International Journal of Pure and Applied Mathematics
Volume 104 No. 4 2015, 599-619
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v104i4.11
ijpam.eu

GENERALIZATION OF m-PARTIAL ISOMETRIES ON A HILBERT SPACE

Ould Ahmed Mahmoud Sid Ahmed

Department of Mathematics
College of Science
Aljouf University
Al Jouf, 2014, SAUDI ARABIA

Abstract: In this paper we introduce a generalization of the class of m-partial isometries operators recently studied in [24]. A bounded linear operator T on a Hilbert space $\mathcal{H}$ is called an m- partial isometry of order q for a positive integers m and q, if

$$T^q\left(T^{*m}T^m - \binom{m}{1}T^{*m-1}T^{m-1} + \binom{m}{2}T^{*m-2}T^{m-2} - \cdots + (-1)^m I\right) = 0.$$

AMS Subject Classification: 47B48
Key Words: Hilbert space, isometry, partial isometry, m-isometric operator, reducing subspace, the single valued extension property

1. Introduction and Preliminaries Results

Let $\mathcal{H}$ denotes on a complex a separable infinite dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$ into itself. For $T \in \mathcal{B}(\mathcal{H})$, T^* denotes the adjoint of T, $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote the range and the null-space of T, respectively, $I = I_{\mathcal{H}}$ being the identity operator.

Received: August 8, 2015

One of the most important subclasses, of the algebra of all bounded linear operators acting on a Hilbert space, the class of partial isometries operators. The operator theory of partial isometries has been studied by several authors ([12], [18]).

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be self-adjoint if $T^* = T$, isometry if $T^*T = I$ and partial isometry if $TT^*T = T$. In recent years this classes has been generalized, in some sense, to the larger sets of operators so-called m-self-adjoint,m-isometry and m-partial isometry.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called m-self- adjoint for some integer $m \geq 1$ if

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{*k} T^{m-k} = 0 \tag{1.1}$$

and it is called m-isometry for some integer $m \geq 1$ if

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{*m-k} T^{m-k} = 0, \tag{1.2}$$

where $\binom{m}{k}$ be the binomial coefficient. In [19],J.W.Helton initiated the study of operator T which satisfy the identity (1.1) and in [1], J. Agler and M.Stankus studied operator T which satisfy (1.2). The development of the theory of m-self-adjoint operators(and the related classes of m-isometries was motivated largely by striking connections with differential equations.

A simple manipulation proves that (1.2) is equivalent to

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \|T^{m-k}x\|^2 \tag{1.3}$$

for all $x \in \mathcal{H}$. Evidently, an 1-self-adjoint operator (resp. a 1-isometric operator) is m-self-adjoint (resp. m-isometric) for all integers $m \geq 1$. Indeed the class of m-self-adjoint operators (resp. m-isometric operators) is a generalization of the class of self-adjoint operators (resp. isometric operators). Major work on m-isometries has been done in a long paper consisting of three parts by Agler and Stankus ([1, 2, 3]) and have since then attracted the attention of several other authors (see for example [6], [7], [8], [10], [11], [13], [14], [23]). More recently a generalization of these operators to Banach spaces has been studied in the paper of Botelho [9], Sid Ahmed [22] , Bayart [4], Bermudez et al. [5], Hoffmann et al. [20] and P.P. Duggal [15]. The equation (1.3) was used to define m-isometries on a Banach space by Sid Ahmed [22] and by Botelho [9]. Bayart [4] has replaced the exponent 2 in (1.3) by an $p \in [1, \infty)$ and was introduced the following definition: a bounded linear operator $T : X \longrightarrow X$, on a Banach spaces X is an (m,p)-isometry ($m \geq 1$ integer, $p \geq 1$ real) if

$$\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\|T^{m-k}x\|^p = 0 \qquad (x \in X). \tag{1.4}$$

Hoffman et al. [20] considered the above definition with $p > 0$ real and studied the role of the second parameter p and also discussed the case $p = \infty$.

In [24], the authors considered an extension of the notion of partial isometries to m-partial isometries. We say that $T \in \mathcal{B}(\mathcal{H})$ is an m-partial isometry if T satisfies

$$T\Delta_{T,\,m} = T\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k} = 0, \tag{1.5}$$

where $\Delta_{T,\,m}$ is obtained formally from the binomial expansion of $\Delta_{T,\,m} = (T^*T - I)^m$ by understanding $(T^*T)^{m-k} = T^{*m-k}T^{m-k}$. The case when $m = 1$ is the partial isometries class. The class of m-partial isometries properly contains class of m-isometries.

Agler and Stankus proved that if T is an m-isometry, then $\Delta_{T,\,m-1} \geq 0$ (Proposition 1.5, [1]).

In the present paper we will give a generalization of m-partial isometries and m-isometries to (m,q)-partial isometries on Hilbert spaces. More precisely we will study the bounded linear operators T on a complex Hilbert space $\mathcal{H}$ that satisfy the identity

$$T^q\left(T^{*m}T^m - \binom{m}{1}T^{*m-1}T^{m-1} + - .... + (-1)^m I\right) = 0. \tag{1.6}$$

We will define an operator satisfying (1.6) to be an m-partial isometry of order q on $\mathcal{H}$. The case when $q = m = 1$, represent the partial isometries class. If T is injective and it verifies (1.6) is called an m-isometry that is deeply studied by J. Agler and M. Stankus in [1]. If $q = 1$,$(m,1)$-partial isometry becomes m-partial isometry.

