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DUGDALE MODEL FOR THREE UNEQUAL COLLINEAR STRAIGHT CRACKS WITH COALESCED YIELD ZONES: A COMPLEX VARIABLE APPROACH

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Abstract: The paper aims to provide an analytical solution of the problem of three unequal collinear straight cracks with coalesced plastic/yield zones weakening an infinite elastic perfectly plastic plate. These cracks are located on the real axis in such a manner that the two cracks exist very close to each other. Cracks are open in mode-I type deformation on the application of uniform stresses at the infinite boundary of the plate. Hence, yield zones are developed at each crack tip. Stresses applied at the infinite boundary of the plate increased to such a limit that the yield zones developed between two closely located cracks are coalesced. Assume that the yield stress of the plate is enough to detain further opening of cracks. The problem is solved using complex variable method and analytical expressions are derived for stresses applied at infinite boundary of the plate, yield zone length. A comparative study is carried out for load bearing capacity with the results of two collinear straight cracks, which is the limiting case of the cracks configuration considered in this paper.

AMS Subject Classification: 74R05, 74R10
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1. Introduction

Engineering materials generally faced crack/defect problems on the application of mechanical loads. Sometimes the structure fails at the stress which is well below the yield stress of the materials [1]. Structure with defects will survive but the residual strength of the material used will decreases [2] in the presence of these defects. Therefore, it is imperative to know the residual strength of the materials in the presence of cracks or crack like defects. In order to determine the residual strength of the material, Dugdale[3] proposed a model under general yielding conditions. Due mathematical simplicity the model has been widely used and modified for various metals, different cracks configurations and loading conditions. Theocaris[4] modify Dugdale's model for various metal configurations. Parabolic stress distribution was assumed on the rims of yield zones by Harrop[5] to evaluate relation between stress intensity factor and crack opening displacement. Mix-mode Dugdale model has been proposed by Bowie[6] and for circular arc cracks by Bhargava et.al. [7].

After the successful use of classical Dugdale model for single crack problem. The idea was extended by Theocaris[8] for two unequal symmetric collinear straight cracks under general yielding conditions. Bhargava et.al.[9] modified Dugdale's model for two unequal asymmetric cracks under quadratically varying stress distribution and for coalesced yield zones in [10]. Collins et.al.[11] further solved two collinear symmetrical straight cracks problem under the assumption that the plastic zones be subjected to yield stress distribution. The complex variable method is widely used to solve the multiple crack problems for example [12], [13], [14], [15], [16], [17]. Other methods were also used to solve multiple crack problems like two collinear symmetrical cracks was solved by Zhou[18] using Fourier transform method. Numerical technique was used by Chang[19] to solve a two collinear cracks problem using Dugdale hypothesis. Weight function approach was adopted to solve multi-site damage (MSD) problem by Wu[20].

The paper deals with a limiting case of two equal collinear straight cracks lying on the real axis. In other words, the problem considered in this paper is assuming a predecessor to the problem of two equal collinear straight cracks. Consider three unequal collinear straight cracks lying on the real axis. Rims of the cracks open in mode-I type deformation when uniform forces are applied at the infinite boundary of the plate. As a result, yield zones develop at each crack tip. Out of the three cracks, the yield zones developed between two closely located cracks are coalesced. To determine load bearing capacity of the plate also to detain cracks from further opening assume that the rims of the yield zones are subjected to yield stress distribution. Graphical illustration of the numerical results obtained for yield zone length, load required ratio has been presented.

Nomenclature

$C_i(i=0,1,2)$	constants of the problem
E	Young's modulus
$F(\theta,k), E(\theta,k), \Pi(\theta,\alpha^2,k)$	incomplete elliptic integral of first, second and third kind respectively
$L_i(i=1,2,3)$	cracks
$P_n(z)$	polynomial of degree n
$\delta(x)$	crack-tip-opening displacement at the crack tip x
$\pm a_1, \pm b_1, c_1, d_1$	crack tips
$\pm b, \pm a$	tips of the developed yield zones
$p(t), q(t)$	applied stresses on the yield zones
$z = x + iy$	complex variable
Γ'	$-\frac{1}{2}(N_1 - N_2)e^{-2i\alpha}, N_1$ and N_2 are the values of principal stresses at infinity, α be the angle between N_1 and the ox-axis
$p_i(i=1,2,...,5)$	developed yield zones
$\Omega(z)=\omega'(z), \Phi(z)=\phi'(z)$	complex stress functions
γ	Poisson's ratio
μ	shear modulus
κ	$=\frac{3-\gamma}{1+\gamma}$ for the plane-stress, $= 3-4\gamma$ for the plane-strain
$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$	components of stress
σ_∞	remotely applied stress at infinite boundary of the plate

