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ON SUBMANIFOLDS OF GENERALIZED RECURRENT MANIFOLDS

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Abstract: In this paper we investigate some properties of submanifolds of generalized recurrent manifold (briefly, GRM_n). Firstly, we show that a totally geodesic hypersurface of GRM_n is a GRM_{n-1}. Secondly, if GRM_n is a Riemannian product manifold, then either one decomposition manifold is locally symmetric or the other decomposition manifold is a space of constant curvature. Thirdly, a Riemannian product manifold of a space of constant curvature with itself is a GRM_n.

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Key Words: totally umbilic hypersurface, totally geodesic hypersurface, generalized recurrent manifold, Riemannian product manifold, associated 1-forms, a space of constant curvature

1. Introduction

As a generalization of the notion of a space of constant curvature, the notion of (locally) symmetric manifolds was introduced by Cartan [4] who obtained a classification of such manifolds. During the last six decades, the notion of

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locally symmetric manifolds was weakened by many authors in various ways to different extent such as recurrent manifolds [10], conformally recurrent manifolds [1], 2-recurrent manifolds [7], Ricci recurrent manifolds [9], concircularly recurrent manifolds [8] and projectively recurrent manifolds [2]. As an extending notion of recurrent manifolds, the notion of generalized recurrent manifolds was introduced by Dubey [6] and this manifold has received a great deal of attention. In [3] Arslan et al studied such a manifold in considerable detail. A Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be generalized recurrent if its curvature tensor R of type (0,4) satisfies the condition

$$(\nabla_X R)(Y, Z, V, W) = A(X)R(Y, Z, V, W) + B(X)(g \bullet g)(Y, Z, V, W), \quad (1.1)$$

where ∇ denotes the Levi-Civita connection and A, B are the associated 1-forms. Here the symbol $\bullet$ is the Nomizu-Kulkarni product of symmetric (0,2)-tensors generating a curvature type tensor:

$$(h \bullet k)(X, Y, Z, W) = h(X, Z)k(Y, W) + h(Y, W)k(X, Z) \\ - h(X, W)k(Y, Z) - h(Y, Z)k(X, W).$$

From now on, in this paper, an n-dimensional generalized recurrent manifold is denoted by GRM_n. In particular, if $B = 0$ in (1.1), then the manifold reduces to a recurrent manifold. The purpose of this paper is to investigate some properties of a Riemannian product GRM_n and hypersurfaces of GRM_n.

2. Preliminaries

Let (M^n, g) be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; y^\alpha\}$ and $(\bar{M}^{n-1}, \bar{g})$ a hypersurface of (M^n, g) covered by a system of coordinate neighborhoods $\{V; x^i\}$. Let $y^\alpha = y^\alpha(x^i)$ be the parametric representation of the hypersurface $\bar{M}^{n-1}$ in M^n, where Greek indices take the values $1, 2, ..., n$ and Latin indices take the values $1, 2, ..., n-1$. Then we have

$$\bar{g}_{ij} = g_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}.$$

Here we adopt the Einstein convention, that is, when an index variable appears once in an upper and once in a lower position in a term, it implies summation of that term over all the values of the index. Let N^α be a local unit normal to

$(\bar{M}^{n-1}, \bar{g})$. Then we have the relations

$$g_{\alpha\beta}N^{\alpha}\frac{\partial y^{\beta}}{\partial x^{j}} = 0, g_{\alpha\beta}N^{\alpha}N^{\beta} = 1, g^{\alpha\beta} = \bar{g}^{ij}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}} + N^{\alpha}N^{\beta}.$$

