_\alpha, x_\beta\right), \delta\left(T^{q-p}x_\alpha, T^{q-p}x_\beta\right)\right\} & \\ \leqslant \varphi\left(T^{h+q-p}x_\alpha\right) - \varphi\left(T^{h+q-p}x_\beta\right) & \\ \leqslant \varphi\left(T^{h+q-p}x_\alpha\right) - t & \end{aligned}$$

so taking the limit with respect to β yields

$$\begin{aligned} \max\left\{\delta\left(x_\alpha, \bar{x}\right), \delta\left(T^{q-p}x_\alpha, T^{q-p}\bar{x}\right)\right\} & \\ \leqslant \varphi\left(T^{h+q-p}x_\alpha\right) - t & \\ \leqslant \varphi\left(T^{h+q-p}x_\alpha\right) - \varphi\left(T^{h+q-p}\bar{x}\right) & \end{aligned}$$

This proves that $x_\alpha \preceq \bar{x}$ for each $\alpha \in I$.

Having thus shown that every chain in $(X_1, \preceq)$ has a upper bound we can appeal Zorn's lemma to conclude that $(X_1, \preceq)$ has a maximal element, say x. But by (8) $x \preceq Tx$; hence $Tx = x$. □

Theorem 18. *Let (X, δ) be a complete metric space, $q = \prod_{i=1}^{r} p_i^{\alpha_i} \in \mathbb{N}^\star$, $1 < p_1 < p_2 < \cdots < p_r$, and T a self map of X satisfying:*

1. There exists $k \in \{1, 2, \cdots, r\}$ such that $\alpha_k \neq 0$ and T^{p_k} is continuous mapping,

2. for all $x \in X$:

$$\begin{aligned} \max\left\{\delta\left(x, Tx\right), \delta\left(T^{p_1}x, T^{p_1+1}x\right), \cdots, \delta\left(T^{p_r}x, T^{p_r+1}x\right)\right\} & \\ \leqslant \varphi\left(T^q x\right) - \varphi\left(T^{q+1}x\right) & \quad (9) \end{aligned}$$

3. $\varphi : T^{p_k}(X) \longrightarrow \mathbb{R}_+$ is lower semicontinous,

Then T has at least one fixed point.

Proof. First note that from (9) we have for all x in X

$$\max\left\{\delta\left(x,Tx\right),\delta\left(T^{p_k}x,T^{p_k+1}x\right)\right\}\leqslant\varphi\left(T^qx\right)-\varphi\left(T^{q+1}x\right) \tag{10}$$

we introduce the partial order $\preceq$ in X as follows. For $x,y\in X$ say that

$$x\preceq y\Longleftrightarrow\max\left\{\delta\left(x,y\right),\delta\left(T^{p_k}x,T^{p_k}y\right)\right\}\leqslant\varphi\left(T^qx\right)-\varphi\left(T^qy\right)$$

Let $\{x_\alpha\}_{\alpha\in I}$ be any chain in $(X,\preccurlyeq)$, and for $\alpha,\beta\in I$ set $\alpha\leqslant\beta\Leftrightarrow x_\alpha\preccurlyeq x_\beta$. Analysis similar to that in the proof of Theorem 17 shows that the real sequence $\{\varphi\left(T^qx_\alpha\right)\}_\alpha$ is nonincreasing so there exists $t\in\mathbb{R}_+$ such that

$$\lim\varphi\left(T^qx_\alpha\right)=t$$

Let $\epsilon>0$. Then there exists α_0 such that $\alpha\geqslant\alpha_0$ implies

$$t\leqslant\varphi\left(T^qx_\alpha\right)\leqslant t+\epsilon$$

and so for $\beta\geqslant\alpha\geqslant\alpha_0$,

$$\max\left\{\delta\left(x_\alpha,x_\beta\right),\delta\left(T^{p_k}x_\alpha,T^{p_k}x_\beta\right)\right\}\leqslant\varphi\left(T^qx_\alpha\right)-\varphi\left(T^qx_\beta\right)\leqslant\epsilon$$

Thus $\{T^{p_k}x_\alpha\}_{\alpha\in I}$ is a Cauchy net while at the same time $\{x_\alpha\}_{\alpha\in I}$ is a Cauchy net in X. It follows that there exists $\bar{x}$ in X such that $\lim x_\alpha=\bar{x}$ and $\lim T^{p_k}x_\alpha=T^{p_k}\bar{x}$ since T^{p_k} is a continous mapping so it is T^q because $q=p_k\times p_k^{\alpha_k-1}\times\prod_{i\neq k}p_i^{\alpha_i}$ if $\alpha_k>1$ or $q=p_k\times\prod_{i\neq k}p_i^{\alpha_i}$ if $\alpha_k=1$, then $T^qx_\alpha\longrightarrow T^q\bar{x}$ and the lower semicontinuity of φ yields $\varphi\left(T^q\bar{x}\right)\leqslant t$.

Therefore $\{x_\alpha\}_\alpha$ is an increasing net so for all $\alpha,\beta\in I$ with $\beta\geqslant\alpha$ implies

$$\max\left\{\delta\left(x_\alpha,x_\beta\right),\delta\left(T^{p_k}x_\alpha,T^{p_k}x_\beta\right)\right\}\leqslant\varphi\left(T^qx_\alpha\right)-\varphi\left(T^qx_\beta\right)\leqslant\varphi\left(T^qx_\alpha\right)-t$$

so taking the limit with respect to β yields

$$\max\left\{\delta\left(x_\alpha,\bar{x}\right),\delta\left(T^{p_k}x_\alpha,T^{p_k}\bar{x}\right)\right\}\leqslant\varphi\left(T^qx_\alpha\right)-t\leqslant\varphi\left(T^qx_\alpha\right)-\varphi\left(T^q\bar{x}\right)$$

This proves that $x_\alpha\preceq\bar{x}$ for each $\alpha\in I$.

Then every chain in $(X,\preceq)$ has a upper bound we can applied Zorn's lemma to conclude that $(X,\preceq)$ has a maximal element, say x. But by (10) $x\preceq Tx$; hence $Tx=x$. □

Using Theorem 2 and Theorem 5 we shell prove the following theorem.

Theorem 19. *Let (X,δ) be a complete metric space, φ any mapping from X into a nonnegative numbers and two coprime positive integers p and q. $T : X \to X$ and $x \mapsto \delta(x,Tx)$ a lower semicontinuous function, suppose there exist $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ continuous, strictly increasing, concave downward, and vanishes at 0, satisfies for all x in X:*

$$\max\{\xi(\delta(x,T^p x)), \xi(\delta(x,T^q x))\} \leqslant \varphi(x) - \varphi(T^{pq}x)$$

Then T has a fixed point.

Acknowledgment

The authors would to thank all members of the Applied Functional Analysis team of L3A Laboratory of the Department of Mathematics and Computer Sciences at Ben M'sik Faculty of Sciences Casablanca.

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