International Journal of Pure and Applied Mathematics
Volume 105 No. 2 2015, 153-162
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v105i2.3

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OPERATIONS ON S-GRAPHS

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Abstract: In our earlier paper (see [4]), we have introduced the notion of semiring-valued graphs and (see [3]) the notion of regularity on S-graphs. In this paper, we introduce and study some operations such as union and sum of two S-graphs.

AMS Subject Classification: 05C25, 16Y60
Key Words: S-graphs, regular S-grpahs, operations on S-graphs

1. Introduction

Eventhough the concept of semiring was first introduced by H.S.Vandiver [5] in 1934, the developments of the theory in semirings and ordered semirings have been taking place since 1950. Jonathan S.Golan [2] has introduced the notion of S-graph where he considers a function $g : V \times V \to S$ such that $g(v_1, v_2) \neq 0$. But nothing more has been dealt. This motivated us to study graphs whose vertices and edges are assigned values from the semiring S [4]. Golan considers the S-graph by assigning values to edges only. However we assign values to every vertex of the graph and the weights of an edge is assigned in relation to the weights of the vertices incident with the edges. Since every semiring

Received: August 5, 2015

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possesses a canonical pre-order, for any edge $e = (v_i, v_j)$, we can assign the weight of e as the minimum weights of v_i and v_j. Such a graph we called a S-graph. In our paper [3], we studied the notion of regularity on S-graphs. In this paper, we introduce and study the notion of operations such as union and sum of two S-graphs.

2. Preliminaries

In this section, we recall the basic definitions on operations on crisp graphs, such as union and sum of two graphs. For graph theoretical concepts, we refer [1].

Definition 2.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \phi$. We define the Union $G_1 \cup G_2 = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.

Definition 2.2. The sum $G_1 + G_2$ as $G_1 \cup G_2$ together with all the lines joining points of V_1 to points of V_2.

Definition 2.3. A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set S together with two binary operations $+$ and $\cdot$ such that

1. $(S, +, 0)$ is a monoid.
2. $(S, \cdot)$ is a semigroup.
3. For all $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
4. $0 \cdot x = x \cdot 0 = 0 \ \forall \ x \in S$.

Definition 2.4. Let S be a semiring. $\preceq$ is said to be a Canonical Pre-order if for $a, b \in S$, $a \preceq b$ if and only if there exists $c \in S$ such that $a + c = b$.

Definition 2.5. [4] Let $G = (V, E \subset V \times V)$ be the underlying graph with both $V, E \neq \phi$. For any semiring $(S, +, \cdot)$, a Semiring-valued graph (or an S-valued graph) G^S is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \to S$ and $\psi : E \to S$ is defined to be

$$\psi(x, y) = \begin{cases} min\{\sigma(x), \sigma(y)\} & if \ \sigma(x) \preceq \sigma(y) \ or \ \sigma(y) \preceq \sigma(x) \\ 0 & otherwise \end{cases}$$

for every unordered pair (x, y) of $E \subset V \times V$. We call σ, a S-vertex set and ψ, a S-edge set of S-valued graph G^S.

Henceforth we call a S-valued graph simply as a S-graph.

Definition 2.6. [4] If $\sigma(x) = a, \forall\ x \in V$ and some $a \in S$ then the corresponding S-graph G^S is called a vertex regular S-graph (or simply vertex regular).

Definition 2.7. [4] An S-graph G^S is said to be an edge regular S-graph (or simply edge regular) if $\psi(x, y) = a$ for every $(x, y) \in E$ and some $a \in S$.

Definition 2.8. [4] An S-graph G^s is said to be S-regular if it is both vertex regular and edge regular .

Definition 2.9. [3] Let G^S be an S-graph corresponding to an underlying graph G. G^S is said to be $(a, k)-$regular S-graph if the following conditions are true.

1. The crisp graph G is $k-$regular.
2. $\sigma(v) = a$ for every vertex v in G.

3. Operations on S-Graph

In this section, we define the notion of union and sum of two S-graphs and prove some simple results.

Definition 3.1. Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ be two S-graphs corresponding to the underlyings graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \phi$ respectively. Then their union, denoted by $G_1^S \cup G_2^S$, is defined as $G_1^S \cup G_2^S = (V, E, \sigma, \psi)$ where$V = V_1 \cup V_2$; $E = E_1 \cup E_2$ and for $v \in V$,

$$\sigma(v) = \begin{cases} \sigma_1(v) & if v \in V_1 \\ \sigma_2(v) & if v \in V_2 \end{cases};$$

For $(v_i, v_j) \in E = E_1 \cup E_2$,$\psi(v_i, v_j) = \begin{cases} \psi_1(v_i, v_j) & if\ (v_i, v_j) \in E_1 \\ \psi_2(v_i, v_j) & if\ (v_i, v_j) \in E_2 \end{cases}$

Remark 3.2. Clearly $G_1^S \cup G_2^S$ is an S-graph. In general, $G_1^S \cup G_2^S \neq (G_1 \cup G_2)^S$.