The contents of the paper are the following. In Section 1 we set up notation and terminology. Furthermore, we collect some facts about m-isometries. In Section 2, we will study some properties of (m,q)-partial isometries operators. Exactly we will give conditions under which:

- an operator T is (m,q)-partial isometry.
- (m,q)-partial isometry operator it becomes m-partial isometry.

- (m,q)-partial isometry operator it becomes partial isometry.
- (m,q)-partial isometry operator it becomes $(m+1,q)$-partial isometry.
- the product and sum of two (m,q)-partial isometries operators are (m,q)-partial isometry.
- a power of (m,q)-partial isometry is an (m,q)-partial isometry.
- (m,q)-partial isometry operator has the single valued extension property.

  In order to answer these questions we will briefly review some basic facts about m-isometries.

Definition 1.1. *A subspace $\mathcal{M}$ of $\mathcal{H}$ is called*

1. *invariant for T or T-invariant if $T(\mathcal{M}) \subset \mathcal{M}$.*
2. *a reducing subspace for T if both $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are $T-$invariant or equivalently if $\mathcal{M}$ is invariant for both T and T^*.*

Theorem 1.1. ([1]) *Let $T \in \mathcal{B}(\mathcal{H})$ be an m-isometry for some $m \geq 1$. Then*

$$T^{*n}T^n = \sum_{0 \leq k \leq m-1} n^{(k)} \beta_k(T) \tag{1.7}$$

where

$$\beta_k(T) = \frac{1}{k!} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j} T^{*j} T^j$$

and

$$n^{(k)} = \begin{cases} 0, \quad \text{if } \ n = 0 \\ \\ 0 \ \text{if}\ n > 0 \ \text{ and } \ k > n \\ \\ \binom{n}{k} k! \ \text{ if } \ n > 0 \quad \text{and } \ k \leq n. \end{cases}$$

Proposition 1.1. *([4, Theorem 2.2]) and [22, Proposition 2.3]). An (m,p)-isometry $T \in \mathcal{B}(X)$ is an $(m+1,p)$-isometry.*

Theorem 1.2. *([20], Proposition 2.1) Let $T \in \mathcal{B}(X)$ be an (m,p)-isometry such that for all $x \in X$ there exists a real number $C(x) > 0$ such that*

$$\|T^n(x)\| \leq C(x) \ \ \forall n \in \mathbb{N}.$$

Then T is an isometry.

2. m-Partial Isometries of Order q

Definition 2.1. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an (m,q)-partial isometry or m-partial isometry of order q if and only if T satisfies the identity*

$$T^q\left(T^{*m}T^m - \binom{m}{1}T^{*m-1}T^{m-1} + ... - .... + (-1)^m I\right) = 0. \tag{2.1}$$

In particular, if T is a $(2,q)$-partial isometry or a $(3,q)$-partial isometry, then it must satisfy the operator equation

$$T^q\big(T^{*2}T^2 - 2T^*T + I\big) = 0 \tag{2.2}$$

or

$$T^q\big(T^{*3}T^3 - 3T^{*2}T^2 + 3T^*T - I\big) = 0 \tag{2.3}$$

Remark 2.1. 1. $(1,1)$-partial isometry operator is an partial isometry.

2. $(m,1)$-partial isometry operator is an m-partial isometry.

3. $(1,q)$-partial isometry is an partial isometry of order q i.e., $T^qT^*T = T^q$.

4. Every m-partial isometry is an (m,q)-partial isometry.

5. Every (m,q)-partial isometry is an $(m,q+1)$-partial isometry.

The following example shows that there exists an operator which is (m,q)-partial isometry but not $(m,1)$-partial isometry.

Example 2.1. Let $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$. A simple computation shows that T is a $(2,2)$-partial isometry but not a $(2,1)$-partial isometry.

Example 2.2. We now consider [24, Theorem 4.3]. Let us fix an orthonormal basis $\big(e_n\big)_{n\geq 1}$ of $\mathcal{H}$. For a sequence of complex numbers $\big(\omega_n\big)_{n\geq 1}$, the associated weighted operator on $\mathcal{H}$ with

$$Te_n = \omega_n e_{n+1} \quad \text{for all} \quad n \geq 1.$$

Ii is well know that T is bounded operator if and only if the weighted sequence $\big(\omega_n\big)_{n\geq 1}$ is bounded.We assume that T is bounded weighted shift operator.

Since $Te_n = \omega_n e_{n+1}$ for all $n \geq 1$, we see that $T^k e_n = (\prod_{n\leq j\leq k+n-1} \omega_j)e_{n+k}$.
Consequently

$$T^{*k}e_n = \begin{cases} 0, \text{ if } n \leq k \\ (\prod_{n-k\leq j\leq n-1} \overline{\omega_j})e_{n-k} \text{ if } n > k+1. \end{cases}$$

Therefore

$$T^{*k}T^k e_n = (\prod_{n\leq j\leq k+n-1} |\omega_j|^2)e_n.$$

T is a (m,q)-partial isometry if and only if for any integer $n \geq 1$

$$(\prod_{n\leq j\leq q+n-1} \omega_j)\Big((-1)^m + \sum_{1\leq k\leq m} (-1)^{m-k}\binom{m}{k}(\prod_{n\leq j\leq k+n-1} |\omega_j|^2)\Big) = 0$$

Remark 2.2. If $T \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{N}(T^q)$ is a reducing subspace for T, then

$$T^{*k}T^k\big(\mathcal{N}(T^q)^\perp\big) \subset \mathcal{N}(T^q)^\perp,\ k = 1, 2, ..., ..$$

The following theorem characterizes some (m,q)-partial isometries operators

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{N}(T^q)$ is a reducing subspace for T. Then the following properties are equivalent.*

(1) T is an (m,q)-partial isometry.

