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BOUNDS FOR A TOADER-TYPE MEAN BY ARITHMETIC AND CONTRAHARMONIC MEANS

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Abstract: In this paper, we present the best possible parameters α_i and β_i with $i = 1, 2, 3, 4$ such that the double inequalities

$$\alpha_1 A(a,b) + (1-\alpha_1)\, C(a,b) < T[A(a,b), C(a,b)] < \beta_1 A(a,b) + (1-\beta_1)\, C(a,b),$$

$$A^{\alpha_2}(a,b) C^{1-\alpha_2}(a,b) < T[A(a,b), C(a,b)] < A^{\beta_2}(a,b) C^{1-\beta_2}(a,b),$$

$$\frac{\alpha_3}{A(a,b)} + \frac{1-\alpha_3}{C(a,b)} < \frac{1}{T[A(a,b), C(a,b)]} < \frac{\beta_3}{A(a,b)} + \frac{1-\beta_3}{C(a,b)},$$

$$C[\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a] < T[A(a,b), C(a,b)] < C[\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a]$$

hold for all $a, b > 0$ with $a \neq b$, as consequences, we provide several new bounds for the complete elliptic integral $\mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2\theta)^{1/2} d\theta$ $(r \in (0, \sqrt{3}/2)$ of the second kind, where $T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta$, $A(a,b) = (a+b)/2$ and $C(a,b) = (a^2+b^2)/(a+b)$ are the Toader, arithmetic and contraharmonic means of a and b, respectively.

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1. Introduction

For $a, b > 0$ with $a \neq b$, let $p \in [0,1]$ and $q \in \mathbb{R}$, then the p-th generalized Seiffert mean $S_p(a,b)$, q-th Gini mean $G_q(a,b)$, q-th power mean $M_q(a,b)$, q-th Lehmer mean $L_q(a,b)$, harmonic mean $H(a,b)$, geometric mean $G(a,b)$, quadratic mean $Q(a,b)$, centroidal mean $\overline{C}(a,b)$, arithmetic mean $A(a,b)$ and contraharmonic mean $C(a,b)$ are respectively defined by

$$S_p(a,b) = \begin{cases} \frac{p(a-b)}{\arctan\left[\frac{2p(a-b)}{a+b}\right]}, & 0 < p \leq 1, \\ \frac{a+b}{2}, & p = 0, \end{cases}$$

$$G_q(a,b) = \begin{cases} \left(\frac{a^{q-1}+b^{q-1}}{a+b}\right)^{1/(q-2)}, & q \neq 2, \\ \left(a^a b^b\right)^{1/(a+b)}, & q = 2, \end{cases}$$

$$M_q(a,b) = \begin{cases} \left(\frac{a^q+b^q}{2}\right)^{1/q}, & q \neq 0, \\ \sqrt{ab}, & q = 0, \end{cases}$$

$$L_q(a,b) = \frac{a^{q+1}+b^{q+1}}{a^q+b^q}, \quad H(a,b) = \frac{2ab}{a+b}, \quad G(a,b) = \sqrt{ab},$$

$$Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}, \quad \overline{C}(a,b) = \frac{2(a^2+ab+b^2)}{3(a+b)}$$

$$A(a,b) = \frac{a+b}{2}, \quad C(a,b) = \frac{a^2+b^2}{a+b}. \tag{1.1}$$

It is well known that $S_p(a,b)$, $G_q(a,b)$, $M_q(a,b)$ and $L_q(a,b)$ are continuous and strictly increasing with respect to $p \in [0,1]$ and $q \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, and the inequality chain

$$H(a,b) = M_{-1}(a,b) = L_{-1}(a,b) < G(a,b) = M_0(a,b) = L_{-1/2}(a,b)$$

$$< A(a,b) = M_1(a,b) = L_0(a,b) < T(a,b) < \overline{C}(a,b)$$

$$< Q(a,b) = M_2(a,b) < C(a,b) = L_1(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

In [1], Toader introduced the Toader mean $T(a,b)$ of two positive numbers a and b as follows:

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta$$

$$= \begin{cases} \frac{2a}{\pi}\mathcal{E}\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right), & a > b, \\ \frac{2b}{\pi}\mathcal{E}\left(\sqrt{1-\left(\frac{a}{b}\right)^2}\right), & a < b, \\ a, & a = b, \end{cases} \tag{1.2}$$

where $\mathcal{E}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta$ $(r \in [0,1])$ is the complete elliptic integral of the second kind.

