International Journal of Pure and Applied Mathematics
Volume 104 No. 4 2015, 509-516
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v104i4.2

ijpam.eu

STABILITY AND LYAPUNOV EXPONENTS OF A q-DEFORMED MAP

Bhagwati Prasad[1 §], Kuldip Katiyar[2]
[1,2]Department of Mathematics
Jaypee Institute of Information Technology
A-10, Sector-62, Noida, UP-201307 INDIA

Abstract: The aim of this paper is to study the stability analysis of q-deformed chaotic map under Mann iterative scheme. We also plot the time series and Lyapunov exponent for changing values of the parameters.

AMS Subject Classification: 37B25, 37C75, 65P40
Key Words: logistic map, q-deformation, Mann iteration, Lyapunov exponent

1. Introduction

Verhulst postulated that the growth rate at any time should be proportional to the fraction of the environment that is not yet used up by the population at that time. The discrete version of the Verhlusts model was further expressed by R. May [12] by the following difference equation

$$x_{n+1} = rx_n(1 - x_n), \tag{1}$$

where x_n (a real number between 0 and 1) represents population density at time $n = 1, 2, 3, ...$ and r is used for the combined rate for reproduction and starvation (see also [13] and [15]).

Received: April 17, 2015

§Correspondence author

This is one of the most popular nonlinear models used to describe various physical and natural systems. For a details discussions and applications of logistic map one may refer [2], [3], [5], [6], [7], [8], [10], [14], [16], [17] and several references therein. A deformation of the logistic map has been proposed by Banerjee and Parthasarathy [1]. Some of the physical problems show quantum group structures therefore the q-deformed physical systems has been the subject of intense research in the recent past (see, for e.g., [4] and references therein). In q-deformation of a function there is some modification in function such that in the limiting case $q \to 1$ the modified function changes to original function. Theory of quantum integrable systems has initiated a new type of symmetry and the mathematical objects associated with it are called quantum groups. These groups are related to the usual Lie groups in the same manner as quantum mechanics is related to its classical limit. It is believed that the various interactions in many physical systems are taken into account through q-deformation more effectively. The q-deformed logistic map may also be used to show the exceptional phenomena of the co-existence of the chaotic and normal behaviour of a system. In this paper, we study the stability of this map through time series analysis and visualize the Lyapunov exponent patterns of such map for varying values of the parameters under Mann iteration scheme.

2. Preliminaries

In this section, we present the basic concepts required for our work.

Definition 1. **[2].** Let X be a non empty space and $f : X \to X$. A point $p \in X$ is called a periodic point of f of period $n \geq 1, n \in N$, iff $f^n(p) = p$ and $f^k(p) \neq p$ for all $k = 1, 2, ..., n-1$, where $f^k(p) := \underbrace{f(f(...(f}_{k\text{times}}(p))...))$.

A periodic point of f of period 1 is simply a fixed point of f.

Definition 2. **[2].** Let (X, d) be a metric space and $f : X \to X$. The orbit of a point x in X under the transformation f is defined as a sequence $\{f^n(x) : n = 0, 1, 2, \ldots\}$. This transformation f may also be called a dynamical system denoted by $\{X, f\}$.

Let X be a non empty set and $f : X \to X$. Then for any point x_0 in X, the iterative scheme $x_{n+1} = f(x_n), n = 0, 1, 2, \ldots$ is called Picard iterate and an Picard orbit is defined as

$$O(f, x_0) = \{x_n : x_n = f(x_{n-1}), n = 0, 1, 2, \ldots\} \tag{2}$$

Now we define a two step feedback scheme essentially due to Mann [11].

Definition 3. Let X be a non empty set and $f : X \to X$. For a point x_0 in X, construct a sequence $\{x_n\}$ in the following manner:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)x_n \tag{3}$$

for $n = 0, 1, 2, \ldots$, where $0 < \alpha_n \leq 1$. Then sequence $\{x_n\}$ is called Mann iterate of a point x_0 and it is denoted by $MO(f, x_0, \alpha_n)$.

It is remarked that (3) with $\alpha_n = 1$ is the Picard iteration (2).