(2)

$$\sum_{0\leq k\leq m} (-1)^m\binom{m}{k}\|T^{m-k}T^{*q}x\|^2 = 0, \quad \textit{for all } x \in \mathcal{H}.$$

Proof. First , assume that T is an (m,q)-partial isometry. We have that for all $x \in \mathcal{H}$

$$T^q \sum_{0\leq k\leq m} (-1)^k\binom{m}{k}T^{*m-k}T^{m-k}T^{*q}x = 0$$

$$\Longrightarrow \ \langle T^q \sum_{0\leq k\leq m} (-1)^k\binom{m}{k}T^{*m-k}T^{m-k}T^{*q}x,\ x\rangle = 0$$

$$\Longrightarrow \ \sum_{0\leq k\leq m} (-1)^k\binom{m}{k}\|T^{m-k}T^{*q}x\|^2 = 0.$$

Conversely assume that $\sum_{0\leq k\leq m}(-1)^m\binom{m}{k}\|T^{m-k}T^{*q}x\|^2=0,$ for all $x\in\mathcal{H}$. It follows that

$$\begin{aligned}
&\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\|T^{m-k}T^{*q}x\|^2=0\\
\Longrightarrow\quad &\langle T^q\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}T^{*q}x,\ x\rangle=0,\ \forall\ x\in\mathcal{H}\\
\Longrightarrow\quad &T^q\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}T^{*q}x=0,\forall\ x\in\mathcal{H}.
\end{aligned}$$

We deduce that

$$T^q\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}=0\ \text{ on }\ \overline{\mathcal{R}(T^{*q})}=\mathcal{N}(T^q)^{\perp}.$$

As $\mathcal{N}(T^q)$ is a reducing subspace, we have

$$T^q\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}=0\ \text{ on }\ \mathcal{N}(T^q)$$

and hence,

$$T^q\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}=0.$$

□

Corollary 2.1. *Let $T\in\mathcal{B}(\mathcal{H})$ such that $\mathcal{N}(T^q)$ is a reducing subspace for T, then the following properties are equivalent*

1. *T is an (m,q)-partial isometry.*

2. *$T|_{\mathcal{N}(T^q)^{\perp}}$ is an m-isometry.*

In the following theorem we show that by imposing certain conditions on (m,q)-partial isometry operator it becomes m-partial isometry.

Theorem 2.2. *If T is an (m,q)-partial isometry such that $\mathcal{N}(T)=\mathcal{N}(T^2)$ then T is an m-partial isometry.*

Proof. By the assumption we have that $\mathcal{N}(T) = \mathcal{N}(T^n)$ for all positive integer n. Hence

$$T^q\left(\sum_{0\le k\le m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}\right) = 0.$$

implies

$$T\left(\sum_{0\le k\le m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}\right) = 0.$$

□

Proposition 2.1. *If T is an (m,q)-partial isometry such that T^k is an partial isometry for $k = 1, 2, 3, ...m-1$ then T^m is a partial isometry for $m \ge q$.*

Proof. Since T is an (m,q)-partial isometry we have

$$T^q \sum_{0\le k\le m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k} = 0$$

Multiplying the above equation from the left by T^m we get

$$T^q\left(T^mT^{*m}T^m + \sum_{1\le k\le m}(-1)^k\binom{m}{k}T^mT^{*m-k}T^{m-k}\right) = 0.$$

Since T^k is a partial isometry for $k = 1, ..., m-1$ we deduce that

$$T^q\left(T^mT^{*m}T^m + \sum_{1\le k\le m}(-1)^k\binom{m}{k}T^m\right) = 0.$$

Thus,

$$T^q\left(T^mT^{*m}T^m - T^m\right) = 0$$

or equivalently

$$\left(T^{*m}T^mT^{*m} - T^{*m}\right)T^{*q} = 0$$

Hence,

$$T^mT^{*m}T^m - T^m = 0 \quad \text{on} \quad \overline{\mathcal{R}(T^{*q})} = \mathcal{N}(T^q)^{\perp}.$$

On the other hand, since $m \geq q$,

$$T^m T^{*m} T^m - T^m = 0 \quad \text{on} \quad \mathcal{N}(T^q).$$

$\square$

Proposition 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be an (m, q)-partial isometry such that T^k is a partial isometry for $k = 2, 3, ..., m$. Then $T^{m+q} = T^{m+q} T^* T$ i.e., T is an $(1, m+q)$-partial isometry.*

Proof. Using the fact that T is an (m, q)-partial isometry, we get

$$T^q \left(\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{*m-k} T^{m-k} \right) = 0.$$

Multiplying the above equation from the left by T^m we get the identity

$$T^q \left(\sum_{0 \leq k \leq m-2} (-1)^k \binom{m}{k} T^m T^{*m-k} T^{m-k} + (-1)^{m-1} m T^m T^* T + (-1)^m T^m \right) = 0.$$

By the assumption we get

$$T^q \left(\sum_{0 \leq k \leq m-2} (-1)^k \binom{m}{k} T^m + (-1)^{m-1} m T^m T^* T + (-1)^m T^m \right) = 0.$$

A calculation shows that $T^{m+q}(I - T^* T) = 0$. Hence, the desired result. $\square$

In the following corollary we show that by imposing certain conditions on (m, q)-partial isometry operator it becomes partial isometry.