Recently, the Toader mean $T(a,b)$ has been the subject of intensive research. Vuorinen [2] conjectured that the inequality

$$M_{3/2}(a,b) < T(a,b)$$

holds for all $a, b > 0$ with $a \neq b$. This conjecture was proved by Qiu and Shen [3], and Barnard *et al.* [4], respectively.

Alzer and Qiu [5] presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/(\log \pi - \log 2)}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Neuman [6], and Kazi and Neuman [7] proved that the inequalities

$$\frac{(a+b)\sqrt{ab}-ab}{AGM(a,b)} < T(a,b) < \frac{4(a+b)\sqrt{ab}+(a-b)^2}{8AGM(a,b)},$$

$$T(a,b) < \frac{1}{4}\left(\sqrt{(2+\sqrt{2})a^2+(2-\sqrt{2})b^2}+\sqrt{(2+\sqrt{2})b^2+(2-\sqrt{2})a^2}\right)$$

hold for all $a, b > 0$ with $a \neq b$, where $AGM(a,b)$ is the arithmetic-geometric mean of a and b.

In [8-10], the best possible parameters $\lambda_1, \mu_1 \in [0,1]$ and $\lambda_2, \mu_2, \lambda_3, \mu_3 \in \mathbb{R}$ such that the double inequalities $S_{\lambda_1}(a,b) < T(a,b) < S_{\mu_1}(a,b)$, $G_{\lambda_2}(a,b) < T(a,b) < G_{\mu_2}(a,b)$ and $L_{\lambda_3}(a,b) < T(a,b) < L_{\mu_3}(a,b)$ hold for all $a, b > 0$ with $a \neq b$ were presented.

In [11-15], the authors proved that the double inequalities

$$\alpha_1 Q(a,b) + (1-\alpha_1)A(a,b) < T(a,b) < \beta_1 Q(a,b) + (1-\beta_1)A(a,b),$$

$$Q^{\alpha_2}(a,b)A^{(1-\alpha_2)}(a,b) < T(a,b) < Q^{\beta_2}(a,b)A^{(1-\beta_2)}(a,b),$$

$$\alpha_3 C(a,b) + (1-\alpha_3)A(a,b) < T(a,b) < \beta_3 C(a,b) + (1-\beta_3)A(a,b),$$

$$\frac{\alpha_4}{A(a,b)}+\frac{1-\alpha_4}{C(a,b)}<\frac{1}{T(a,b)}<\frac{\beta_4}{A(a,b)}+\frac{1-\beta_4}{C(a,b)},$$

$$\alpha_5 C(a,b)+(1-\alpha_5)H(a,b)<T(a,b)<\beta_5 C(a,b)+(1-\beta_5)H(a,b),$$

$$\alpha_6[C(a,b)-H(a,b)]+A(a,b)<T(a,b)<\beta_6[C(a,b)-H(a,b)]+A(a,b),$$

$$\alpha_7\overline{C}(a,b)+(1-\alpha_7)A(a,b)<T(a,b)<\beta_7\overline{C}(a,b)+(1-\beta_7)A(a,b),$$

$$\frac{\alpha_8}{A(a,b)}+\frac{1-\alpha_8}{\overline{C}(a,b)}<\frac{1}{T(a,b)}<\frac{\beta_8}{A(a,b)}+\frac{1-\beta_8}{\overline{C}(a,b)},$$

$$\alpha_9 Q(a,b)+(1-\alpha_9)H(a,b)<T(a,b)<\beta_9 Q(a,b)+(1-\beta_9)H(a,b),$$

$$\frac{\alpha_{10}}{H(a,b)}+\frac{1-\alpha_{10}}{Q(a,b)}<\frac{1}{T(a,b)}<\frac{\beta_{10}}{H(a,b)}+\frac{1-\beta_{10}}{Q(a,b)}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\le 1/2$, $\beta_1\ge(4-\pi)/[(\sqrt{2}-1)\pi]$, $\alpha_2\le 1/2$, $\beta_2\ge 4-2\log\pi/\log 2$, $\alpha_3\le 1/4$, $\beta_3\ge 4/\pi-1$, $\alpha_4\le\pi/2-1$, $\beta_4\ge 3/4$, $\alpha_5\le 5/8$, $\beta_5\ge 2/\pi$, $\alpha_6\le 1/8$, $\beta_6\ge 2/\pi-1/2$, $\alpha_7\le 3/4$, $\beta_7\ge 12/\pi-3$, $\alpha_8\le\pi-3$, $\beta_8\ge 1/4$, $\alpha_9\le 5/6$, $\beta_9\ge 2\sqrt{2}/\pi$, $\alpha_{10}\le 0$ and $\beta_{10}\ge 1/6$.