2. Mathematical Formulation

According to Muskhelishvili[21] the components of stresses, $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ may be expressed in terms of two complex potential functions $\Phi(z)$ and $\Omega(z)$ as

$$\sigma_{xx} + \sigma_{yy} = 2[\Phi(z) + \overline{\Phi(z)}], \quad (1)$$

$$\sigma_{yy} - i\sigma_{xy} = \Phi(z) + \Omega(\bar{z}) + (z-\bar{z})\overline{\Phi'(z)}, \quad (2)$$

Under the assumption that for all t on L_i, $\lim_{y\to 0} y\Phi'(t+iy) = 0$, eqs.2 may be expressed in terms of two Hilbert problems

$$\Phi^+(t) + \Omega^-(t) = \sigma_{yy}^+ - i\sigma_{xy}^+, \tag{3}$$

$$\Phi^-(t) + \Omega^+(t) = \sigma_{yy}^- - i\sigma_{xy}^-, \quad on \quad \bigcup_{i=1}^{n} L_i. \tag{4}$$

Stress distributions $\sigma_{yy}^{\pm}$, $\sigma_{xy}^{\pm}$ are prescribed on the rims of straight cuts $L_i (i = 1, 2, ..., n)$, where superscript (+) denotes upper and (-) lower faces of the cuts. The problems defined in eqs. (3 & 4) are solved using methodology given in [21]. Hence, the desired Complex potential functions $\Phi(z)$ and $\Omega(z)$ may be expressed as

$$\Phi(z) = \Phi_0(z) + \frac{P_n(z)}{X(z)} - \frac{1}{2}\overline{\Gamma}', \tag{5}$$

$$\Omega(z) = \Omega_0(z) + \frac{P_n(z)}{X(z)} + \frac{1}{2}\overline{\Gamma}', \tag{6}$$

where:

$$\Phi_0(z) = \frac{1}{2\pi i X(z)} \int\limits_{\bigcup_{i=1}^{n} L_i} \frac{X^+(t)p(t)}{t-z} dt + \frac{1}{2\pi i} \int\limits_{\bigcup_{i=1}^{n} L_i} \frac{q(t)}{t-z} dt, \tag{7}$$

$$\Omega_0(z) = \frac{1}{2\pi i X(z)} \int\limits_{\bigcup_{i=1}^{n} L_i} \frac{X^+(t)p(t)}{t-z} dt - \frac{1}{2\pi i} \int\limits_{\bigcup_{i=1}^{n} L_i} \frac{q(t)}{t-z} dt, \tag{8}$$

$$p(t) = \frac{1}{2}(\sigma_{yy}^+ + \sigma_{yy}^-) - \frac{i}{2}(\sigma_{xy}^+ + \sigma_{xy}^-), q(t) = \frac{1}{2}(\sigma_{yy}^- - \sigma_{yy}^-) - \frac{i}{2}(\sigma_{xy}^- - \sigma_{xy}^-), \tag{9}$$

$$X(z) = \prod_{k=1}^{n} \sqrt{z-a_k}\sqrt{z-b_k}, P_n(z) = C_0 z^n + C_1 z^{n-1} + ... + C_n. \tag{10}$$

Constants $C_i (i = 0, 1, 2, .., n)$ shown in eq.10 are evaluated using loading condition at an infinite boundary of the plate and single-valuedness condition of displacement around the rims of the cracks or cuts,

$$2(\kappa+1)\int\limits_{L_i} \frac{P_n(t)}{X(t)} dt + \kappa \int\limits_{L_i} [\Phi_0^+(t) - \Phi_0^+(t)] dt + \int\limits_{L_i} [\Omega_0^+(t) - \Omega_0^-(t)] dt = 0. \tag{11}$$

The mathematical formulation given above is taken from Muskhelishvili[21] for making the paper self-sufficient.