Corollary 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be an (m, q)-partial isometry such that T^k is a partial isometry for $k = 2, 3, ..., m$. If $\mathcal{N}(T) = \mathcal{N}(T^2)$, then T is an partial isometry.*

Proof. It is a consequence of Proposition 2.2 and the fact that $\mathcal{N}(T) = \mathcal{N}(T^n)$ for all positive integer n. $\square$

Theorem 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be an (m, q)-partial isometry such that $\mathcal{N}(T^q)$ is a reducing subspace for T. Assume that there exists a constant $M > 0$ satisfying*

$$\|T^n|_{\mathcal{N}(T^q)^\perp}\| \leq M, \ \forall \, n \in \mathbb{N},$$

then T is a $(1, q)$- partial isometry.

Proof. If T is an (m,q)-partial isometry, then $T|_{\mathcal{N}(T^q)^\perp}$ is an m-isometry, and by Theorem 1.2 applied to the operator $T|_{\mathcal{N}(T^q)^\perp}$ we have that $T|_{\mathcal{N}(T^q)^\perp}$ is an isometry. In particular Corollary 2.1 gives $T^qT^*T = T^q$. □

The following example shows that a (m,q)-partial isometry operator need not be a $(m+1,q)$-partial isometry and vice versa.

Example 2.3. Let $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$ and $S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{\frac{1+\sqrt{5}}{2}} & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$ then a direct computation shows that

- T is a $(1,q)$-partial isometry but is not a (2,1)-partial isometry.
- S is a $(2,q)$-partial isometry but is not a $(1,1)$-partial isometry.

It is well know that every m-self-adjoint(resp. m-isometry) operator is $(m+1)$-self-adjoint (resp.$(m+1)$-isometry) operator.

In the following theorem we show that by imposing certain conditions on (m,q)-partial isometry operator it becomes $(m+1,q)$-partial isometry.

Theorem 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be an (m,q)-partial isometry such that $\mathcal{N}(T^q)$ is a reducing subspace for T. Then T is an $(m+n,q)$-partial isometry for $n = 1, 2, ...$*

Proof. Two proofs for this theorem will be given.

The First Proof. Since T is an (m,q)-partial isometry and $T(\mathcal{N}(T^q)^\perp) \subset \mathcal{N}(T^q)^\perp$ it follows that $T/_{\mathcal{N}(T^q)^\perp}$ is an m-isometry. By Proposition 1.1 applied to the operator $T/_{\mathcal{N}(T^q)^\perp}$ we obtain that $T/_{\mathcal{N}(T^q)^\perp}$ is an $(m+n)$-isometry and hence T is an $(m+n,q)$-partial isometry.

The second Proof. The standard formula $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$ for binomial coefficients gives that

$$\begin{aligned} & \sum_{0\le k\le m+1} (-1)^k \binom{m+1}{k} \|T^{m+1-k}T^{*q}x\|^2 \\ = & \sum_{0\le k\le m+1} (-1)^k \left(\binom{m}{k} + \binom{m}{k-1}\right) \|T^{m+1-k}T^{*q}x\|^2 \\ = & \sum_{0\le k\le m} (-1)^k \binom{m}{k} \|T^{m-k}TT^{*q}x\|^2 - \sum_{0\le k\le m} (-1)^k \binom{m}{k} \|T^{m-k}T^{*q}x\|^2 \end{aligned}$$

$$= \quad 0.$$

□

The following example shows that Theorem 2.4 is not necessarily true if $\mathcal{N}(T)$ is not reducing subspace for T.

Example 2.4. ([24]) Let $T = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$. We not that $\mathcal{N}(T)$ is not reducing for T and T is a 1-partial isometry but T is not a 2-partial isometry.

Example 2.5. The operator $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$ is a 1-partial isometry and a 2-partial isometry and $\mathcal{N}(T)$ is a reducing subspace for T.

Proposition 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be an (m,q)-partial isometry. Then T $(m+1,q)$-partial isometry if and if T is an m-isometry on $\mathcal{R}(TT^{*q})$.*

Proof. First assume that T is an (m,q)-partial isometry and an $(m.+1,q)$-partial isometry we have by (2.1)

$$T^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} T^{*m-k}T^{m-k} = 0 \tag{2.4}$$

and

$$T^q \sum_{0\leq k\leq m+1} (-1)^k \binom{m+1}{k} T^{*m+1-k}T^{m+1-k} = 0 \tag{2.5}$$

Combining (2.4) and (2.5) we obtain

$$T^qT^*\Big(\sum_{0\leq k\leq m} (-1)^k \binom{m}{k} T^{*m-k}T^{m-k}\Big)T = 0.$$

Thus implies that

$$T^qT^*\Big(\sum_{0\leq k\leq m} (-1)^k \binom{m}{k} T^{*m-k}T^{m-k}\Big)TT^{*q} = 0.$$

the above inequality means that we can write

$$\sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \|T^{m-k}TT^{*q}x\|^2 = 0, \quad \text{for all } x \in \mathcal{H}$$

and hence the desired result.

Conversely assume that T is an (m,q)-partial isometry and an m-isometry on $\mathcal{R}(TT^{*q})$. We have that

$$T^q \sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k} T^{*k}T^k = 0 \tag{2.6}$$

and

$$\sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k} T^{*k}T^kTT^{*q} = 0 \tag{2.7}$$

The equation (2.7) implies

$$T^q \sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k} T^{*k+1}T^{k+1} = 0 \tag{2.8}$$

or equivalently

$$T^q \sum_{1\le k\le m+1} (-1)^{m-k}\binom{m}{k-1} T^{*k}T^k = 0. \tag{2.9}$$

Combining (2.6) and (2.9), we obtain

$$T^q\left((-1)^m I + \sum_{1\le k\le m} (-1)^{m-k}\left(\binom{m}{k} + \binom{m}{k-1}\right)T^{*k}T^k - T^{*m+1}T^{m+1}\right) = 0.$$

The binomial coefficient identity $\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$ for $k = 1, 2, ...m$ gives

$$T^q \sum_{0\le k\le m+1} (-1)^{m+1-k}\binom{m+1}{k} T^{*k}T^k = 0.$$

This completes the proof. □

In [22, Theorem 2.2], it is proved that if T and S commuting bounded linear operators on a Banach space X such that T is a 2-isometry and S is an m-isometry, then ST is an $(m+1)$-isometry. This result was improved in [6, Theorem 3.3]: if $TS = ST$, T is an (m,q)-isometry and S is an (n,q)-isometry, then ST is an $(m+n-1,q)$-isometry.

