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ON CLASSIFICATION OF SETS IN CLUSTER TOPOLOGICAL SPACE

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Abstract: In this paper we deals with a general concept of classification of sets in a topological space X with respect to a given nonempty system $\mathcal{E}$ of nonempty subsets of X. Using system $\mathcal{E}$ a generalization of the classical topological notions of closed, perfect, scattered, dense in itself and nowhere dense sets is introduced and studied.

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1. Introduction

In the past few years it is observed that a large number of papers is devoted to a unification of some generalized topological approaches. Considering a family of subsets of X, many basic topological notions can be defined as in standard topological spaces. For example, a generalized topology was introduced at the end of the twentieth century by Á. Császár ([4], see also [2], [12], [20]) as a family containing the empty set and closed under unions. A minimal structure containing the empty set and X was introduced in [13], [22], [23]. Again, Á. Császár ([5]) introduced and studied so-called a weak structure, which is a family of subsets of X containing the empty set. Perhaps the most general

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family was introduced by J. Ávila and F. Molina ([1]). Their family called a general weak structure is any nonempty collection of subsets of X. Owing to the fact that the corresponding definitions of generalized open sets have many features in common, a considerable part of the properties can be deduced from suitable more general definitions.

For example, if $w \subset 2^X = \{A : A \subset X\}$, then w-closure and w-interior of $A \subset X$ are defined by $c_w(A) = \cap\{F : A \subset F, X \setminus F \in w\}$ and $i_w(A) = \cup\{U : U \subset A, U \in w\}$, respectively. Mimicking the known definitions of some subsets in a topological space α-open $A \subset \overline{A^\circ}^\circ$, semi-open $A \subset \overline{A^\circ}$, pre-open $A \subset \overline{A}^\circ$, β-open $A \subset \overline{\overline{A}^\circ}$ and using the operators c_w and i_w, we can define similar types of subsets by inclusions $A \subset i_w c_w i_w(A)$, $A \subset c_w i_w(A)$, $A \subset i_w c_w(A)$, $A \subset c_w i_w c_w(A)$, respectively. Apart from a concept based on an operator from 2^X to 2^X gives a large possibility to generalize many topological notions. A typical representant of this concept is a topological space (X, τ) with an ideal $\mathcal{I}$ in X ([7], [9], [10], [26]) which covers also a grill approach ([24], [25]). A grill $\mathcal{G}$ on X is a nonempty subset of 2^X satisfying the following conditions:

(1) $\emptyset \notin \mathcal{G}$,

(2) $A \subset B$, $A \in \mathcal{G}$ implies $B \in \mathcal{G}$,

(3) $A \cup B \in \mathcal{G}$ implies $A \in \mathcal{G}$ or $A \in \mathcal{G}$

and $\mathcal{S}$ is called a stack if it satisfies the conditions (1) and (2) above ([21]). It is necessary to recall that a collection $\{A : A \notin \mathcal{G}\}$ is an ideal. So the ideal topological spaces $(X, \tau, \mathcal{I})$ cover the grill topological spaces $(X, \tau, \mathcal{G})$.

The generalized structures (usually weaker then a topology) and ideals on a topological space have been investigated from many aspects and applications and a focus of the research is to examine basic properties of newly defined subsets and to find some analogies with the classical topological notions (Banach category theorem, Baire classification of sets, compactness, connectedness, continuity and so on).

In this paper we will consider a family $\mathcal{E}$ (called a cluster system) which covers the approaches mentioned above, it is sufficiently general (we suppose only one assumption, namely the sets from $\mathcal{E}$ are nonempty), it has many applications in the continuity and selection theorems ([3], [8], [11], [14], [15]) and we will find some analogies and relations with the topological notions.

2. Definitions, Properties, Examples

In the sequel, (X, τ) is a nonempty topological space. By $\overline{A}$, A° we denote the closure, the interior of A in X, respectively and $\mathcal{U}(z)$ is a base of the open neighborhoods of a point z. By N, we denote the natural numbers.