The structure equations of Gauss and Codazzi for a hypersurface $(\bar{M}^{n-1}, \bar{g})$ of (M^n, g) can be respectively written as

$$\bar{R}_{ijkl} = R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}}\frac{\partial y^{\gamma}}{\partial x^{k}}\frac{\partial y^{\delta}}{\partial x^{l}} + \bar{\omega}_{il}\bar{\omega}_{jk} - \bar{\omega}_{ik}\bar{\omega}_{jl}, \tag{2.1}$$

$$R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}}\frac{\partial y^{\gamma}}{\partial x^{k}}N^{\delta} = \bar{\omega}_{jk;i} - \bar{\omega}_{ik;j}, \tag{2.2}$$

where $\bar{R}_{ijkl}$ and $R_{\alpha\beta\gamma\delta}$ are the curvature tensors of $(\bar{M}^{n-1}, \bar{g})$ and (M^n, g) respectively, and $\bar{\omega}_{ij}$ is the second fundamental form of $(\bar{M}^{n-1}, \bar{g})$.

The hypersurface $(\bar{M}^{n-1}, \bar{g})$ is said to be a totally umbilic hypersurface of (M^n, g) [5] if its second fundamental form $\bar{\omega}_{ij}$ satisfies

$$\bar{\omega}_{ij} = H\bar{g}_{ij}, (\frac{\partial y^{\alpha}}{\partial x^{i}})_{;j} = \bar{g}_{ij}HN^{\alpha}, \tag{2.3}$$

where H denotes the mean curvature of $(\bar{M}^{n-1}, \bar{g})$ defined by $H = \frac{1}{n-1}\bar{g}^{ij}\bar{\omega}_{ij}$, and semicolon ";" indicates covariant differentiation. In particular, if H=0, then the totally umbilic hypersurface $(\bar{M}^{n-1}, \bar{g})$ is called a totally geodesic hypersurface of (M^n, g) [5]. The equations of Weingarten, Gauss and Codazzi for a totally umbilic hypersurface $(\bar{M}^{n-1}, \bar{g})$ of (M^n, g) are respectively obtained as

$$N^{\alpha}_{;i} = -H\frac{\partial y^{\alpha}}{\partial x^{i}}, \tag{2.4}$$

$$\bar{R}_{ijkl} = R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}}\frac{\partial y^{\gamma}}{\partial x^{k}}\frac{\partial y^{\delta}}{\partial x^{l}} + H^2(\bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}), \tag{2.5}$$

$$R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}}\frac{\partial y^{\gamma}}{\partial x^{k}}N^{\delta} = H_{;i}\bar{g}_{jk} - H_{;j}\bar{g}_{ik}. \tag{2.6}$$

3. Hypersurfaces of GRM_n

In this section we deal with some hypersurfaces of GRM_n. At first, concerning the covariant derivative of curvature tensor, we obtain

Lemma 3.1. *Let $(\bar{M}^{n-1}, \bar{g})$ be a totally umbilic hypersurface of (M^n, g). Then we have*

$$\bar{R}_{ijkl;p} = R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} \frac{\partial y^\mu}{\partial x^p} +$$
$$+ HH_{;i}(\bar{g}_{lp}\bar{g}_{jk} - \bar{g}_{kp}\bar{g}_{jl}) + HH_{;j}(\bar{g}_{il}\bar{g}_{kp} - \bar{g}_{ik}\bar{g}_{lp}) + HH_{;k}(\bar{g}_{jp}\bar{g}_{li} - \bar{g}_{ip}\bar{g}_{lj}) +$$
$$+ HH_{;l}(\bar{g}_{ip}\bar{g}_{kj} - \bar{g}_{jp}\bar{g}_{ki}) + 2HH_{;p}(\bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}), \tag{3.1}$$

$$R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} N^\delta \frac{\partial y^\mu}{\partial x^p} + H(R_{\alpha\beta\gamma\delta} N^\alpha \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} N^\delta \bar{g}_{ip} +$$
$$+ R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} N^\beta \frac{\partial y^\gamma}{\partial x^k} N^\delta \bar{g}_{jp} + R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} N^\gamma N^\delta \bar{g}_{kp} +$$
$$- R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^p}) = H_{;ip}\bar{g}_{jk} - H_{;jp}\bar{g}_{ik}. \tag{3.2}$$