Moreover, Chu *et al.* [16] and Hua and Qi [17] proved that the double inequalities

$$C[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a]<T(a,b)<C[\mu a+(1-\mu)b,\mu b+(1-\mu)a],$$

$$\overline{C}[\alpha a+(1-\alpha)b,\alpha b+(1-\alpha)a]<T(a,b)<\overline{C}[\beta a+(1-\beta)b,\beta b+(1-\beta)a]$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\lambda\le 3/4$, $\mu\ge 1/2+\sqrt{\pi(4-\pi)}/(2\pi)$, $\alpha\le 1/2+\sqrt{3}/4$ and $\beta\ge 1/2+\sqrt{12/\pi-3}/2$.

The main purpose of this paper is to present the best possible parameters α_1, α_2, α_3, α_4 and β_1, β_2, β_3, β_4 such that the double inequalities

$$\alpha_1 A(a,b)+(1-\alpha_1)\,C(a,b)<T[A(a,b),C(a,b)]<\beta_1 A(a,b)+(1-\beta_1)\,C(a,b),$$

$$A^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b)<T[A(a,b),C(a,b)]<A^{\beta_2}(a,b)C^{1-\beta_2}(a,b),$$

$$\frac{\alpha_3}{A(a,b)}+\frac{1-\alpha_3}{C(a,b)}<\frac{1}{T[A(a,b),C(a,b)]}<\frac{\beta_3}{A(a,b)}+\frac{1-\beta_3}{C(a,b)},$$

$$C[\alpha_4 a+(1-\alpha_4)b,\alpha_4 b+(1-\alpha_4)a]<T[A(a,b),C(a,b)]$$
$$<C[\beta_4 a+(1-\beta_4)b,\beta_4 b+(1-\beta_4)a]$$

hold for all $a,b>0$ with $a\neq b$.

2. Basic Knowledge and Lemmas

In order to prove our main results we need several formulas and lemmas, which we present in this section.

For $r \in (0,1)$ and $r' = \sqrt{1-r^2}$, the well-known complete elliptic integrals of the first and second kinds are defined by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta, \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r') \\ \mathcal{K}(0) = \pi/2, \quad \mathcal{K}(1) = +\infty, \end{cases}$$

and

$$\begin{cases} \mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta, \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r') \\ \mathcal{E}(0) = \pi/2, \quad \mathcal{E}(1) = 1, \end{cases}$$

respectively, and the following formulas were presented in [18, Appendix E, pp. 474-475]:

$$\begin{aligned} \frac{d\mathcal{K}}{dr} &= \frac{\mathcal{E}-r'^2\mathcal{K}}{rr'^2}, \quad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E}-\mathcal{K}}{r}, \\ \frac{d(\mathcal{E}-r'^2\mathcal{K})}{dr} &= r\mathcal{K}, \quad \frac{d(\mathcal{K}-\mathcal{E})}{dr} = \frac{r\mathcal{E}}{r'^2}, \end{aligned} \tag{2.1}$$

$$\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E}-r'^2\mathcal{K}}{1+r}. \tag{2.2}$$

In what follows, two special values $\mathcal{E}(1/3)$ and $\mathcal{K}(1/3)$ will be used. By numerical computations, these are given by

$$\mathcal{E}(1/3) = 1.52621\cdots, \qquad \mathcal{K}(1/3) = 1.61739\cdots.$$

Lemma 2.1. *(See [18, Theorem 1.25]). For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b), let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. *Let $r \in (0,1)$, then* itemindent=-2em *the function* $r \to (\mathcal{E} - r'^2\mathcal{K})/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, 1)$*;*

iitemiindent=-2em *the function* $r \to 2\mathcal{E} - r'^2\mathcal{K}$ *is increasing and log-convex from* $(0,1)$ *onto* $(\pi/2, 2)$*.*

Proof. Parts (1) and (2) follow from [18, Theorem 3.21(1) and Exercise 3.43(13)]. □

Lemma 2.3. *Let* $r \in (0, 1/3)$, $\lambda_1 = 2[1 - \frac{3}{\pi}\mathcal{E}(1/3) + \frac{4}{3\pi}\mathcal{K}(1/3)] = 0.458\cdots$ *and*

$$f(r) = \frac{1 + r - \frac{2}{\pi}(2\mathcal{E} - r'^2\mathcal{K})}{2r},$$

then $f(r)$ *is strictly decreasing from* $(0, 1/3)$ *onto* $(\lambda_1, 1/2)$*.*