Any nonempty system $\mathcal{E} \subset 2^X \setminus \{\emptyset\}$ will be called a cluster system in X. If G is a nonempty open set and any nonempty open subset of G contains a set from $\mathcal{E}$, then $\mathcal{E}$ is called a π-network in G. For a cluster system $\mathcal{E}$ and a subset A of a topological space X, we define the set $\mathcal{E}(A)$ of all points $x \in X$ such that for any neighborhood U of x, the intersection $U \cap A$ contains a set form $\mathcal{E}$. A triplet $(X, \tau, \mathcal{E})$ is called a cluster topological space. The next properties of the $\mathcal{E}$-operator are clear and the proof of the following lemma is omitted.

Lemma 2.1.

(1) $\mathcal{E}(\emptyset) = \emptyset$,

(2) $\mathcal{E}(A)$ *is closed,*

(3) $\mathcal{E}(A) \subset \overline{A}$,

(4) if $A_1 \subset A_2$, *then* $\mathcal{E}(A_1) \subset \mathcal{E}(A_2)$,

(5) $\mathcal{E}$ *is a* π*-network in an open set* $G \neq \emptyset$ *iff* $\mathcal{E}(G) = \mathcal{E}(\overline{G}) = \overline{G}$.

The next definition introduces some basic notions derived from $\mathcal{E}$-operator reminding a local function which is known from an ideal topological space.

Definition 2.1. A set A is called $\mathcal{E}$-scattered ($\mathcal{E}$-closed, $\mathcal{E}$-open, $\mathcal{E}$-dense in itself, $\mathcal{E}$-perfect) if A contains no set from $\mathcal{E}$ ($\mathcal{E}(A) \subset A$, $X \setminus A$ is $\mathcal{E}$-closed, $\mathcal{E}(A) \supset A$, $\mathcal{E}(A) = A$). A set A is locally $\mathcal{E}$-scattered at a point $x \in X$ if there is $U \in \mathcal{U}(x)$ such that $U \cap A$ is $\mathcal{E}$-scattered (i.e., $x \notin \mathcal{E}(A)$) and A is locally $\mathcal{E}$-scattered if A is $\mathcal{E}$-scattered at any point from A (i.e., $A \cap \mathcal{E}(A) = \emptyset$). The family of all $\mathcal{E}$-scattered sets is denoted by $\mathcal{S}_{\mathcal{E}}$ and a set is called σ-$\mathcal{E}$-scattered if it is a countable union of $\mathcal{E}$-scattered sets.

Before giving a few specific examples we would like to mention some cluster systems which have been studied recently for their special applications. Put $\mathcal{E} = \{A : A \text{ is not nowhere dense and } G_\delta\}$. This cluster system was used in [18], [19] for a characterization of the weakly Volterra and Volterra spaces.

Studying some generalized continuities of multifunctions and finding a quasi continuous selection, a cluster system containing the sets of second category with the Baire property has been intensively studied in [16], [17].

As it was mentioned above, there is a large scale of results studied in an ideal topological space $(X, \tau, \mathcal{I})$. In this case a cluster system $\mathcal{E}_{\mathcal{I}} = \{A : A \notin \mathcal{I}\}$ covers the ideal topological approach and A is $\mathcal{E}_{\mathcal{I}}$-scattered iff $A \in \mathcal{I}$. If $\mathcal{S}$ is a stack, then $\mathcal{E}_{\mathcal{S}} = \{A : A \text{ contains a set from } \mathcal{S}\}$ covers the stack topological approach ([21]). In this case A is $\mathcal{E}_{\mathcal{S}}$-scattered iff $A \notin \mathcal{S}$. From this point of view a cluster setting seems to be sufficiently general providing reasonable applications.

Now we will give a few examples of the cluster systems illustrating the notions introduced in the definition above and their relations to some topological notions.

Example 2.1.

(1) Consider a cluster system $\mathcal{E}_{\mathsf{op}} = \{G : G \text{ is nonempty open}\}$. Then $\mathcal{E}_{\mathsf{op}}$ is a π-network in any nonempty open set, $\mathcal{E}_{\mathsf{op}}(A) = \overline{A^\circ}$ and $\mathcal{E}_{\mathsf{op}}(A) = \emptyset$ iff $A^\circ = \emptyset$ (A is a boundary set). A set A is $\mathcal{E}_{\mathsf{op}}$-dense in itself iff $A \subset \mathcal{E}_{\mathsf{op}}(A) = \overline{A^\circ}$, equivalently A is semi-open. Similar, A is semi-closed iff $X \setminus A$ is $\mathcal{E}_{\mathsf{op}}$-dense in itself. Further, A is $\mathcal{E}_{\mathsf{op}}$-perfect iff $\mathcal{E}_{\mathsf{op}}(A) = \overline{A^\circ} = A$, equivalently A is regular closed. A set is $\mathcal{E}_{\mathsf{op}}$-scattered iff it is locally $\mathcal{E}_{\mathsf{op}}$-scattered iff $A^\circ = \emptyset$.