Example 3.3. Let $(S = \{0, a, b\}, +, \cdot)$ be a semiring with the following Cayley Tables:

$+$	0	a	b
0	0	a	b
a	a	o	b
b	b	b	b

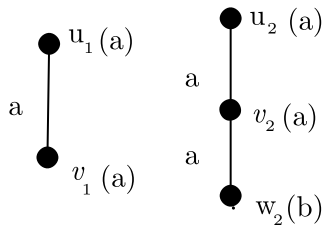
$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

Clearly $\preceq$ is a canonical pre-order in S, where

$$0 \preceq 0, 0 \preceq a, 0 \preceq b, a \preceq a, a \preceq 0, a \preceq b, b \preceq b.$$

Let G_1 and G_2 be two crisp graphs given by :

Here $V_1 = \{u_1, v_1\}$ and $V_2 = \{u_2, v_2, w_2\}$
$E_1 = \{(u_1, v_1)\}$ and $E_2 = \{(u_2, v_2), (v_2, w_2)\}$
$\sigma_1 : V_1 \to S$is defined by $\sigma_1(u_1) = \sigma_1(v_1) = a$and $\psi_1(u_1, v_1) = a$.
And $\sigma_2 : V_2 \to S$ is defined by $\sigma_2(u_2) = \sigma_2(v_2) = a$ and $\sigma_2(w_2) = b$.
Then $\psi_2(u_2, v_2) = \psi_2(v_2, w_2) = a$.
Therefore the union of the S-graphs G_1^S and G_2^S is the S-graph $G_1^S \cup G_2^S$:



Here $G_1^S \cup G_2^S = (V, E, \sigma, \psi)$ where $V = V_1 \cup V_2$; $E = E_1 \cup E_2$ such that,
$\sigma(u_1) = \sigma_1(u_1)$ and $\sigma(v_1) = \sigma_1(v_1)$ $(\because u_1, v_1 \in V_1)$
$\sigma(u_2) = \sigma_2(u_2)$; $\sigma(v_2) = \sigma_2(v_2)$; $\sigma(w_2) = \sigma_2(w_2)$ $(\because u_2, v_2, w_2 \in V_2)$

$$Similarly\, \psi(u_1, v_1) = \psi_1(u_1, v_1), \psi(u_2, v_2) = \psi_2(u_2, v_2),\ and\, \psi(v_2, w_2) = \psi_2(v_2, w_2)$$

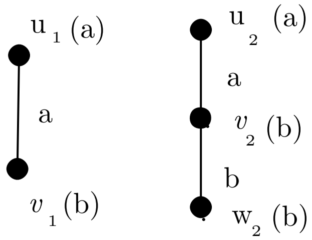
Now $G_1 \cup G_2 = (V = V_1 \cup V_2\ ,\ E = E_1 \cup E_2)$
where $V = \{u_1, v_1, u_2, v_2, w_2\}$ and $E = \{(u_1, v_1), (u_2, v_2), (v_2, w_2)\}$

Since the existance of an S-graph corresponding to its underlying graph is not unique, we define $\sigma_3 : V \to S$ such that
$\sigma_3(u_1) = a$; $\sigma_3(v_1) = b$; $\sigma_3(u_2) = a$; $\sigma_3(v_2) = b$; $\sigma_3(w_2) = b$;
and $\psi_3 : E \to S$ such that

$$\psi_3(u_1, v_1) = min\,\{\sigma_3(u_1), \sigma_3(v_1)\} = min\,\{a, b\} = a \quad (\because a \preceq b)$$

Similarly $\psi_3(u_2, v_2) = a$, $\psi_3(v_2, w_2) = b$
$\Rightarrow \sigma_3(v) \neq \sigma(v)$ for every $v \in V = V_1 \cup V_2$

$\psi_3(v_i, v_j) \neq \psi(v_i, v_j)$ for every $(v_i, v_j) \in E = E_1 \cup E_2$.
Therefore $\sigma_3 \neq \sigma$ and $\psi_3 \neq \psi$. Then $(G_1 \cup G_2)^S$ is



We observe that $G_1^S \cup G_2^S \neq (G_1 \cup G_2)^S$ and the equality holds onlyif $\sigma = \sigma_3$.