Prasad and Katiyar [18], [19], [20], [21] and [22] studied the map (1) for different iterative schemes and interesting fractal patterns are generated in [19], [20] and [22] .

Definition 4. Let f be a continuous map of the real line R. The Lyapunov exponent $LE(x_1)$ of the orbit $\{x_1, x_2, x_3, \ldots\}$ is dened as

$$LE(x_1) = \lim_{n\to\infty} (1/n)[\ln |f'(x_1)| + \ln |f'(x_2)| + \ldots + \ln |f'(x_n)|],$$

if this limit exists. The orbit $\{x_n\}$ is generated by the rule (3).

3. Stability Analysis

Banerjee and Parthasarathy [1] proposed the following Quantumgroup (Qu-group) type deformation, called q-deformation of the logistic map.

$$[x_{n+1}] = r[x_n](1 - [x_n]) \tag{4}$$

where $[x] = \frac{1-q^{x}}{1-q}$. such that $[x] \to x$ when $q \to 1$. Here q is real and x is in the interval $[0, 1]$.

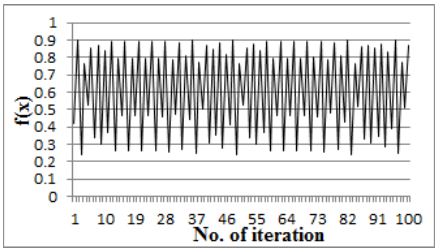
The q-deformed logistic map with Mann iteration (3) with $\alpha_n = \alpha$ is given by

$$[x_{n+1}] = \alpha r[x_n](1 - [x_n]) + (1 - \alpha)[x_n], \tag{5}$$

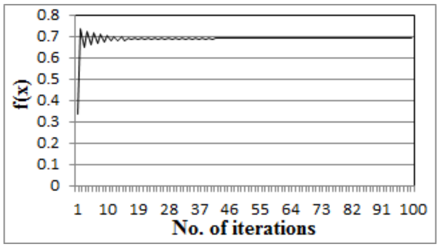
where $0 < \alpha \leq 1$ and $[x] = \frac{1-q^{x}}{1-q}$ Notice that the deformed map (4) is concave in parts of x-space while (1) is always convex.

3.1. Stability Analysis through Time Series

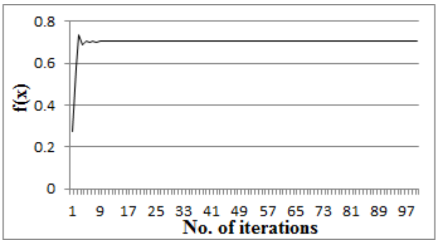
In this section, we attempt to investigate the behaviour of q-deformed logistic map by time series analysis. We first study the behavior of q-deformed logistic



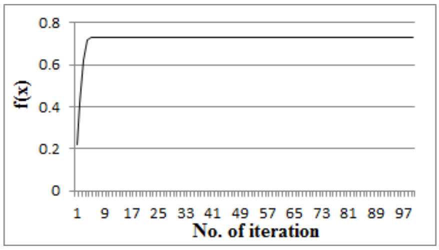
(**a**). $(r, \alpha) = (3.6, 1)$



(**b**). $(r, \alpha) = (3.5, 0.7)$



(**c**). $(r, \alpha) = (3.6, 0.5)$



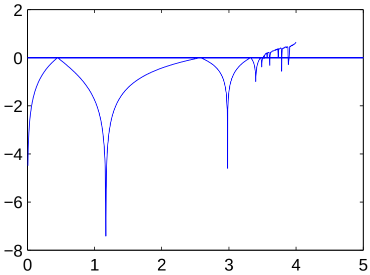
(**d**). $(r, \alpha) = (3.5, 0.3)$

Figure 1: Time-Series for $x_0 = 0.1$ and $q = 0.5$.