(2) Let $\mathcal{E}^B = 2^B \setminus \{\emptyset\}$, $\emptyset \neq B \subset X$. Then $\mathcal{E}^B(A) = \overline{A \cap B}$. That means A is $\mathcal{E}$-perfect iff $\mathcal{E}^B(A) = \overline{A \cap B} = A$ iff A is closed and $A \cap B$ is dense in A. If $A \subset B$, then A is $\mathcal{E}$-closed ($\mathcal{E}^B(A) = \overline{A} \subset A$) iff A is closed. If $B \subset A$, then A is $\mathcal{E}$-closed iff $\mathcal{E}^B(A) = \overline{B} \subset A$. Specially $\mathcal{E}^X(A) = \overline{A}$ and $\mathcal{E}^{\{b\}}(A) = \overline{\{b\}}$ if $b \in A$ and $\mathcal{E}^{\{b\}}(A) = \emptyset$ otherwise.

(3) Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$ with the usual topology, $\mathcal{E}_1 = \{E : E \text{ is infinite}\}$ and $\mathcal{E}_2 = \{E : E \text{ is cofinite}\}$. Then A is infinite iff $\mathcal{E}_1(A) = \{0\}$ and A is cofinite iff $\mathcal{E}_2(A) = \{0\}$. So, an infinite (cofinite) set A is $\mathcal{E}_1$-closed ($\mathcal{E}_2$-closed) iff $0 \in A$. Any finite set is $\mathcal{E}_i$-closed and there are no nonempty $\mathcal{E}_i$-perfect sets, $i = 1, 2$. A set A is $\mathcal{E}_1$-scattered iff A is finite and $\mathcal{E}_2$-scattered iff $X \setminus A$ is infinite. For example, $B = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$ is locally $\mathcal{E}_i$-scattered but not $\mathcal{E}_i$-scattered.

(4) A set A is called pre-open if there is an open set G such that $A \subset G$ and A is dense in G. Let $\mathcal{E}_{\mathsf{po}}$ be a cluster system containing all nonempty pre-open sets. Then $\mathcal{E}_{\mathsf{po}}(A) = \overline{\overline{A}^\circ}$. That means A is nowhere dense iff $\mathcal{E}_{\mathsf{po}}(A) = \emptyset$ iff A is $\mathcal{E}_{\mathsf{po}}$-scattered. A set is σ-$\mathcal{E}_{\mathsf{po}}$-scattered iff it is of first category. Moreover, $A \setminus \mathcal{E}_{\mathsf{po}}(A)$ is nowhere dense. A set A is $\mathcal{E}_{\mathsf{po}}$-dense in

itself ($A \subset \mathcal{E}_{\mathsf{po}}(A) = \overline{\overline{A}^{\circ}}$) iff A is β-open. Similar, A is β-closed iff $X \setminus A$ is $\mathcal{E}_{\mathsf{po}}$-dense in itself. Further, A is $\mathcal{E}_{\mathsf{po}}$-perfect iff $\mathcal{E}_{\mathsf{po}}(A) = \overline{\overline{A}^{\circ}} = A$, equivalently A is regular closed.

(5) Let X contain at least two points and put $\mathcal{E} = \{A : A$ contains at least two points$\}$. In this case $\mathcal{E}(X)$ is the set of all accumulation points of X and $X \setminus \mathcal{E}(X)$ is the set of all isolated points of X. A set A is $\mathcal{E}$-scattered iff A is a singleton and A is σ-$\mathcal{E}$-scattered iff it is a countable set.

Lemma 2.2.

(1) if A_t is $\mathcal{E}$-dense in itself for any $t \in T$, then $\cup_{t\in T} A_t$ is so,

(2) if A is $\mathcal{E}$-dense in itself, then $\overline{A}$ is so.