Lemma 3.4. Union of two vertex regular S-graphs is a vertex regular S-graph if and only if their corresponding S-vertex sets are constant and assigns the same value in S.

Proof. Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ be two vertex regular S-graphs, corresponding to the underlying graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively.
Since G_1^S and G_2^S are vertex regular, $\sigma_1(v) = a$ for every $v \in V_1$, and some $a \in S$ and $\sigma_2(v) = b$ for every $v \in V_2$, and some $b \in S$,respectively. Hence σ_1 *and* σ_2 are constant. Let $G_1^S \cup G_2^S = (V, E, \sigma, \psi)$ be the union of given S-graphs and let it be vertex regular.
Therefore $\sigma(v) = c$ for every $v \in V$, and some $c \in S$.

Claim. *σ_1 and σ_2 assigns the same value in S.*

Let $v \in V_1 \Rightarrow \sigma_1(v) = a$. Since $v \in V_1 \subset V \Rightarrow \sigma(v) = c$. But $\sigma(v) = \sigma_1(v)$ $\Rightarrow c = a$.

Similarly, if $v \in V_2 \Rightarrow \sigma_2(v) = b$. Since $v \in V_2 \subset V \Rightarrow \sigma(v) = c$. But $\sigma(v) = \sigma_2(v) \Rightarrow c = b$. Therefore $a = b$.

Hence σ_1 and σ_2 are constant and assigns the same value in S.
Conversely, Let σ_1 and σ_2 be constant and assigns the same value in S. Let $a \in S$. and $\sigma_1(v) = a$ *for all* $v \in V_1$, $\sigma_2(v) = a$ for all $v \in V_2$.

Let $v \in V$ be arbitrary.Then

$$\sigma(v) = \begin{cases} \sigma_1(V) & if \;\; v \in V_1 \\ \sigma_2(V) & if \;\; v \in V_2 \end{cases} = \begin{cases} a & if \;\; v \in V_1 \\ a & if \;\; v \in V_2 \end{cases} = a.$$

Hence $\sigma(v) = a$, $\forall v \in V$.
Therefore $G_1^S \cup G_2^S$ is a vertex regular S-graph.

Corollary 3.5. Union of two vertex regular S-graphs is S-regular if and only if their corresponding S-vertex sets are constant and assigns the same value in S.

Lemma 3.6. Union of two edge regular S-graphs is an edge regular S-graph if and only if their corresponding S-edge sets are constant and assigns the same value in S.

Proof. Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ be two edge regular
S-graphs, corresponding to the underlying graphs $G_1 = (V_1, E_1)$ and
$G_2 = (V_2, E_2)$ respectively.
Since G_1^S and G_2^S are edge regular,
$\psi_1(v_i, v_j) = a$, for every $(v_i, v_j) \in E_1$, and some $a \in S$ and $\psi_2(v_i, v_j) = b$ for every $(v_i, v_j) \in E_2$ and some $b \in S$. then both ψ_1 and ψ_2 are constant. Let $G_1^S \cup G_2^S = (V, E, \sigma, \psi)$ be the union of the given S-graphs and let it be an edge regular S-graph.

Then $\psi(v_i, v_j) = c$ for every $(v_i, v_j) \in E$, and some $c \in S$.

Claim. *ψ_1 and ψ_2 assigns the same value in S.*

Let $(v_i, v_j) \in E_1 \Rightarrow \psi_1(v_i, v_j) = a$. Since $(v_i, v_j) \in E_1 \subset E \Rightarrow \psi(v_i, v_j) = c$.
But $\psi(v_i, v_j) = \psi_1(v_i, v_j)$. Hence $c = a$.
Similarly, if $(v_i, v_j) \in E_2$, we can prove that $c = b$. Therefore $a = b$. That is, ψ_1 and ψ_2 are constant and assigns the same value in S.
Conversely, assume that ψ_1 and ψ_2 are constant and assigns the same value in S. Let $a \in S$. $\psi_1(v_i, v_j) = a, \forall (v_i, v_j) \in E_1$; $\psi_2(v_i, v_j) = a, \forall (v_i, v_j) \in E_2$.

Let $(v_i, v_j) \in E$ be arbitrary.Then

$$\psi(v_i, v_j) = \begin{cases} \psi_1(v_i, v_j) & if \ \ (v_i, v_j) \in E_1 \\ \psi_2(v_i, v_j) & if \ \ (v_i, v_j) \in E_2 \end{cases} = \begin{cases} a & if \ \ (v_i, v_j) \in E_1 \\ a & if \ \ (v_i, v_j) \in E_2 \end{cases} = a$$

That is $\psi(v_i, v_j) = a$ for all $(v_i, v_j) \in E$

Therefore $G_1^S \cup G_2^S$ is an edge regular S-graph.