It is natural to ask whether the product and sum of two (m,q)-partial isometries operators are (m,q)-partial isometry. In general they need not be. The following Theorems give an affirmative answer under some conditions.

Theorem 2.5. *Let $T, S \in \mathcal{B}(\mathcal{H})$ are (m,q)-partial isometries. The following properties hold:*

1. *If $ST = TS$ and $\mathcal{R}(S) \subset \mathcal{N}(T)$ or $\mathcal{R}(T) \subset \mathcal{N}(S)$, then TS is an (m,q)-partial isometry.*
2. *If $ST = TS = S^*T = TS^* = 0$, then $T + S$ is an (m,q)-partial isometry.*

Proof. The proof follows from the Definition 2.1. □

Proposition 2.4. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that T is an (m,q)-partial isometry and S is an (n,q)-partial isometry . The following properties hold:*

1. *If $ST = TS$ and $\mathcal{R}(S) \subset \mathcal{N}(T)$ or $\mathcal{R}(T) \subset \mathcal{N}(S)$, then TS is an $(m+n,q)$-partial isometry.*
2. *If $ST = TS = S^*T = TS^* = 0$, $\mathcal{N}(T)$ is a reducing subspace for T and $\mathcal{N}(S)$ is a reducing subspace for S, then $T + S$ is an $(m+n,q)$-partial isometry.*

Proof. 1. Clear.

2. Since T is it follows that

$$
\begin{aligned}
& (TS)^q \sum_{0 \le k \le n+m} (-1)^{n+m-k} \binom{m+n}{k} (T+S)^{*k} (T+S)^k \\
= \; & (TS)^q \sum_{0 \le k \le n+m} (-1)^{n+m-k} \binom{m+n}{k} \left(T^{*k}T^k + S^{*k}S^k \right) \\
= \; & 0 \;\; \text{(by Theorem 2.4)}.
\end{aligned}
$$

□

The following example shows that the product of (m,q)-partial isometries is not necessarily an (m,q)-partial isometry.

Example 2.6. Let $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$ acting on $\mathbb{C}^3$. It easy to see that T and S are 1-partial isometries but TS is not a 1-partial isometry.

The following example shows that the sum of (m,q)-partial isometries is not necessarily an (m,q)-partial isometry.

Example 2.7. Let $T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ acting on $\mathbb{C}^3$. It easy to see that T and S are $(2,2)$-partial isometries but $T+S$ is not a $(2,2)$-partial isometry.

We will use the following remark in the proof of Theorem 2.6.

Remark 2.3.

1. The following characterization of 3-isometry operator follows from the identity (1.7). An operator $T \in \mathcal{B}(\mathcal{H})$ is a 3-isometry if and only if there exist operators $B_1(T^*,T)$ and $B_2(T^*,T)$ such that for all natural numbers n,

$$T^{*n}T^n = I + nB_1(T^*,T) + n^2B_2(T^*,T), \tag{2.10}$$

where

$$B_1(T^*,T) = \frac{1}{2}\big(-T^{*2}T^2 + 4T^*T - 3I\big)$$

and

$$B_2(T^*,T) = \frac{1}{2}\big(T^{*2}T^2 - 2T^*T + I\big).$$

2. From the identity (1.7) the following characterization of 2-isometry holds. For an $T \in \mathcal{B}(\mathcal{H})$,then T is an 2-isometry if and only if

$$T^{*k}T^k = kT^*T - (k-1)I, \quad k = 1, 2, \ldots.$$

Theorem 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be an (m,q)-partial isometry and let $S \in \mathcal{B}(\mathcal{H})$ for which $TS = ST$ and $TS^* = S^*T$. The following properties hold:*

1. *if S is an isometry, then TS is an (m,q)-partial isometry.*
2. *if $\mathcal{N}(T^q)$ is a reducing subspace for T and S is an 2-isometry,then TS is an (m+1,q)-partial isometry.*
3. *if $\mathcal{N}(T^q)$ is a reducing subspace for T and S is an 3- isometry, then TS is an $(m+2,q)$-partial isometry.*

Proof. 1. Let $x \in \mathcal{H}$, we have

$$\begin{aligned} & (TS)^q \sum_{0\le k\le m} (-1)^k \binom{m}{k} (TS)^{*m-k}(TS)^{m-k}(x) \\ = \; & (TS)^q \sum_{0\le k\le m} (-1)^k \binom{m}{k} T^{*m-k}T^{m-k}(S^{*m-k}S^{m-k}x) \end{aligned}$$

$$
\begin{aligned}
&= T^q \sum_{0\le k\le m} (-1)^k \binom{m}{k} T^{*m-k}T^{m-k}(S^q x) \\
&= 0
\end{aligned}
$$