3. Formulation of the Problem

An infinite isotropic elastic perfectly plastic plate is weakened by three unequal collinear straight cracks, L_1, L_2 and L_3 lying along ox-axis and occupy the intervals $[-a_1, -b_1], [b_1, c_1]$ and $[d_1, a_1]$, respectively. Plastic zones are developed at each crack tip when uniform stresses σ_∞ act at the boundary of the plate. Developed plastic zones between two closely situated cracks L_2 and L_3 get coalesced on increasing stresses at the boundary of the plate. These plastic zones are denoted by p_1, p_2, p_3, p_4 and p_5 and occupy the intervals $(-a, -a_1), (-b_1, -b), (b, b_1), (c_1, d_1)$ and (a_1, a) on the real axis. Moreover, plastic zones are subjected to a yield stress distribution, σ, to arrest further opening of cracks. The entire configuration of the problem is depicted in fig.1.

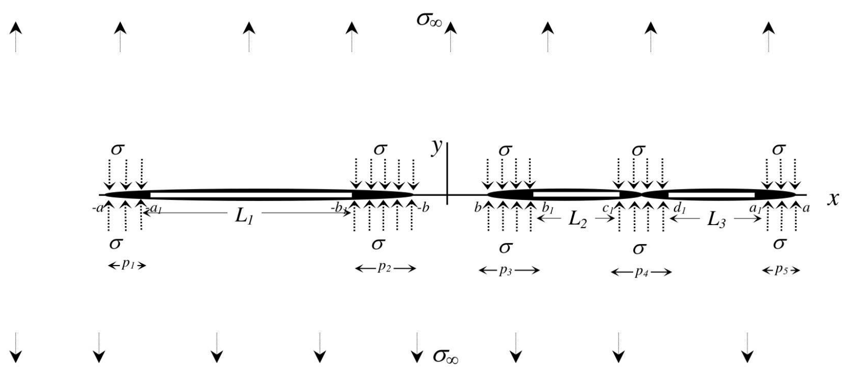

Figure 1: Configuration of the problem

4. Solution of the Problem

The stated problem is divided into two component problems and solved separately using the methodology given in Section 2. First problem is considered as applied case in which forces applied at the infinite boundary of the plate to open the cracks in mode-I type deformation. On the other hand, second case is considered as closing case when yield stress of the plate, σ, is assumed to be distributed on the rims of the yield zones. The solution of the problem define in the Section 3 is obtained by superposing the solutions of two component problems.

4.1. Case-I: Applied Case

Two equal collinear straight cracks weakened an infinite isotropic plate are assumed to be exists on the real axis as shown in fig.2. The cracks open in mode-I type deformation under the action of uniform stress, σ_∞, at the boundary of the plate. As a results, plastic zones developed at each crack tip. Assume that the yield zones are inline with the cracks and considered as a part of physical cracks. Furthermore, σ_∞ increases to such a limit that the yield zones developed between two closely situated cracks get coalesced. The boundary conditions of the problem are

$$\sigma_{yy} = \sigma_\infty, \sigma_{xy} = 0, \quad \text{when} \quad y \to \pm\infty, \tag{12}$$

$$\sigma_{yy} = 0, \sigma_{xy} = 0, \quad \text{when} \quad y \to 0. \tag{13}$$

The desired complex potential function is obtained easily using boundary conditions given in eqs.12&13 or may be taken directly from [11],

$$\Phi_{applied}(z) = \frac{\sigma_\infty}{2\sqrt{z^2-a^2}\sqrt{z^2-b^2}}\left[z^2 - a^2\lambda^2\right] - \frac{\sigma_\infty}{4}. \tag{14}$$

where $k^2 = \frac{a^2-b^2}{a^2}$,$\lambda^2 = \frac{E(k)}{K(k)}$ and $E(k), F(k)$ are the complete elliptic integral of first and second kind respectively. The notations are same as given in [22].

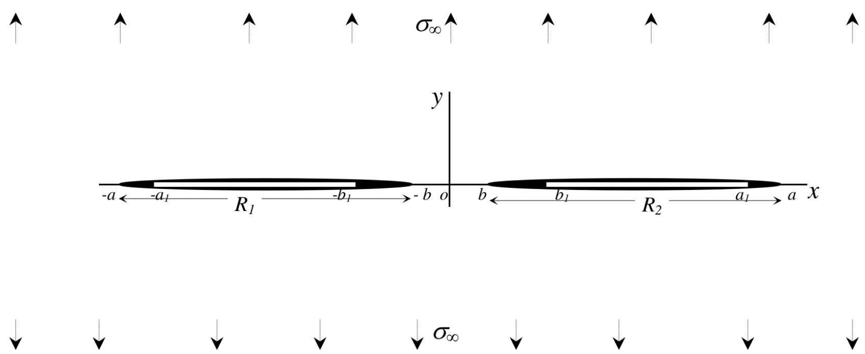

Figure 2: Configuration of the elastic case of the problem

4.2. Case-II: Yield Case

An infinite isotropic plate is weakened by three unequal collinear straight cracks with coalesced yield zones as depicted in fig.3. Infinite boundary is stress free. Rims of the developed yield zones, $p_i (i = 1, 2, \ldots, 5)$ are subjected to yield stress distribution, σ, to detain further opening of cracks. The problem is subjected to following boundary conditions