Proof. (1) follows from the inclusions

$$\cup_{t\in T} A_t \subset \cup_{t\in T}\mathcal{E}(A_t) \subset \mathcal{E}(\cup_{t\in T} A_t).$$

(2) Since $A \subset \mathcal{E}(A)$, $\overline{A} \subset \overline{\mathcal{E}(A)} = \mathcal{E}(A) \subset \mathcal{E}(\overline{A})$.

□

The next lemma characterizes the nonempty $\mathcal{E}$-open sets.

Lemma 2.3. *A nonempty set A is $\mathcal{E}$-open if and only if for any $x \in A$ there is $U \in \mathcal{U}(x)$ such that $U \setminus A$ is $\mathcal{E}$-scattered.*

Proof. " $\Rightarrow$ " Let $x \in A$. Since $X \setminus A$ is $\mathcal{E}$-closed, $\mathcal{E}(X \setminus A) \subset X \setminus A$ and $x \notin \mathcal{E}(X \setminus A)$. That means there is $U \in \mathcal{U}(x)$ such that $U \cap (X \setminus A) = U \setminus A$ does not contain a set from $\mathcal{E}$, so $U \setminus A$ is $\mathcal{E}$-scattered.

" $\Leftarrow$ " We will show $X \setminus A$ is $\mathcal{E}$-closed, i.e., $\mathcal{E}(X \setminus A) \subset X \setminus A$. Let $x \in \mathcal{E}(X \setminus A)$ and suppose $x \notin X \setminus A$. Since $x \in A$ and by the assumption there is $U \in \mathcal{U}(x)$ such that $U \setminus A$ is $\mathcal{E}$-scattered. On the other hand, $x \in \mathcal{E}(X \setminus A)$, hence $U \cap (X \setminus A) = U \setminus A$ contains a set from $\mathcal{E}$, a contradiction. □

Remark 2.1. Since X is $\mathcal{E}$-closed ($\mathcal{E}(X) \subset X$), $\emptyset$ is $\mathcal{E}$-open and by lemma above, the family of all $\mathcal{E}$-open sets in X forms a general topology.

3. Main Results

A subset A is dense in itself if A contains no isolated point of A. For a topological space X, the Cantor-Bendixson derivative of X is the set of all nonisolated points of X. If $\mathcal{E}^2$ is the family of all sets containing at least two points, then $\mathcal{E}^2(X)$ is the Cantor-Bendixon derivative of X. A subset A of X is $\mathcal{E}^2$-dense in itself if and only if A is dense in itself. A topological space X is called scattered if it contains no nonempty subset which is dense-in-itself ([6]) or equivalently every nonempty subset has an isolated point or using the cluster system $\mathcal{E}^2$, $A \setminus \mathcal{E}^2(A) \neq \emptyset$ for any $A \neq \emptyset$. So, the classical topological notions can be characterized by an appropriate cluster system.

The next central notion of the paper is a generalization of the known definition of nowhere dense set which is defined by $\overline{A}^\circ = \emptyset$.

Definition 3.1. A set A is $\mathcal{E}$-nowhere dense if for any nonempty open set U there is a nonempty open subset H of U such that $H \cap A$ is $\mathcal{E}$-scattered and A is of $\mathcal{E}$-first category if $A = \cup_{n \in N} A_n$ where A_n is $\mathcal{E}$-nowhere dense for any $n \in N$. The family of all $\mathcal{E}$-nowhere dense sets, $\mathcal{E}$-first category sets (nowhere dense sets, first category sets) is denoted by $\mathcal{N}_\mathcal{E}$, $\mathcal{I}_\mathcal{E}$ ($\mathcal{N}$, $\mathcal{I}$), respectively.

Lemma 3.1. *The next conditions are equivalent*

(1) $A \in \mathcal{N}_\mathcal{E}$,

(2) $\mathcal{E}(A) \in \mathcal{N}$,

(3) $(\mathcal{E}(A))^\circ = \emptyset$.