2. From part 2. of Remark 2.4 and Theorem 2.4, it follows that

$$
\begin{aligned}
& (TS)^q \sum_{0\le k\le m+1} (-1)^k \binom{m+1}{k} (TS)^{*m+1-k}(TS)^{m+1-k} \\
&= (TS)^q \sum_{0\le k\le m+1} (-1)^k \binom{m+1}{k} T^{*m+1-k}T^{m+1-k}(S^{*m+1-k}S^{m+1-k}) \\
&= (TS)^q \sum_{0\le k\le m+1} (-1)^k \binom{m+1}{k} T^{*m+1-k}T^{m+1-k}\big((m+1-k)S^*S\big) \\
& -(TS)^q \sum_{0\le k\le m+1} (-1)^k \binom{m+1}{k} T^{*m+1-k}T^{m+1-k}(m-k)I \\
&= (TS)^q \sum_{0\le k\le m+1} (-1)^k \binom{m+1}{k} T^{*m+1-k}T^{m+1-k}k\big(I - S^*S\big) \\
&= S^qT^q \sum_{0\le k\le m} (-1)^k \binom{m}{k} T^{*m-k}T^{m-k}\big(I - S^*S\big) \\
&= 0.
\end{aligned}
$$

3. Since S is a 3-isometry and $TS = ST$,$TS^* = S^*T$, we have from equation (2.10) and Theorem 2.4 that

$$
\begin{aligned}
& (ST)^q \sum_{0\le k\le m+2} (-1)^k \binom{m+2}{k} (ST)^{*m+2-k}(ST)^{m+2-k} \\
&= S^qT^q \sum_{0\le k\le m+2} (-1)^k \binom{m+2}{k} (T)^{*m+2-k}(T)^{m+2-k}\big(S^{*m+2-k}S^{m+2-k}\big) \\
&= S^qT^q \Big\{ \sum_{0\le k\le m+2} (-1)^k \binom{m+2}{k} (T)^{*m+2-k}(T)^{m+2-k} I + \\
& \sum_{0\le k\le m+2} (-1)^k \binom{m+2}{k} (T)^{*m+2-k}(T)^{m+2-k}(m+2-k)B_1(S^*, S) + \\
& \sum_{0\le k\le m+2} (-1)^k \binom{m+2}{k} (T)^{*m+2-k}(T)^{m+2-k}(m+2-k)^2 B_2(S^*, S) \Big\}
\end{aligned}
$$

$$= \underbrace{S^qT^q \sum_{1\leq k\leq m+2}(-1)^k\binom{m+2}{k}k(T)^{*m+2-k}(T)^{m+2-k}A_m(S^*,S)}_{I}$$

$$+ \underbrace{S^qT^q \sum_{k=1}^{m+2}(-1)^k\binom{m+2}{k}k^2(T)^{*m+2-k}(T)^{m+2-k}\big(B_2(S^*,S)\big)}_{J}$$

where

$$A_m(S^*,S) = \Bigg(-B_1(S^*,S)+(-2(m+2))B_2(S^*,S)\Bigg)$$

$$\begin{aligned} I =& S^qT^q \sum_{1\leq k\leq m+2}(-1)^k(m+2)\binom{m+1}{k-1}(T)^{*m+2-k}(T)^{m+2-k}A_m(S^*,S) \\ =& -(m+2)S^qT^q \sum_{0\leq k\leq m+1}(-1)^k\binom{m+1}{k}(T)^{*m+1-k}(T)^{m+1-k} \\ & \Bigg(-B_1(S^*,S)+(-2(m+2))B_2(S^*,S)\Bigg) \\ =&0. \end{aligned}$$

$$\begin{aligned} J =&S^qT^q \sum_{1\leq k\leq m+2}(-1)^k\binom{m+2}{k}k^2(T)^{*m+2-k}(T)^{m+2-k}\big(B_2(S^*,S)\big) \\ =&S^qT^q \sum_{k=1}^{m+2}(-1)^k\binom{m+2}{k}(k(k-1)+k)(T)^{*m+2-k}(T)^{m+2-k}\big(B_2(S^*,S)\big) \\ =&S^qT^q \sum_{1\leq k\leq m+2}(-1)^k\binom{m+2}{k}k(k-1)(T)^{*m+2-k}(T)^{m+2-k}\big(B_2(S^*,S)\big) \\ =&(m+2)(m+1)S^qT^q \sum_{k=2}^{m+2}(-1)^k\binom{m}{k-2}(T)^{*m+2-k}(T)^{m+2-k}\big(B_2(S^*,S)\big) \\ =&(m+2)(m+1)S^qT^q \sum_{0\leq k\leq m}(-1)^k\binom{m}{k}(T)^{*m-k}(T)^{m-k}\big(B_2(S^*,S)\big) \\ =&0. \end{aligned}$$

Hence $I+J=0$. Thus TS is a $(m+2,q)$-partial isometry.

□

The following corollary is an immediate consequence of Theorem 2.6.

Corollary 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be an (m, q)-partial isometry and let $S \in \mathcal{B}(\mathcal{H})$ for which $TS = ST$ and $TS^* = S^*T$. the following properties hold:*

1. *if S is an isometry, then TS^n is an (m, q)-partial isometry for all positive integer n.*
2. *if $\mathcal{N}(T^q)$ is a reducing subspace for T and S is an 2-isometry,then TS^n is an (m+1,q)-partial isometry for all positive integer n.*
3. *if $\mathcal{N}(T^q)$ is a reducing subspace for T and S is an 3- isometry, then TS^n is an $(m + 2, q)$-partial isometry for all positive integer n.*

It is clear that if T is an isometry, then T^r is also an isometry. Saddi and Sid Ahmed in [24, Theorem 2.1] prove that any power of a $(2, 1)$-partial isometry if it has a nontrivial reducing sub space $\mathcal{N}(T)$ is again a $(2, 1)$-partial isometry. In [16] it was showed that any power of a $(m, 1)$-partial isometry if it has a nontrivial reducing subspace $\mathcal{N}(T)$ is again a $(m, 1)$-partial isometry. Now we generalize it to (m, q)-partial. As the proof is very similar to [24, Theorem 2.1] and ([16], Theorem 2.17) , we omit it.