$$\sigma_{yy} = \sigma, \sigma_{xy} = 0 \quad \text{for} \quad y \to 0, x \in \bigcup_{n=1}^{4} \Gamma_n, \tag{15}$$

$$\sigma_{yy} = 0, \sigma_{xy} = 0 \quad \text{for} \quad -\infty < x < \infty, y \to 0. \tag{16}$$

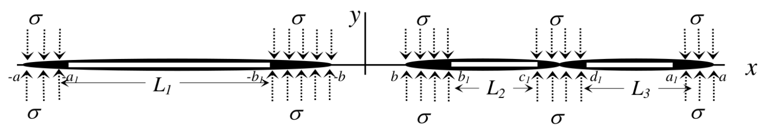

Figure 3: Configuration of the yield case of the problem

Complex potential function $\Phi(z)$ due to yield stress distribution σ over the rims of the yield zones is evaluated using methodology given in Section 2 and using the boundary conditions given in eqs.15 & 16

$$\Phi_{yield}(z) = \frac{\sigma}{2\pi X(z)}\Big(P - z^2 Q + zR + \frac{z(z^2 - b^2)}{a} G(z) + 2X(z)F(z)\Big), \tag{17}$$

where:

$$\begin{aligned}
P &= a^2 k^2 (\sin 2\phi_{b_1} - \sin 2\phi_{a_1} - 0.5 \sin 2\phi_{c_1} + 0.5 \sin 2\phi_{d_1}) \\
&\quad + \lambda^2 a^2 Q + 2ab_1(\lambda^2 F(\phi_{b_1}, k) - E(\phi_{b_1}, k)) + \\
&\quad ac_1(\lambda^2 F(\phi_{c_1}, k) - E(\phi_{c_1}, k)) + ad_1(\lambda^2 F(\phi_{d_1}, k) - E(\phi_{d_1}, k)) \\
&\quad - 2aa_1(\lambda^2 F(\phi_{a_1}, k) - E(\phi_{a_1}, k)), \\
Q &= \pi - 2\phi_{b_1} + 2\phi_{a_1} + \phi_{c_1} - \phi_{d_1}, \\
R &= a(F(\phi_{d_1}, k) - F(\phi_{c_1}, k) + E(\phi_{c_1}, k) - E(\phi_{d_1}, k)), \\
G(z) &= II(\phi_{d_1}, \alpha^2, k) - F(\phi_{d_1}, k) - II(\phi_{c_1}, \alpha^2, k) + F(\phi_{c_1}, k)
\end{aligned}$$

$$
\begin{aligned}
F(z) &= \frac{\pi}{2} - \tan^{-1}\Big(\frac{1+i\tan\phi_z\cot\phi_{b_1}}{1-i\tan\phi_z\cot\phi_{b_1}}\Big) + \tan^{-1}\Big(\frac{1+i\tan\phi_z\cot\phi_{a_1}}{1-i\tan\phi_z\cot\phi_{a_1}}\Big) + \\
& 0.5\tan^{-1}\Big(\frac{1+i\tan\phi_z\cot\phi_{c_1}}{1-i\tan\phi_z\cot\phi_{c_1}}\Big) + 0.5\tan^{-1}\Big(\frac{1+i\tan\phi_z\cot\phi_{d_1}}{1-i\tan\phi_z\cot\phi_{d_1}}\Big). \\
\sin^2\phi_t &= \frac{a^2-t^2}{a^2-b^2}, \alpha^2 = \frac{a^2-b^2}{a^2-z^2}
\end{aligned}
$$

5. Stress Intensity Factors and Yield Zone Length

Mode-I stress intensity factor at each crack tip $z = \pm a, \pm b$ is obtained using formula given in [11],

$$
K_I = 2\sqrt{2\pi} \lim_{z \to z_1} \sqrt{z - z_1}(\Phi_{applied}(z) + \Phi_{yield}(z)). \quad (18)
$$