Proof. (1) ⇒ (2): Suppose $\mathcal{E}(A) \notin \mathcal{N}$. Since $\mathcal{E}(A)$ is closed, $(\mathcal{E}(A))^\circ \neq \emptyset$. That means there is a nonempty open set G $(= (\mathcal{E}(A))^\circ)$ such that for any nonempty open subset H of G, $H \cap A$ contains a set from $\mathcal{E}$, so $\mathcal{E}(A) \notin \mathcal{N}_\mathcal{E}$, a contradiction.

The implication (2) ⇒ (3) is clear.

(3) ⇒ (1): Since $(\mathcal{E}(A))^\circ = \emptyset$, $G \cap (X \setminus \mathcal{E}(A))$ contains a point x for any nonempty open set G. That means there is $H \in \mathcal{U}(x)$ subset of G such that $H \cap A$ contains no set from $\mathcal{E}$, so $H \cap A$ is $\mathcal{E}$-scattered. That means $A \in \mathcal{N}_\mathcal{E}$. □

Theorem 3.1.

(1) $\mathcal{N} \subset \mathcal{N}_\mathcal{E}$*. Consequently, if* $A \in \mathcal{N}$*, then* $\overline{A} \in \mathcal{N}_\mathcal{E}$,

(2) if $A \in \mathcal{N}_\mathcal{E}$ $(A \in \mathcal{I}_\mathcal{E})$ *and* $B \subset A$*, then* $B \in \mathcal{N}_\mathcal{E}$ $(B \in \mathcal{I}_\mathcal{E})$,

(3) $A \setminus \mathcal{E}(A) \in \mathcal{N}_\mathcal{E}$ *and* $A \setminus \mathcal{E}(A)$ *is locally* $\mathcal{E}$*-scattered,*

(4) any locally $\mathcal{E}$-scattered set is from $\mathcal{N}_{\mathcal{E}}$,

(5) if $A \in \mathcal{N}$ and $B \in \mathcal{N}_{\mathcal{E}}$, then $A \cup B \in \mathcal{N}_{\mathcal{E}}$,

(6) if $G_t \in \mathcal{N}_{\mathcal{E}}$ and G_t is open for any $t \in T$, then $\cup_{t\in T} G_t \in \mathcal{N}_{\mathcal{E}}$,

(7) $A \in \mathcal{N}_{\mathcal{E}}$ iff A is a sum of a locally $\mathcal{E}$-scattered set and a set from $\mathcal{N}$.

Proof. (1) Let $A \in \mathcal{N}$. By Lemma 2.1, $\mathcal{E}(A) \subset \overline{A} \in \mathcal{N}$, so $\mathcal{E}(A) \in \mathcal{N}$. By Lemma 3.1, $A \in \mathcal{N}_{\mathcal{E}}$.

(2) It follows from Lemma 3.1 and from the implications $B \subset A \Rightarrow \mathcal{E}(B) \subset \mathcal{E}(A) \Rightarrow (\mathcal{E}(B))^\circ \subset (\mathcal{E}(A))^\circ = \emptyset$.

(3) Let G be nonempty open. If $G \cap (A \setminus \mathcal{E}(A))$ is empty, there is nothing to prove. Let $x \in G \cap (A \setminus \mathcal{E}(A))$. Then there is $H \in \mathcal{U}(x)$ subset of G, such that $H \cap A$ contains no set from $\mathcal{E}$. Then $H \cap (A \setminus \mathcal{E}(A))$ contains no set from $\mathcal{E}$, so $H \cap (A \setminus \mathcal{E}(A))$ is $\mathcal{E}$-scattered. That means $A \setminus \mathcal{E}(A) \in \mathcal{N}_{\mathcal{E}}$. The second part is clear.

(4) Since $A \cap \mathcal{E}(A) = \emptyset$, $A = A \setminus \mathcal{E}(A)$ is $\mathcal{E}$-nowhere dense by (3).

(5) Let G be nonempty open. Since A is nowhere dense and B is $\mathcal{E}$-nowhere dense, there are two nonempty open sets $G_0 \subset G$ and $H \subset G_0$ such that $A \cap G_0 = \emptyset$ and $B \cap H$ is $\mathcal{E}$-scattered. Hence $(A \cup B) \cap H = B \cap H$ is $\mathcal{E}$-scattered, so $A \cup B$ is $\mathcal{E}$-nowhere dense set.