Theorem 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be an (m, q)-partial isometry such that $\mathcal{N}(T^q)$ is a reducing subspace for T. Then any power of T is also an (m, q)-partial isometry.*

Lemma 2.1. ([21]) *Let $n \geq 1$ be an integer, and let $T \in \mathcal{B}(\mathcal{H})$ an operator such that $r(T) \leq 1$. Then the following equality hold*

$$\begin{aligned} &\sum_{0\leq k\leq n} \binom{n}{k} \varphi_\alpha(T)^{*k}\varphi_\alpha(T)^k \\ =\ & (1-|\alpha|^2)^n\big(I-\alpha T^*\big)^{-n}\Big(\sum_{0\leq k\leq n}(-1)^k\binom{n}{k}T^{*k}T^k\Big)\big(I-\overline{\alpha}T\big)^{-n} \end{aligned}$$

holds for every conformal automorphism φ_α of the unit disc of the form $\varphi_\alpha(z) = \dfrac{z-\alpha}{1-\overline{\alpha}z}$ for all $z \in \mathbb{D}$ and $\alpha \in \mathbb{D}$.

Let $Aut(\mathbb{D})$ be the group of all conformal mapping from $\mathbb{D}$ onto itself (also called disk automorphisms of $\mathbb{D}$). It is well known that $Aut(\mathbb{D})$ coincides with the set of all Möbius transformations of $\mathbb{D}$ onto itself:

$$Aut(\mathbb{D}) = \{\lambda\varphi_\alpha : |\lambda| = 1, \alpha \in \mathbb{D}\}.$$

We can now conclude that the conformal automorphisms operate on the class of m-isometries.

Corollary 2.4. *If $T \in \mathcal{B}(\mathcal{H})$ is an m-isometry, then so is $\varphi(T)$ for every $\varphi \in Aut(\mathbb{D})$.*

Proof. It is a consequence of the above lemma. □

Put

$$\begin{aligned} S_{T^q} &:= T^{*q}\Delta_{T^q,\, m-1}T^q \\ &= T^{*q}\Big(\sum_{0\le k\le m-1}(-1)^k\binom{m-1}{k}\big(T^{*q}\big)^{m-1-k}\big(T^q\big)^{m-1-k}\Big)T^q \end{aligned}$$

Proposition 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be an (m,q)-partial isometry such that $\mathcal{N}(T^q)$ is a reducing subspace for T, then $S_{T^q} \geq 0$ i.e.; $\langle S_{T^q}x,\, x\rangle \geq 0 \quad \forall\, x \in \mathcal{H}$.*

Proof. For $x \in \mathcal{H}$, we have $\langle S_{T^q}x,\ x\rangle = \langle \Delta_{T^q,\, m-1}T^q x,\ T^q x\rangle$. According to Corollary 2.1 ,Proposition 1.1 and ([1] Proposition 1.5) we have $\Delta_{T^q|_{\mathcal{N}(T^q)^\perp},\, m-1} \geq 0$. Since T reduces $\mathcal{N}(T^q)$ then $Tx \in \mathcal{N}(T^q)^\perp$ and

$$\langle S_{T^q}x\ ,x\rangle = \langle \Delta_{T^q,\, m-1}T^q x,\ T^q x\rangle \geq 0.$$

Hence the result. □

Definition 2.2. *An operator T is said to have the single valued extension properties if, for every open subset $\mathcal{U}$ of $\mathbb{C}$,an analytic function $f : \mathcal{U} \longrightarrow \mathcal{H}$ satisfies $(T-\lambda)f(\lambda) = 0 \ \forall\, \lambda \in \mathcal{U}$, then $f(\lambda) = 0 \ \forall\ \lambda \in \mathcal{U}$.*

Theorem 2.8. ([11]) *An m-isometric operator T has the single valued extension property.*

In the following theorem, we extend this result to some (m,q)-partial isometries.

Theorem 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$ be an (m,q)-partial isometry such that $\mathcal{N}(T^q)$ is a reducing subspace for T. Then T has the single valued extension properties.*

Proof. Assume that T is (m,q)-partial isometry for some positive integer m. Let $\lambda \in \mathbb{C}$ and let $\mathcal{U}$ be any open neighborhood of λ in $\mathbb{C}$. Assume that f is an analytic $\mathcal{H}$-valued function defined on $\mathcal{U}$ such that

$$\big(T-\lambda\big)f(\lambda) \equiv 0 \ \text{ on } \mathcal{U}. \tag{2.11}$$

Let $f(\lambda) = f_1(\lambda) \oplus f_2(\lambda) \in \mathcal{N}(T^q) \oplus \mathcal{N}(T^q)^{\perp}$, then we have

$$\big(T-\lambda\big)f(\lambda) \equiv 0 \Longleftrightarrow (T-\lambda)f_1(\lambda) + (T-\lambda)f_2(\lambda) \equiv 0 \text{ on } \mathcal{U}.$$