Proof. Let $f_1(r) = 1 + r - 2(2\mathcal{E} - r'^2\mathcal{K})/\pi$ and $f_2(r) = 2r$. Then $f_1(0) = f_2(0) = 0$ and $f(r) = f_1(r)/f_2(r)$. Differentiation of $f_1(r)$ and $f_2(r)$ yields

$$\frac{f_1'(r)}{f_2'(r)} = \frac{1}{2} - \frac{r}{\pi} \cdot \frac{\mathcal{E} - r'^2\mathcal{K}}{r^2}. \tag{2.3}$$

It follows from Lemma 2.2(1) that $f_1'(r)/f_2'(r)$ is strictly decreasing in $(0, 1)$.

Therefore, Lemma 2.3 follows from Lemma 2.1 and the limiting values $f(0^+) = \frac{1}{2}$ and $f(\frac{1}{3}^-) = \lambda_1$. □

Lemma 2.4. *Let* $r \in (0, 1/3)$, $\lambda_2 = \log[3\pi/(9\mathcal{E}(1/3) - 4\mathcal{K}(1/3))]/\log 2 = 0.375\cdots$ *and*

$$g(r) = \frac{\log(4\mathcal{E} - 2r'^2\mathcal{K}) - \log[\pi(1 + r)]}{\log(1 - r) - \log(1 + r)},$$

then $g(r)$ *is strictly decreasing from* $(0, 1/3)$ *onto* $(\lambda_2, 1/2)$*.*

Proof. Let $g_1(r) = \log(4\mathcal{E} - 2r'^2\mathcal{K}) - \log[\pi(1 + r)]$ and $g_2(r) = \log(1 - r) - \log(1 + r)$ Then $g_1(0) = g_2(0) = 0$, $g(r) = g_1(r)/g_2(r)$ and

$$\frac{g_1'(r)}{g_2'(r)} = g_3(r)g_4(r). \tag{2.4}$$

where

$$g_3(r) = \frac{(1 - r)^2[(1 + r)\mathcal{K} - \mathcal{E}]}{r}, \quad g_4(r) = \frac{1}{2(2\mathcal{E} - r'^2\mathcal{K})}.$$

A simple calculation yields

$$g_3'(r) = -\frac{(1 - r)[(1 + 2r)(\mathcal{K} - \mathcal{E}) + r^2\mathcal{K}]}{r^2} < 0 \tag{2.5}$$

for $r \in (0,1)$.

It follows from (2.5) and Lemma 2.2(2) that $g_3(r)$ is strictly decreasing from $(0,1)$ onto $(0,\pi/2)$ and $g_4(r)$ is strictly decreasing from $(0,1)$ onto $(1/4,1/\pi)$. This conjunction with (2.4) implies that $g_1'(r)/g_2'(r)$ is strictly decreasing in $(0,1)$.

Therefore, Lemma 2.4 follows from Lemma 2.1 and the limiting values $g(0^+) = \frac{1}{2}$ and $g(\frac{1}{3}^-) = \lambda_2$. □

Lemma 2.5. *Let $r \in (0,1/3)$, $\lambda_3 = [3\pi - 9\mathcal{E}(1/3) + 4\mathcal{K}(1/3)]/[9\mathcal{E}(1/3) - 4\mathcal{K}(1/3)] = 0.297\cdots$ and*

$$h(r) = \frac{\pi r'^2 - 2(1-r)(2\mathcal{E} - r'^2\mathcal{K})}{4r(2\mathcal{E} - r'^2\mathcal{K})},$$

then $h(r)$ is strictly decreasing from $(0,1/3)$ onto $(\lambda_3, 1/2)$.

Proof. Let $\widehat{h}(r) = [\pi r'^2 - 2(1-r)(2\mathcal{E} - r'^2\mathcal{K})]/r$, then $h(r) = \widehat{h}(r)/[4(2\mathcal{E} - r'^2\mathcal{K})]$. Following from Lemma 2.2(2), it suffices to prove that $\widehat{h}(r)$ is strictly decreasing in $(0,1/3)$.