(6) Let $\{H_s\}_{s\in S}$ be a maximal family of pairwise disjoint open sets such that any H_s is a subset of some set from $\{G_t\}_{t\in T}$. Then $A := \cup_{t\in T} G_t \setminus \cup_{s\in S} H_s$ is nowhere dense and it is clear that $B := \cup_{s\in S} H_s$ is $\mathcal{E}$-nowhere dense. By item (5), $\cup_{t\in T} G_t = A \cup B$ is $\mathcal{E}$-nowhere dense.

(7) " $\Rightarrow$ " It follows from equation $A = (A \setminus \mathcal{E}(A)) \cup (A \cap \mathcal{E}(A))$, item (3) and Lemma 3.1. The opposite implication follows from the items (4) and (5). □

The next theorem deals with a generalization of the Cantor-Bendixon theorem which states that an uncountable closed set in a Polish space is the disjoint union of a countable set and a perfect set.

Theorem 3.2. *For any nonempty closed set $Z \subset X$ there is its decomposition $Z = A \cup B$, where A is $\mathcal{E}$-nowhere dense and B is $\mathcal{E}$-perfect (one of A, B can be empty). Moreover, if any $E \in \mathcal{E}$ contains at least two points, then B is perfect.*

Proof. Put $B_0 = \cup\{C \subset Z : C$ is $\mathcal{E}$-dense in itself$\}$ and $B := \overline{B_0} \subset Z$. By Lemma 2.2, B is $\mathcal{E}$-dense in itself, so $B \subset \mathcal{E}(B)$. Since $B \subset \mathcal{E}(B) \subset \overline{B} = B$, B is $\mathcal{E}$-perfect. Let $A := Z \setminus B$. We will prove, A is $\mathcal{E}$-nowhere dense. Let G be nonempty open. We can suppose $G \cap A \neq \emptyset$. Since $G \cap A$ is not $\mathcal{E}$-dense in itself (otherwise it would be a subset of B_0), there is $x \in (G \cap A) \setminus \mathcal{E}(G \cap A)$. So, there is a nonempty open set $H \subset G$, $x \in H$, such that $H \cap (G \cap A) = H \cap A$ contains no set from $\mathcal{E}$. We have proved, A is $\mathcal{E}$-nowhere dense.

If any $E \in \mathcal{E}$ contains at least two points, then for any $x \in B$ and any $U \in \mathcal{U}(x)$ the intersection $U \cap B$ contains a set from $\mathcal{E}$, so $U \cap B$ contains at least two points. Consequently B is perfect. □

Remark 3.1. The decompositions $Z = (Z \setminus \mathcal{E}(Z)) \cup \mathcal{E}(Z)$ and $Z = A \cup B$ from theorem above are different. The set $Z \setminus \mathcal{E}(Z)$ is locally $\mathcal{E}$-scattered, while A is only $\mathcal{E}$-nowhere dense. The set B is $\mathcal{E}$-perfect (and closed), while $\mathcal{E}(Z)$ is only closed. Compare with Example 2.1 item (3): $X = (X \setminus \mathcal{E}_1(X)) \cup \mathcal{E}_1(X) = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\} \cup \{0\}$ while the decomposition from theorem above is $X = X \cup \emptyset$. The next theorem deals with a case when both decompositions are the same.

Theorem 3.3. *Suppose any locally $\mathcal{E}$-scattered set is $\mathcal{E}$-scattered and $\mathcal{S}_{\mathcal{E}}$ is an ideal. If Z is a closed subset of X, then the decomposition from theorem above is $Z = (Z \setminus \mathcal{E}(Z)) \cup \mathcal{E}(Z)$, where $Z \setminus \mathcal{E}(Z)$ is an $\mathcal{E}$-scattered set and $\mathcal{E}(Z)$ is an $\mathcal{E}$-perfect set. Moreover, if any $E \in \mathcal{E}$ contains at least two points, then $\mathcal{E}(Z)$ is perfect.*