Using Dugdale hypothesis that the stresses remains finite at each cracks. Hence, four non-linear equations are obtained to determine yield zone length at each crack tip. Equations at the tips $z = a, -a, b, -b$ are given below, respectively.

$$
\begin{aligned}
\pi a^2(1-\lambda^2)\big(\frac{\sigma_\infty}{\sigma}\big)_a + (P - a^2Q + aR) &= 0, & (19)\\
\pi a^2(1-\lambda^2)\big(\frac{\sigma_\infty}{\sigma}\big)_{-a} + (P - a^2Q - aR) &= 0, & (20)\\
\pi(b^2 - a^2\lambda^2)\big(\frac{\sigma_\infty}{\sigma}\big)_b + (P - b^2Q + bR) &= 0, & (21)\\
\pi(b^2 - a^2\lambda^2)\big(\frac{\sigma_\infty}{\sigma}\big)_{-b} + (P - b^2Q - bR) &= 0. & (22)
\end{aligned}
$$

These equations are enable to determine the yield zone length at each crack tip with respect to remotely applied stresses at the infinite boundary of the plate.

6. Validation of Analytical Expressions

The analytical expressions given in eqs.17, 19 & 21 are validated with the results of two equal cracks given in [11] taking $c_1 = d_1$.

7. Application: Yield Zone Length and Crack Tip Opening Displacement

A case study has been carried out to study the behaviour of yield zone length due to increasing stresses acting at the infinite boundary of the plate, which is damaged by three unequal collinear straight cracks with coalesced yield zones. Assume that the problem considered in the paper is a priori of the problem of two equal collinear straight cracks. Therefore, the results obtained be compared with the existing results of two equal collinear straight cracks given by collins[11].

Variations has been plotted between applied load ratio $\frac{\sigma_\infty}{\sigma}$ and yield zone length $\frac{p_i}{p_0}(i = 2, 3)$ in fig.4 & 5. The results obtained are normalized with the yield zone length of a single Dugdale crack $[p_0 = 0.5(a_1 - b_1)\big(\sec(\frac{\pi\sigma_\infty}{2\sigma}) - 1\big)]$. Ratio $\frac{A}{D}$ denotes the distance between two systems $S_1 = p_1 \cup L_1 \cup p_2$ and