Differentiation of $\widehat{h}(r)$ leads to

$$\widehat{h}'(r) = -\frac{\widehat{h}_1(r)}{r^2}, \tag{2.6}$$

$$\widehat{h}_1(r) = \pi(1+r^2) + 2rr'^2\mathcal{K} - 2(1+r)\mathcal{E}, \tag{2.7}$$

$$\widehat{h}_1(0) = 0, \tag{2.8}$$

$$\widehat{h}_1'(r) = \frac{2}{r}[r^2(\pi - 2r\mathcal{K}) + (1+r)(\mathcal{K} - \mathcal{E})]. \tag{2.9}$$

It is well-known that the function $\pi - 2r\mathcal{K}$ is strictly decreasing in $(0,1)$ and then $\pi - 2r\mathcal{K} > \pi - \frac{2}{3}\mathcal{K}(1/3) = 2.063\cdots > 0$ for $0 < r < 1/3$. It is easy to see that $\widehat{h}_1'(r) > 0$ for $0 < r < 1/3$ following from (2.9). This conjunction with (2.6)-(2.8) implies that $\widehat{h}(r)$ is strictly decreasing in $(0,1/3)$.

Therefore, Lemma 2.5 follows from the limiting values $h(0^+) = \frac{1}{2}$ and $h(\frac{1}{3}^-) = \lambda_3$. □

Lemma 2.6. *Let $r \in (0,1/3)$, $\lambda_4 = [18\mathcal{E}(1/3) - 8\mathcal{K}(1/3)]/3\pi - 1 = 0.5419\cdots$ and*

$$\varphi(r) = \frac{\frac{2}{\pi}(2\mathcal{E} - r'^2\mathcal{K}) + r - 1}{2r},$$

then $\varphi(r)$ is strictly increasing from $(0,1/3)$ onto $(1/2, \lambda_4)$.

Proof. The proof completes easily from the following facts

$$\varphi'(r) = \frac{\frac{\pi}{2} - \mathcal{E}}{\pi r^2} > 0$$

for $r \in (0, 1/3)$ and the limiting values $\varphi(0^+) = 1/2$ and $\varphi(\frac{1}{3}^-) = \lambda_4$. □

3. Main Results

Theorem 3.1. *The double inequality*

$$\alpha_1 A(a,b) + (1-\alpha_1)C(a,b) < T[A(a,b), C(a,b)] < \beta_1 A(a,b) + (1-\beta_1)C(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_1 \geq 1/2$ *and* $\beta_1 \leq \lambda_1 = 2[1 - \frac{3}{\pi}\mathcal{E}(1/3) + \frac{4}{3\pi}\mathcal{K}(1/3)] = 0.458\cdots$.

Proof. It is well-known that $A(a,b)$, $T(a,b)$ and $C(a,b)$ are symmetric and homogeneous of degree one. Without loss of generality, we assume that $a > b$. Let $r = (a-b)^2/(3a^2 + 2ab + 3b^2) \in (0, 1/3)$. From (1.1), (1.2) and (2.2), we see that

$$A(a,b) = C(a,b)\frac{1-r}{1+r}, \tag{3.1}$$

$$T[A(a,b), C(a,b)] = \frac{2}{\pi}C(a,b)\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2}{\pi}C(a,b)\frac{2\mathcal{E} - r'^2\mathcal{K}}{1+r}, \tag{3.2}$$

and

$$\frac{C(a,b) - T[A(a,b), C(a,b)]}{C(a,b) - A(a,b)} = \frac{1 - \frac{2}{\pi}\frac{2\mathcal{E} - r'^2\mathcal{K}}{1+r}}{1 - \frac{1-r}{1+r}} = f(r), \tag{3.3}$$

where $f(r)$ is defined as in Lemma 2.3.

Therefore, Theorem 3.1 follows easily from Lemma 2.3 and (3.3). □

Theorem 3.2. *The double inequality*

$$A^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b) < T[A(a,b), C(a,b)] < A^{\beta_2}(a,b)C^{1-\beta_2}(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_2 \geq 1/2$ *and* $\beta_2 \leq \lambda_2 = \log[3\pi/(9\mathcal{E}(1/3) - 4\mathcal{K}(1/3))]/\log 2 = 0.375\cdots$.

Proof. Without loss of generality, we assume that $a > b$. Let $r = (a-b)^2/(3a^2+2ab+3b^2) \in (0, 1/3)$. Then (3.1) and (3.2) lead to

$$\frac{\log C(a,b) - \log T[A(a,b), C(a,b)]}{\log C(a,b) - \log A(a,b)} = g(r), \tag{3.4}$$

where $g(r)$ is defined as in Lemma 2.4.