Proof. Firstly we will prove the inclusion $\mathcal{E}(A) \subset \mathcal{E}(\mathcal{E}(A))$, where A is an arbitrary subset of X. Let $x \in \mathcal{E}(A) \setminus \mathcal{E}(\mathcal{E}(A))$. Then there is $U \in \mathcal{U}(x)$ such that $U \cap A$ contains a set from $\mathcal{E}$ and $U \cap \mathcal{E}(A)$ is $\mathcal{E}$-scaterred. By Theorem 3.1 item (3), $A \setminus \mathcal{E}(A)$ is locally $\mathcal{E}$-scaterred and by assumption it is $\mathcal{E}$-scattered, so $U \cap (A \setminus \mathcal{E}(A))$ is $\mathcal{E}$-scattered. Since $\mathcal{S}_{\mathcal{E}}$ is an ideal, $[U \cap (A \setminus \mathcal{E}(A))] \cup [U \cap \mathcal{E}(A)] \supset U \cap A$ is $\mathcal{E}$-scaterred. It is a contradiction, since $U \cap A$ contains a set from $\mathcal{E}$.

From the inclusion $\mathcal{E}(Z) \subset \mathcal{E}(\mathcal{E}(Z))$ follows that $\mathcal{E}(Z)$ is $\mathcal{E}$-dense in itself. That means, $\mathcal{E}(Z) \subset B_0 \subset B$ where B_0 and B are from theorem above. Since $B \subset Z$ and B is $\mathcal{E}$-perfect (see theorem above), $B = \mathcal{E}(B) \subset \mathcal{E}(Z)$, so $B = \mathcal{E}(Z)$. That means the decomposition from theorem above is $Z = (Z \setminus \mathcal{E}(Z)) \cup \mathcal{E}(Z)$. The set $Z \setminus \mathcal{E}(Z)$ is locally $\mathcal{E}$-scattered, so $\mathcal{E}$-scattered and $\mathcal{E}(Z) = B$ is $\mathcal{E}$-perfect. □

Remark 3.2. If X is a hereditarily Lindelöf topological space, then any locally $\mathcal{E}$-scattered set A is σ-$\mathcal{E}$-scattered (for any $a \in A$ there is $U_a \in \mathcal{U}(a)$

such that $A \cap U_a$ is $\mathcal{E}$-scattered. Since $\{A \cap U_a : a \in A\}$ is an open cover of the subspace A, it has a countable subcover, so A is σ-$\mathcal{E}$-scattered).

If Z is a closed uncountable subset of a hereditarily Lindelöf topological space, then the Cantor-Bendixon theorem follows from lemma above considering $\mathcal{E} = \{A : A \text{ is uncountable}\}$ (in this case any σ-$\mathcal{E}$-scattered set is $\mathcal{E}$-scattered and any $\mathcal{E}$-scattered set is countable).

Now, we can ask under which conditions $\mathcal{N}_{\mathcal{E}}$ is an ideal (see Example 2.1 (1) where $\mathcal{N}_{\mathcal{E}}$ may not be an ideal. If X contains two disjoint dense subsets A, B, then $A, B \in \mathcal{N}_{\mathcal{E}}$ but $A \cup B \notin \mathcal{N}_{\mathcal{E}}$).

It is known that A is nowhere dense iff $\overline{A}$ is nowhere dense. An analogous equivalence for the $\mathcal{E}$-nowhere dense sets leads to the fact that $\mathcal{N}_{\mathcal{E}}$ is an ideal.

Theorem 3.4. *If $\overline{A}$ is $\mathcal{E}$-nowhere dense whenever A is $\mathcal{E}$-nowhere dense, then $\mathcal{N}_{\mathcal{E}}$ is an ideal.*

Proof. Let $A, B \in \mathcal{N}_{\mathcal{E}}$. If $A \cup B \notin \mathcal{N}_{\mathcal{E}}$, then there is a nonempty open set G such that for any nonempty open set $H \subset G$ the intersection $H \cap (A \cup B)$ is not $\mathcal{E}$-scattered $^{(1)}$ and there is a nonempty open set $H_0 \subset G$ such that $B \cap H_0$ and $A \cap H_0$ are $\mathcal{E}$-scattered $^{(2)}$. So, $A \cap H_0$ is $\mathcal{E}$-nowhere dense and by the assumption, $\overline{A \cap H_0}$ is $\mathcal{E}$-nowhere dense.