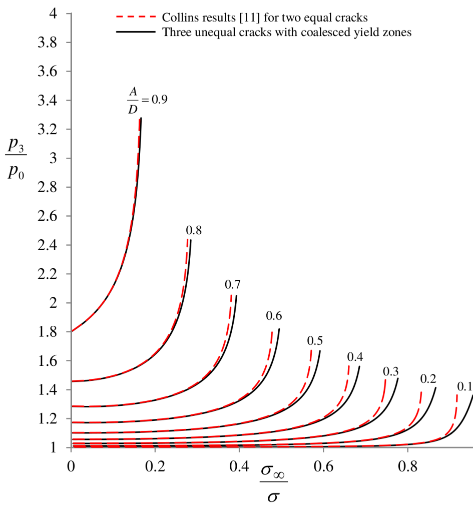

Figure 4: Variation of $\frac{p_3}{p_0}$ to $\frac{\sigma_\infty}{\sigma}$

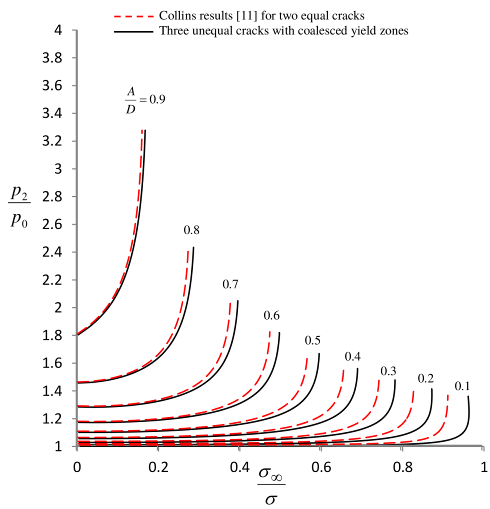

Figure 5: Variation of $\frac{p_2}{p_0}$ to $\frac{\sigma_\infty}{\sigma}$

$S_2 = p_3 \cup L_2 \cup p_4 \cup L_3 \cup p_5$. These systems are situated far away form each other when $\frac{A}{D} = 0.1$ and very close to each other when $\frac{A}{D} = 0.9$. It has been observed that the yield zone p_3 increases gradually when load applied at the infinite boundary increases as shown in fig.4. When load applied at the infinite boundary is lower than the yield stress of the plate then configuration shown in fig.1 behaves like a single crack at $\frac{A}{D} = 0.1$, but a significant difference is seen when σ_∞ is approximately equal to σ. As the systems S_1 & S_2 assumed to be exist closer to each other ($\frac{A}{D} = 0.8, 0.9$) the infinite plate can bear very less load.Moreover, the results obtained are compared with the results of two equal cracks and observed a significant difference in load bearing capacity of the plate in the said comparison at a stress level when σ_∞ and σ are approximately same.

Fig.5 shows the same variation discussed above at the crack tip $z = -b$. Length of yield zone p_2 increases as the load applied at the infinite boundary increases. When $\frac{A}{D} = 0.1$, significant difference is seen between the load bearing

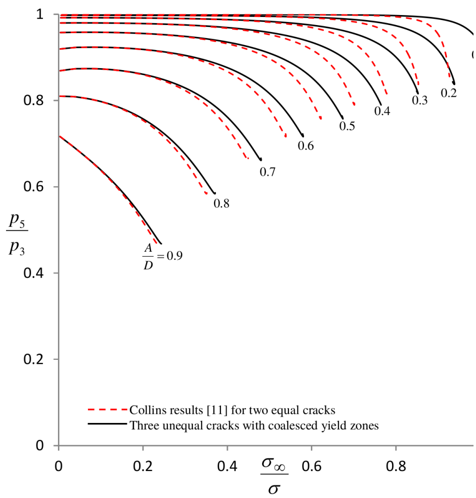

Figure 6: Variation of $\frac{p_5}{p_3}$ to $\frac{\sigma_\infty}{\sigma}$

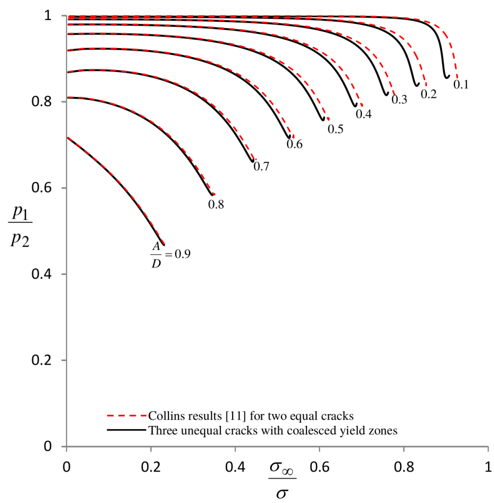

Figure 7: Variation of $\frac{p_1}{p_2}$ to $\frac{\sigma_\infty}{\sigma}$

capacity of plate containing three cracks with coalesced yield zone in comparison to two equal cracks.

Furthermore, variation between applied load ratio $\frac{\sigma_\infty}{\sigma}$ and yield zone ratio $\frac{p_5}{p_3}$ has been plotted in fig.6 to compare the yield zone lengths at the inner and outer crack tips p_3 and p_5. It has been observed that the yield zone p_3 is bigger than p_5 significantly when the systems S_1 and S_2 are located far away from each other and also when σ_∞ is approximately equal to σ. Hence, the inner crack tip is more critical than outer crack tip due to bigger yield zone. Same variation is shown in fig.7 but this time for yield zone ratio $\frac{p_1}{p_2}$. The results shown in figs6 and 7 are compared with the results of two equal cracks and observed a significant difference in load bearing capacity of the plate when σ_∞ is approximately equal to σ.

8. Conclusion

Based on the current study following conclusion has been drawn

- Dugdale model of three unequal cracks with coalesced yield zones has been discussed and analyzed under general yielding conditions.
- Analytical expressions for complex potential function and stress intensity factors are obtained using complex variable method.
- Behaviour of yield zones has been plotted on increasing stresses applied at the infinite boundary of the plate.
- It is seen from the figures that the yield zones p_2, p_3 increase faster and bigger than p_1, p_5 respectively.
- Results obtained are compared with the existing results of two equal cracks problem. A significant difference is seen between these results when stresses applied at the infinite boundary of the plate is approximately equal to the yield stress of the plate.

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