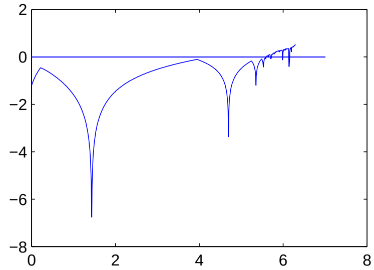
orbits generated by (5) using time series. For comparing the behaviors of the model via new iterative scheme, we consider the same values of x_0, q and r as taken by Banerjee and Parthasarathy [1] and vary α from 1 to 3. It is observed that the cyclic behavior of logistic map as in case of [1] changes to stable behavior as shown in Figures 1(a)-(d) .

3.2. Stability Analysis through Lyapunov Exponent

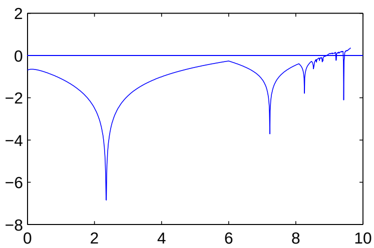
In this section, we investigate the behaviour of q-deformed logistic map through its Lyapunov exponent (LE). Besides measuring the sensitive dependence upon the initial condition, the LE of a function also measures the stretching rate per iteration averaged over the trajectory. It gives an indication of divergence or convergence of the orbits starting close together. Thus it has an important role in identifying the behaviour of a dynamical system. The orbit is attracted towards a stable fixed point or stable periodic orbit when LE is negative for a given r. The system exhibits some sort of steady state mode when LE is zero. The positive value of the LE characterizes an unstable and chaotic behavior of the system for a given r. A number of authors have explored the LE and studied the behaviour of the dynamical systems see for instance, Huberman



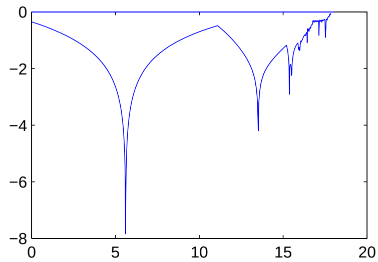
(**a**). $\alpha = 1$



(**b**). $\alpha = 0.7$



(**c**). $\alpha = 0.5$



(**c**). $\alpha = 0.3$

Figure 2: Lyapunov Exponents for $x_0 = 0.1$ and $q = 0.5$.

and Rudnick [7], Giesel et al [9] and McCartney [14] and references thereof. We obtain the LE of the q-deformed logistic map for the same values of the parameters α as taken in section 3.1 by varying the parameter α and fixing x_0 at 0.1 and $q = 0.15$. The Lyapunov exponents and the expected behavior of the orbits at the chosen values of the parameters are shown in Figures 2(a)-(d) and Table 1.

4. Conclusion

The time series and Lyapunov exponent analysis for q-deformed logistic mapping under Mann iterative scheme are studies for same specific choices of the parameters α and q. The chaotic behavior of the mapping shown for Picard iterative scheme is found to be stable when we use the two steps Mann iterative scheme.

α	r	LE	Nature of orbit
1	$3.5106 < r < 3.5218$ $3.5249 < r < 3.3614$ $3.6126 < r < 3.3819$ $3.8959 < r < 3.9999$	Positive	Chaotic orbit
	$0 < r < 3.5106$ $3.5218 < r < 3.5249$ $3.3614 < r < 3.6126$ $3.3819 < r < 3.8959$	Negative	Stable orbit
	$r = 1.1662$	Least negative	More stable orbit
	$r = 4$	—	Unstable orbit
0.7	$5.5962 < r < 5.6938$ $5.7200 < r < 5.9818$ $5.9929 < r < 6.1343$ $6.1544 < r < 6.2923$	Positive	Chaotic orbit
	$0 < r < 5.5962$ $5.6938 < r < 5.7200$ $5.9818 < r < 5.9929$ $6.1343 < r < 6.1544$	Negative	Stable orbit
	$r = 2.6747$	Least negative	More stable orbit
	$r > 6.2923$	—	Unstable orbit
0.5	$8.9173 < r < 9.1929$ $9.2166 < r < 9.4135$ $9.4545 < r < 9.6296$	Positive	Chaotic orbit
	$0 < r < 8.9173$ $9.1929 < r < 9.2166$ $9.4135 < r < 9.4545$	Negative	Stable orbit
	$r = 2.3524$	Least negative	More stable orbit
	$r > 9.6296$	—	Unstable orbit
0.3	$0 < r < 17.8378$	Negative	Stable orbit
	$r = 5.6036$	Least negative	More stable orbit
	$r > 17.8378$	—	Unstable orbit

Table 1: The expected behavior of the orbits for r and α

References

[1] S. Banerjee, R. Parthasarathy, A q-deformed logistic map and its implications, *J. Phys. A: Math. Theor*, **44**, No. 4 (2011), 04510.