Proof. Differentiating (2.6) covariantly, we have

$$\bar{R}_{ijkl;p} = R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\mu}{\partial x^p} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + R_{\alpha\beta\gamma\delta} (\frac{\partial y^\alpha}{\partial x^i})_{;p} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} +$$
$$+ R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} (\frac{\partial y^\beta}{\partial x^j})_{;p} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} (\frac{\partial y^\gamma}{\partial x^k})_{;p} \frac{\partial y^\delta}{\partial x^l} +$$
$$+ R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} (\frac{\partial y^\delta}{\partial x^l})_{;p} + 2HH_{;p}(\bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}).$$

By virtue of (2.4) and the last relation, we obtain

$$\bar{R}_{ijkl;p} = R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\mu}{\partial x^p} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + \bar{g}_{ip} H R_{\alpha\beta\gamma\delta} N^\alpha \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} +$$
$$+ \bar{g}_{jp} H R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} N^\beta \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + \bar{g}_{kp} H R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} N^\gamma \frac{\partial y^\delta}{\partial x^l} +$$
$$+ \bar{g}_{lp} H R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} N^\delta + 2HH_{;p}(\bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}).$$

It follows from (2.7) that the last relation reduces to

$$\bar{R}_{ijkl;p} = R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} \frac{\partial y^\mu}{\partial x^p} +$$
$$+ HH_{;i}(\bar{g}_{lp}\bar{g}_{jk} - \bar{g}_{kp}\bar{g}_{jl}) + HH_{;j}(\bar{g}_{il}\bar{g}_{kp} - \bar{g}_{ik}\bar{g}_{lp}) + HH_{;k}(\bar{g}_{jp}\bar{g}_{li} - \bar{g}_{ip}\bar{g}_{lj}) +$$
$$+ HH_{;l}(\bar{g}_{ip}\bar{g}_{kj} - \bar{g}_{jp}\bar{g}_{ki}) + 2HH_{;p}(\bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}).$$

On the other hand, differentiating (2.7) covariantly, we get

$$\begin{aligned}
&R_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\mu}{\partial x^p}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta \\
&+R_{\alpha\beta\gamma\delta}(\frac{\partial y^\alpha}{\partial x^i})_{;p}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta+R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}(\frac{\partial y^\beta}{\partial x^j})_{;p}\frac{\partial y^\gamma}{\partial x^k}N^\delta \\
&+R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}(\frac{\partial y^\gamma}{\partial x^k})_{;p}N^\delta+R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta_{;p} \\
=&H_{;ip}\bar{g}_{jk}-H_{;jp}\bar{g}_{ik}.
\end{aligned}$$

Taking account of (2.4), (2.5) and the last relation, we have

$$\begin{aligned}
&R_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\frac{\partial y^\mu}{\partial x^p} \\
&+H(R_{\alpha\beta\gamma\delta}N^\alpha\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\bar{g}_{ip}+R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}N^\beta\frac{\partial y^\gamma}{\partial x^k}N^\delta\bar{g}_{jp} \\
&+R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}N^\gamma N^\delta\bar{g}_{kp}-R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^p}) \\
=&H_{;ip}\bar{g}_{jk}-H_{;jp}\bar{g}_{ik}.
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let (M^n, g) be a GRM_n. If $(\bar{M}^{n-1}, \bar{g})$ is a totally geodesic hypersurface of (M^n, g), then the manifold $(\bar{M}^{n-1}, \bar{g})$ is a GRM_{n-1}.*

Proof. By virtue of $H = 0$, we have from (3.8)

$$\bar{R}_{ijkl;p}=R_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p}.$$

Since (M^n, g) is a GRM_n, the last relation yields from (1.1)

$$\bar{R}_{ijkl;p}=A_\mu R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p}+B_\mu(g\bullet g)_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p}.$$

Because of (2.6) and $H = 0$, the last relation reduces to

$$\bar{R}_{ijkl;p}=A_p\bar{R}_{ijkl}+B_p(\bar{g}\bullet\bar{g})_{ijkl},$$

showing that the manifold is a GRM_{n-1}. This completes the proof. □

4. Riemannian Product GRM_n

Let (M^n, g) be a Riemannian product manifold $(M^p \times M^{n-p}, \widehat{g} + \widetilde{g})$. In local coordinates, we adopt the Latin indices (resp. the Greek indices) for tensor components which are constructed on $(M^p, \widehat{g})$ (resp. $(M^{n-p}, \widetilde{g})$). Therefore, the Latin indices take the values from $1, ..., p$ whereas the Greek indices run over the range $p + 1, ..., n$. Now we can state the followings.