Definition 3.7. Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ be two
S-graphs corresponding to the underlyings graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \phi$ respectively. Then their sum, denoted by $G_1^S + G_2^S$, is defined as $G_1^S + G_2^S = (V, E, \sigma, \psi)$ where:

1. $V = V_1 \cup V_2; E = E_1 \cup E_2 \cup \{(v_i, v_j) \mid v_i \in V_1 \ and \ v_j \in V_2\}$

2. for $v \in V$, $\sigma(v) = \begin{cases} \sigma_1(v) & if\ v \in V_1 \\ \sigma_2(v) & if\ v \in V_2. \end{cases}$

3. for$(v_i, v_j) \in E$,

$$\psi(v_i, v_j) = \begin{cases} \psi_1(v_i, v_j) & if\ (v_i, v_j) \in E_1 \\ \psi_2(v_i, v_j) & if\ (v_i, v_j) \in E_2 \\ \min\{\sigma_1(v_i), \sigma_2(v_j)\} & if\ v_i \in V_1\ \ and\ \ v_j \in V_2 \end{cases}$$

Remark 3.8. Clearly sum of two S-graphs is also an S-graph.

Lemma 3.9. Sum of two vertex regular S-graphs is an vertex regular S-graph iff their corresponding S-vertex sets are constant and assigns the same value in S.

Proof. Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ be two vertex regular S-graphs, corresponding to the underlying graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively.
Since G_1^S and G_2^S are vertex regular,
$\sigma_1(v) = a$ for every $v \in V_1$, and some $a \in S$ then σ_1 is constant
and $\sigma_2(v) = b$ for every $v \in V_2$, and some $b \in S$ then σ_2 is constant.

Let $G_1^S + G_2^S = (V, E, \sigma, \psi)$ be the sum of given S-graphs and let it be vertex regular S-graph.

Therefore $\sigma(v) = c$ for every $v \in V$, and some $c \in S$.

Claim. *σ_1 and σ_2 assigns the same value in S.*

Let $v \in V_1 \Rightarrow \sigma_1(v) = a$.
Since $v \in V_1 \subset V \Rightarrow \sigma(v) = c$.
But $\sigma(v) = \sigma_1(v) \Rightarrow c = a$.
Similarly, if $v \in V_2$ we can prove that $c = b$. Therefore $a = b$.
Hence σ_1 and σ_2 assigns the same value in S.
Conversely, assume that σ_1 and σ_2 are constant and assign the same value in S. Let $a \in S$ and $\sigma_1(v) = a$ for all $v \in V_1$, $\sigma_2(v) = a$, for all $v \in V_2$.

Let $v \in V$ be arbitrary. Then

$$\sigma(v) = \begin{cases} \sigma_1(V) & if\ v \in V_1 \\ \sigma_2(V) & if\ v \in V_2 \end{cases} = \begin{cases} a & if\ v \in V_1 \\ a & if\ v \in V_2 \end{cases} = a.$$

That is $\sigma(v) = a$, $\forall v \in V$ and $a \in S$.
Therefore $G_1^S + G_2^S$ is a vertex regular S-graph.

Corollary 3.10. Sum of two vertex regular S-graphs is S-regular if and only if their corresponding S-vertex sets are constant and assigns the same value in S.

Remark 3.11. If G_1^S and G_2^S are edge regular S-graphs then their sum need not be, in general, an edge regular S-graph.

Proof. Let $(S = \{0, a, b\}, +, \cdot)$ be a semiring with the following Cayley Tables:

$+$	0	a	b
0	0	a	b
a	a	o	b
b	b	b	b

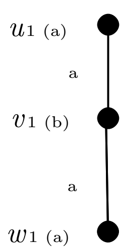
$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

Clearly $\preceq$ is a canonical pre-order in S, where

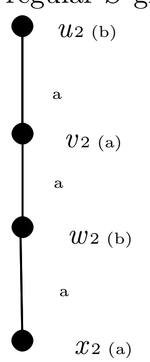
$$0 \preceq 0, 0 \preceq a, 0 \preceq b, a \preceq a, a \preceq 0, a \preceq b, b \preceq b.$$

Let G_1^S and G_2^S be two edge regular but not vertex regular S-graphs given by :

G_1^S :