We will show $H_0 \subset \overline{A \cap H_0}$. If not, $H_0 \setminus \overline{A \cap H_0} := C$ is a nonempty open subset of G and by $^{(1)}$, $C \cap (A \cup B)$ is not $\mathcal{E}$-scattered. That means, there is a set $E \in \mathcal{E}$ such that $E \subset C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$. By $^{(2)}$, $C \cap B$ is $\mathcal{E}$-scattered, so $C \cap A \neq \emptyset$, a contradiction. So $H_0 \subset \overline{A \cap H_0}$.

Since $\overline{A \cap H_0}$ is $\mathcal{E}$-nowhere dense, there is a nonempty open set $H_1 \subset H_0$ such that $H_1 \cap \overline{A \cap H_0} = H_1$ is $\mathcal{E}$-scattered, a contradiction (H_1 is not $\mathcal{E}$-scattered, since $H_1 \subset G$, so $H_1 \cap (A \cup B)$ is not $\mathcal{E}$-scattered, by $^{(1)}$). □

An obvious question is whether the assumption of Theorem 3.4 implies that the families $\mathcal{N}$ and $\mathcal{N}_{\mathcal{E}}$ are the same and if the opposite implication holds. The next examples will give the negative answers.

Example 3.1. Let $X = \{a, b\}$, $\tau = 2^X$, $\mathcal{E} = \{\{a\}\}$. Then $\mathcal{N}_{\mathcal{E}} = \{\emptyset, \{b\}\}$, $\{a\} = \overline{\{a\}}$ and $\{b\} = \overline{\{b\}}$. Finally $\mathcal{N} = \{\emptyset\}$.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$, $\mathcal{E} = \{\{a\}\}$. Then $\mathcal{N}_{\mathcal{E}} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ is an ideal. The set $\{b\} \in \mathcal{N}_{\mathcal{E}}$, but $\overline{\{b\}} = \{a, b, c\} \notin \mathcal{N}_{\mathcal{E}}$.

In the case if $\mathcal{E}$ is a π-network in X, the opposite implication of Theorem 3.4 holds and $\mathcal{N} = \mathcal{N}_{\mathcal{E}}$.

Theorem 3.5. *Let $\mathcal{E}$ be a π-network in X. The next conditions are equivalent.*

(1) $\overline{A}$ is $\mathcal{E}$-nowhere dense if and only if A is $\mathcal{E}$-nowhere dense,

(2) $\mathcal{N} = \mathcal{N}_{\mathcal{E}}$.

Proof. (1) $\Rightarrow$ (2) The inclusion $\mathcal{N} \subset \mathcal{N}_{\mathcal{E}}$ follows from Theorem 3.1 item (1).

Let $A \in \mathcal{N}_{\mathcal{E}}$ and G be nonempty open. Then there is a nonempty open set $H \subset G$ such that $A \cap H$ is $\mathcal{E}$-scattered, hence $A \cap H \in \mathcal{N}_{\mathcal{E}}$. By the assumption $\overline{A \cap H}$ is $\mathcal{E}$-nowhere dense.

We will prove that $A_0 := \overline{A \cap H}$ is nowhere dense. Let G be nonempty open and not disjoint with A_0. Since $\mathcal{E}$ is a π-network in X, $H_0 := G \cap (X \setminus A_0) \neq \emptyset$ (suppose contrary $G \subset A_0$. Since A_0 is $\mathcal{E}$-nowhere dense, there is a nonempty open subset H_1 of G such that $H_1 \cap A_0 = H_1$ contains no set from $\mathcal{E}$, contradiction with assumption that $\mathcal{E}$ is a π-network in X). So H_0 is nonempty open subset of G and disjoint with A_0, so $\overline{A \cap H}$ is nowhere dense.

Since $\overline{A \cap H}$ is nowhere dense, $H \setminus \overline{A} \neq \emptyset$ (if $H \setminus \overline{A} = \emptyset$, then $H \subset \overline{A}$, so $H = H \cap \overline{A} \subset \overline{H \cap A}$, a contradiction). We have found a nonempty open subset G_0 $(= H \setminus \overline{A})$ of G, which is disjoint with $\overline{A}$, so $\overline{A} \in \mathcal{N}$. That means $A \in \mathcal{N}$.

The implication (2) $\Rightarrow$ (1) is clear. □

For further research it is recommended to investigate other conditions under which the system $\mathcal{N}_{\mathcal{E}}$ is an ideal and examine the properties that are known in the Baire spaces.

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