Theorem 4.1. *Let a Riemannian product manifold $(M^p \times M^{n-p}, \widehat{g} + \widetilde{g})$ be a GRM_n. Then either one decomposition manifold $(M^p, \widehat{g})$ is locally symmetric or the other decomposition manifold $(M^{n-p}, \widetilde{g})$ is a space of constant curvature.*

Proof. Since any tensor components of R and its covariant derivatives with both Latin and Greek indices together should be zero, we have from (1.1) and $R_{\alpha\beta\gamma\delta;p} = 0$

$$0 = A_p R_{\alpha\beta\gamma\delta} + B_p (g \bullet g)_{\alpha\beta\gamma\delta}. \tag{4.1}$$

If we assume that $A_p = 0$, then from (4.10) and $(g \bullet g)_{\alpha\beta\gamma\delta} \neq 0$ it follows that $B_p = 0$, which yields from (1.1)

$$R_{ijkl;p} = 0, \tag{4.2}$$

showing that $(M^p, \widehat{g})$ is locally symmetric.

On the other hand, if we assume that $A_p \neq 0$, then it follows from (4.10) that

$$R_{\alpha\beta\gamma\delta} = -\frac{B_p}{A_p} (g \bullet g)_{\alpha\beta\gamma\delta}, \tag{4.3}$$

showing that $(M^{n-p}, \widetilde{g})$ is a space of constant curvature. This completes the proof. □

Theorem 4.2. *Let (M^n, g_c) be a space of constant curvature. Then the Riemannian product manifold (M^{2n}, g) of (M^n, g_c) with itself is a GRM_{2n}.*

Proof. Since (M^n, g_c) is a space of constant curvature, we have

$$R^c_{ijkl} = \frac{s}{2n(n-1)} (g_c \bullet g_c)_{ijkl} \tag{4.4}$$

and

$$R^c_{ijkl;p} = 0. \tag{4.5}$$

Here R^c and s denote the curvature tensor and the scalar curvature respectively on (M^n, g_c). Therefore, from (4.13) and (4.14) it follows that

$$R^c_{ijkl;p} = R^c_{ijkl} - \frac{s}{2n(n-1)}(g_c \bullet g_c)_{ijkl}. \tag{4.6}$$

Now we consider the Riemannian product manifold (M^{2n}, g) of a space of constant curvature (M^n, g_c) with itself. Obviously the Riemannian curvature tensor R of (M^{2n}, g) satisfies

$$R_{ijkl} = R^c_{ijkl} + R^c_{ijkl}$$

and

$$R_{ijkl;p} = 0 = R^c_{ijkl;p} + R^c_{ijkl;p},$$

which yields from (4.15) and the last relations

$$R_{ijkl;p} = R_{ijkl} - \frac{s}{2n(n-1)}(\frac{1}{2})(g \bullet g)_{ijkl}$$

because of

$$(g_c \bullet g_c) + (g_c \bullet g_c) = \frac{1}{2}(g_c + g_c) \bullet (g_c + g_c) = \frac{1}{2} g \bullet g.$$

Therefore we have

$$R_{ijkl;p} = A_p R_{ijkl} + B_p (g \bullet g)_{ijkl},$$

where $A_p = 1$ and $B_p = -\frac{s}{4n(n-1)}$, showing that the Riemannian product manifold (M^{2n}, g) of a space of constant curvature (M^n, g_c) with itself is a GRM_{2n}. This completes the proof. □

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