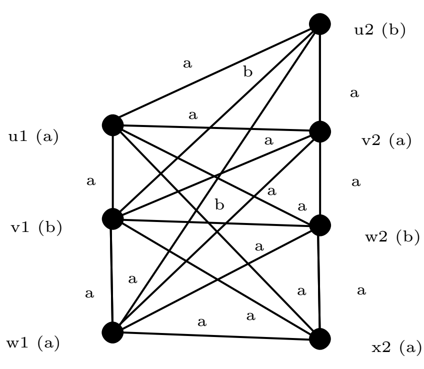


G_2^S :



Here $G_1^S + G_2^S = (V, E, \sigma, \psi)$ where
$V = V_1 + V_2$;$E = E_1 \cup E_2 \cup \{(v_i, v_j) \mid v_i \in V_1 \text{ and } v_j \in V_2\}$
such that,

$G_1^S + G_2^S$:



Here $\psi(v_i, v_j) \neq a$, for all $(v_i, v_j) \in E$.
Therefore $G_1^S + G_2^S$ is not an edge regular S-graph.

Theorem 3.12. Sum of two edge regular S-graphs is an edge regular S-graph only if their corresponding S-vertex sets are constants and assigns the same value in S.

Proof. Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ be two edge regular S-graphs, corresponding to the underlying graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively.

Since G_1^S and G_2^S are edge regular.

$\psi_1(v_i, v_j) = a$ for every $(v_i, v_j) \in E_1$, and some $a \in S$.

$\psi_2(v_i, v_j) = b$ for every $(v_i, v_j) \in E_2$, and some $b \in S$.

Claim. *$G_1^S + G_2^S$ is an edge regular S-graph only if their corresponding S-vertex sets assigns the same value in S.*

Case 1. Suppose the S-vertex sets of $G_1^S$and G_2^S are constants and assigns the same value.

By corollary 3.10 $G_1^S + G_2^S$ is a S-regular graph. Therefore it is an edge regular S-graph.

Case 2. Suppose G_1^S and G_2^S are edge regular and corresponding S-vertex sets are constant and $\sigma_1(v_i) = a\ forsome\ a \in S\ and\ for\ every\ v_i \in V_1$; and $\sigma_2(v_j) = b\ forsome\ b \in S\ and\ for\ every\ v_j \in V_2\ such\ that\ a \neq b$.Then for$(v_i, v_j) \in E$,

$$\begin{aligned} \psi(v_i, v_j) &= \begin{cases} \psi_1(v_i, v_j) & if\ (v_i, v_j) \in E_1 \\ \psi_2(v_i, v_j) & if\ (v_i, v_j) \in E_2 \\ \min\{\sigma_1(v_i), \sigma_2(v_j)\} & if\ v_i \in V_1, v_j \in V_2 \end{cases} \\ &= \begin{cases} a & if\ (v_i, v_j) \in E_1 \\ b & if\ (v_i, v_j) \in E_2 \\ \min\{a, b\} & if\ v_i \in V_1, v_j \in V_2, \end{cases} \end{aligned}$$

Since the edges in E_1 and E_2 assumes different values, we have $\psi(v_i, v_j)$ is not a constant for every $(v_i, v_j) \in E$.

Therefore $G_1^S + G_2^S$ is not an edge regular S-graph.

Case 3. Suppose G_1^S and G_2^S are edge regular and corresponding S-vertex sets are not constant. Then there exists some $v_k \in V_1$ such that $\sigma_1(v_k) = c$ for some $c \in S$, and $c \neq a$.

Similarly there exists some $v_l \in V_2$ such that $\sigma_2(v_l) = d$ for some $d \in S$, and $d \neq b$. Clearly $(v_l, v_k)\ or\ (v_k, v_l) \in E$. Then

$$\psi(v_k, v_l) = \min\{c, d\} = \begin{cases} c & if\ \sigma_1(v_k) \preceq \sigma_2(v_l) \\ d & if\ \sigma_2(v_l) \preceq \sigma_1(v_k) \\ 0 & otherwise \end{cases}$$

In any case, $\psi(v_i, v_j)$ is not a constant for all $(v_i, v_j) \in E$.

Thus the sum of two edge regular S-graphs is an edge regular$S$-graph only if their corresponding S-vertex sets are constants and assigns the same value in S.

Theorem 3.13. Union(Sum) of two $(a, k)-$regular S-graphs is S-regular.

Proof. Since every $(a, k)-$regular S-graph is an S-regular, and union(sum) of two S-regular is an S-regular, we have union(sum) of two $(a, k)-$regular graph is S-regular.

4. Conclusion

Unlike the crisp graph theory, in S-graphs, we define the union and sum of two S-graphs by considering the S-values of both vertices and edges. In our future work, we would like to extend the study of S-graphs subject to other operations such as cartesian products and composition of S-graphs.

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