We deduce that $(T-\lambda)T^q f_2(\lambda) = 0$ on $\mathcal{U}$.
Since T is an m-isometry on $\mathcal{N}(T^q)^{\perp}$ and hence has the single valued extension property (Theorem 2.8) , then $T^q f_2(\lambda) \equiv 0$ and $f_2(\lambda) \equiv 0$. Consequently

$$(T-\lambda)f(\lambda) = 0 \Longleftrightarrow (T-\lambda)f_1(\lambda) = 0.$$

Thus $(T-\lambda)f_1(\lambda) = 0$ implies that $\lambda^q f_1(\lambda) = 0$ and $f_1(\lambda) = 0$ if $\lambda \neq 0$. Since $f_1(\lambda) = 0$ if $\lambda \neq 0$ and f_1 is analytic, $f_1 \equiv 0$. □

Theorem 2.10. *The class of (m,q)-partial isometries is closed subset of $\mathcal{B}(\mathcal{H})$ equipped with the uniform operator (norm) topology.*

Proof. To see that the class of $(m.q)$-partial isometries is closed, we prove that any strong limit $T \in \mathcal{B}(\mathcal{H})$ of a sequence (T_p) in the class of (m,q)]-partial isometry also belongs to the class of (m,q)-partial isometries , i.e., we let (T_p) be a sequence of operators in the class of $(m.q)$ -partial isometries converging to $T \in \mathcal{B}(\mathcal{H})$ in norm:

$$\|T_p x - Tx\| \longrightarrow 0 \ \ as \ \ p \longrightarrow \infty, \text{ for each } x \in H.$$

Hence it follows that

$$\|T_p^* x - T^* x\| = \|(T_p - T)^* x\| \leq \|(T_p - T)^*\|\|x\| = \|T_p - T\|\|x\| \longrightarrow 0,$$

whence (T_p^*) converges strongly to T^*.
Since the product of operators is sequentially continuous in the strong topology (see [17], p.62), one concludes that $T_p^q T_p^{*k} T_p^k$ converge strongly to $T^q T^{*k} T^k$. Hence the limiting case of (2.1) shows that T belongs to the class of (m,q)-partial isometries, completing the proof. □

References

[1] J. Agler and M. Stankus, m-Isometric transformations of Hilbert space I, Integral Equations and Operator Theory, 21 (1995), 383-429.

[2] J. Agler, M. Stankus, m-Isometric transformations of Hilbert space II, Integral Equations Operator Theory 23 (1) (1995) 1-48.

[3] J.Agler, M. Stankus, m-Isometric transformations of Hilbert space III, Integral Equations Operator Theory 24 (4) (1996) 379-421.

[4] F.Bayart, m-isometries on Banach spaces, Math. Nachr. 284 (2011), 2141-2147.

[5] T.Bermúdez, C. Díaz-Mendoza, A.Martinón, Powers of m-isometries, Studia Math. 208 (2012) 249-255.

[6] T.Bermúdez, A.Martinón, J.A.Noda, Products of m-isometries, Linear Algebra and its Applications 438 (2013) 80-86.

[7] T.Bermúdez, A.Martinón, V.Müller, and J. A. Noda, Perturbation of m-isometries by nilpotent operators, Abstr. Appl. Anal., vol.2014. Article ID 745479, 6 pages.

[8] T.Bermúdez, A.Martinón, and E.Negrín, Weighted shift operators which are m-isometries, Integral Equations Operator Theory 68 (2010), No. 3, 301- 312.

[9] F.Botelho, On the existence of n-isometries on l_p- spaces. Acta Sci. Math. (Szeged) 76 (2010), No. 1-2, 183-192.

[10] F.Botelho and J. Jamison, Isometric properties of elementary operators, Linear Algebra Appl. 432 (2010), 357-365.

[11] M. Chõ , S.Ôta, K.Tanahashic, A. Uchiyama, Spectral properties of m-isometric. operators. Functional Analysis, Approximation and Computation.4:2 (2012), 33-39

[12] J. B. Conway, A course in Functional analysis Second Edition. Springer-Verlag 1990.

[13] B.P.Duggal, Tensor product of n-isometries, Linear Alg. Appl. 437(2012), 307-318.

[14] B.P.Duggal, Tensor product of n-isometries II Functional Analysis, Approximation and Computation 4:1(2012), 27-32.

[15] B.P.Duggal, Tensor product of n-isometries III, Functional Analysis, Approximation and Computation 4:2(2012), 61-67.

[16] S.H.Jah, Power of m-Partial isometries on Hilbert spaces.Bulletin of Mathematical Analysis and Applications Volume 5 Issue 3 (2013), Pages 79-89.

[17] P.R. Halmos, A Hilbert Space Problem Book. Springer-Verlag, New York. (1982).

[18] P. R. Halmos and L. J. Wallen, Powers of partial isometries , J. Math. Mech, 19 (1970), 657-663.

[19] J.W.Helton, Operators with a representationby x on a sobolev spaces.Colloquia Math Soc.J.B.5 (1970).

[20] P. Hoffman, M. Mackey and M. Ó Searcóid, On the second parameter of an $(m;p)$-isometry, Integral Equation and Operator Theory71(2011), 389-405.

[21] A.Olofsson, An operator-valued Berezin transform and the class of n-hypercontractions, Integral Equation and Operator Theory 58 (2007) 503-549.

[22] O.A.Mahmoud Sid Ahmed, m-isometric operators on Banach spaces, Asian-European J. Math. 3(2010), 1-19.

[23] S. M. Patel, 2-Isometric Operators, Glasnik Matematicki. Vol. 37(57)(2002), 143-147.

[24] A.Saddi and O. A.Mahmoud Sid Ahmed, m-partial isometries on Hilbert spaces Intern .J. Funct. Anal., Operators Theory Appl. 2 (2010), No. 1, 67-83.