Therefore, Theorem 3.2 follows easily from Lemma 2.4 and (3.4). □

Theorem 3.3. *The double inequality*

$$\frac{\alpha_3}{A(a,b)} + \frac{1-\alpha_3}{C(a,b)} < \frac{1}{T[A(a,b), C(a,b)]} < \frac{\beta_3}{A(a,b)} + \frac{1-\beta_3}{C(a,b)}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq \lambda_3 = [3\pi - 9\mathcal{E}(1/3) + 4\mathcal{K}(1/3)]/[9\mathcal{E}(1/3) - 4\mathcal{K}(1/3)] = 0.297\cdots$ and $\beta_3 \geq 1/2$.

Proof. Without loss of generality, we assume that $a > b$. Let $r = (a-b)^2/(3a^2+2ab+3b^2) \in (0, 1/3)$. Then (3.1) and (3.2) lead to

$$\frac{\frac{1}{T[A(a,b),C(a,b)]} - \frac{1}{C(a,b)}}{\frac{1}{A(a,b)} - \frac{1}{C(a,b)}} = h(r), \tag{3.5}$$

where $h(r)$ is defined as in Lemma 2.5.

Therefore, Theorem 3.3 follows easily from Lemma 2.5 and (3.5). □

Theorem 3.4. *The double inequality*

$$C[\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a] < T[A(a,b), C(a,b)] \\ < C[\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a]$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leq (2+\sqrt{2})/4$ and $\beta_4 \geq (1+\sqrt{\lambda_4})/2 = 0.868\cdots$.

Proof. For positive numbers $a, b > 0$ with $a \neq b$, let $J(x) = C[xa + (1-x)b, xb + (1-x)a]$ be on $[1/2, 1]$, then it is easy to verify that $J(x)$ is continuous and strictly increasing on $[1/2, 1]$. Since $C(a,b)$ and $T(a,b)$ are symmetric and homogenous of degree one, we assume that $a > b$. Let $r = (a-b)^2/(3a^2+2ab+3b^2) \in (0, 1/3)$ and $p \in (1/2, 1)$. Then (3.1) and (3.2) lead to

$$T[A(a,b), C(a,b)] - C[pa + (1-p)b, pb + (1-p)a]$$

$$\begin{aligned} &= \frac{2}{\pi}C(a,b)\frac{2\mathcal{E} - r'^2\mathcal{K}}{1+r} - C(a,b)\frac{1-r+2(2p-1)^2r}{1+r} \\ &= \frac{2rC(a,b)}{1+r}[\varphi(r) - (2p-1)^2], \end{aligned} \tag{3.6}$$

where $\varphi(r)$ is defined as in Lemma 2.6.

Therefore, Theorem 3.4 follows easily from Lemma 2.6 and (3.6).

□

As an application, Corollary 3.5 follows immediately from Theorems 3.1, 3.2, 3.3. We remark that the optimal bounds for the complete elliptic integral of second kind obtained from Theorem 3.4 is the same as those from Theorem 3.1 since $\lambda_1 + \lambda_4 = 1$.

Corollary 3.5. *Let* $\lambda_1 = 2[1 - \frac{3}{\pi}\mathcal{E}(1/3) + \frac{4}{3\pi}\mathcal{K}(1/3)]$, $\lambda_2 = \log[3\pi/(9\mathcal{E}(1/3) - 4\mathcal{K}(1/3))]/\log 2$, $\lambda_3 = [3\pi - 9\mathcal{E}(1/3) + 4\mathcal{K}(1/3)]/[9\mathcal{E}(1/3) - 4\mathcal{K}(1/3)]$ *and* $\lambda_4 = [18\mathcal{E}(1/3) - 8\mathcal{K}(1/3)]/3\pi - 1$. *Then the double inequalities*

$$\frac{\pi}{4}\left(1+\sqrt{1-t^2}\right) < \mathcal{E}(t) < \frac{\pi}{2}\left(1-\lambda_1+\lambda_1\sqrt{1-t^2}\right),$$

$$\frac{\pi}{2}(1-t^2)^{1/4} < \mathcal{E}(t) < \frac{\pi}{2}(1-t^2)^{\lambda_2/2},$$

$$\frac{\pi\sqrt{1-t^2}}{1+\sqrt{1-t^2}} < \mathcal{E}(t) < \frac{\pi\sqrt{1-t^2}}{2(\lambda_3+(1-\lambda_3)\sqrt{1-t^2})}$$

hold for all $t \in (0, \sqrt{3}/2)$.

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