[2] M.F. Barnsley, *Fractals Everywhere*, 2nd ed. Revised with the assistance of and a foreword by Hawley Rising, III. Boston MA. Academic Press Professional (1993).

[3] E. F. Camacho, C. Bordons, *Model Predictive Control*, Springer, Berlin (1999).

[4] M Chaichian, A Demichev, *Introduction to Quantum Groups*, World Scientific, Singapore (1996).

[5] D.R. Dettmer, *Chaos and engineering*, IEE Review, **39**, No. 5 (1993), 199-203.

[6] R.L. Devaney, *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, Addison-Wesley (1992).

[7] B.A. Huberman, J. Rudnick, Scaling Behavior of Chaotic Flows, *Phys. Rev. Lett.*, **45** No. 3, (1980), 154156.

[8] R.A. Holmgren, *A rst Course in Discrete Dynamical Systems*, Springer-Verlag (1994).

[9] T. Giesel, J. Nierwetberg, J. Keller, Critical behavior of the Lyapunov number at the period-doubling onset of chaos, *Phys. Lett. A*, **86**, No. 2 (1981), 7578.

[10] J. Kint, D. Constales, and A. Vanderbauwhede, Pierre-Francois Verhulsts final triumph, In: *The logistic map and the route to chaos: from the beginnings to modern applications*, M. Ausloos, M. Dirickx, Springer-Verlag (2006).

[11] W.R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4**, No. 3 (1953), 506-510.

[12] R.M. May, Simple mathematical models with very complicated dynamics, *Nature*, **261**, No. 459 (1976), 459-475.

[13] R.M. May, and G.F. Oster, Bifurcations and dynamic complexity in simple biological models, *The American Naturalist*, **110**, No. 974 (1976), 573-599.

[14] M. McCartney, Lyapunov exponent for multi-parameter tent and logistic maps, *Chaos*, **21**, No. 4 (2011), 1-7.

[15] H. Pastijn, Chaotic growth with the logistic model of P.-F. Verhulst, In: *The logistic map and the route to chaos: from the beginnings to modern applications*, M. Ausloos, M. Dirickx, Springer-Verlag (2006).

[16] H. Peitgen, H. Jurgens, D. Saupe, *Chaos and Fractals: New Frontiers of Science*, Springer-Verlag (2004).

[17] H. Peitgen, D. Saupe, eds., *The Science of Fractal Images*, Springer-Verlag (1988).

[18] B. Prasad, K. Katiyar, A comparative study of logistic map through function iteration, *Proc. Int. Con. Emerging Trends in Engineering and Technology*, **IETET 2010** (2010), 357-359.

[19] B. Prasad, K. Katiyar, Fractal Patterns of the Noor Iterates of Complex Logistic Map. *Proceedings of the International Conference on Recent Trends in Computing*, **ICRTC-2012** (2012), 229-232.

[20] B. Prasad, K. Katiyar, Dynamics of Julia Sets for Complex Exponential Functions, In: *ICMMSC 2012*, P. Balasubramaniam, R. Uthayakumar, CCIS, Springer, Heidelberg, **283**, No. 19 (2012), 185-192.

[21] B. Prasad, K. Katiyar, A Stability Analysis of Logistic Model, *International Journal of Nonlinear Science*, **17**, No. 1, (2014), 71-79.

[22] B. Prasad, K. Katiyar, Stability and fractal patterns of complex logistic map, *Cybernetics and Information Technologies*, **14**, No. 3 (2014), 14-24.