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THE γ-SPECTRUM OF CYCLE WITH ONE CHORD

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Abstract: Let G be a graph of order n and size m. A γ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces an edge-labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ on G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The value of f is defined as

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e)\,.$$

The γ-spectrum of a graph G is defined as

$$\mathrm{spec}(G) = \{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

In this paper, γ-spectrum of cycle with one chord is determined.

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1. Introduction

In 2005 Chartrand, Erwin, VanderJagt, and Zhang [2] studied, for a graph G of order n and size m, a γ*-labeling* of G defined as a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$

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of the edges of G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. Therefore, a graph G of order n and size m has a γ-labeling if and only if $m \geq n-1$. Each γ-labeling f is assigned a *value* denoted by $\mathrm{val}(f)$ and defined by

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e).$$

Obviously, since f is one-to-one, it follows that $f'(e) \geq 1$, for any edge e, and therefore, $\mathrm{val}(f) \geq m$. Moreover, G has a γ-labeling if and only if $m \geq n-1$ and every connected graph has a γ-labeling.

Figure 1 shows nine γ-labelings $f_1, f_2, \ldots, f_9$ of the path P_5 of order 5 (where the vertex labels are shown above each vertex and the induced edge labels are shown below each edge). The value of each γ-labeling is shown in Figure 1 as well.

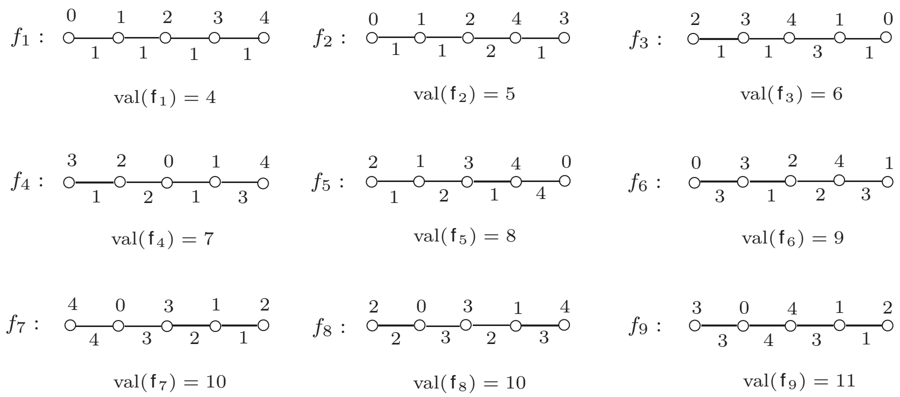

Figure 1: Some γ-labelings of P_5

If the induced edge-labeling f' of a γ-labeling f of a graph is also one-to-one, then f is a *graceful labeling*. Among all labelings of graphs, graceful labelings are probably the best known and most studied. Graceful labelings originated with a paper of Rosa [11], who used the term β-valuations. A few years later, Golomb [9] called these labelings "graceful" and this is the terminology that has been used since then. Gallian [8] has written an extensive survey on labelings of graphs.

The *maximum value* and the *minimum value* of a γ-labeling of G are defined in [2] as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f) \colon \ f \text{ is a } \gamma\text{-labeling of } G\}$$

and

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f)\colon\ f \text{ is a } \gamma\text{-labeling of } G\},$$

respectively. It turns out that $\mathrm{val}_{\max}(P_5) = 11$ and $\mathrm{val}_{\min}(P_5) = 4$.

A γ-labeling g of G is a γ-max *labeling* if $\mathrm{val}(g) = \mathrm{val}_{\max}(G)$ and a γ-labeling h is a γ-min *labeling* if $\mathrm{val}(h) = \mathrm{val}_{\min}(G)$.

The subject of maximum and minimum values of a γ-labeling of path P_n, cycle C_n, complete graph K_n, double star $S_{p,q}$ and complete bipartite graph $K_{r,s}$ and cycle with a triangle $C_n^{\triangle}$ are studied in [1, 2, 3, 5, 6].

Later, the extreme values of γ-labeling of cycle with one chord $C_n + e$, i.e., cycle with a chord e joining two nonadjacent vertices in cycle C_n, are established in [10] as we state next.

Theorem 1.1 ([10]). *Let $C_n + e$ be a cycle with one chord of order $n \geq 4$. Then we have the following.*

(*a*) $\mathrm{val}_{\min}(C_n + e) = 2n - 1$.

(*b*) *For an odd integer n,* $\mathrm{val}_{\max}(C_n + e) = \frac{n^2+6n-3}{2}$.

(*c*) *For an even integer n,*

(*i*) $\mathrm{val}_{\max}(C_n + e) = \frac{n^2+6n+2}{2}$ *where e is a chord joining two vertices with odd distance in even cycle C_n,*

(*ii*) $$\mathrm{val}_{\max}(C_n + e) = \begin{cases} \frac{n^2+5n-2}{2} & \text{if } n = 4, 6, 8 \\ \frac{n^2+6n-10}{2} & \text{if } n \geq 8 \end{cases}$$

where e is a chord joining two vertices with even distance in even cycle C_n.

The γ-*spectrum* of a graph G is defined in [2] as

$$\mathrm{spec}(G) = \{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

Observe that $\mathrm{val}_{\min}(G), \mathrm{val}_{\max}(G) \in \mathrm{spec}(G)$ for every graph G. For integers a and b with $a < b$, let

$$[a, b] = \{a, a + 1, \ldots, b\}$$

be a *consecutive set* of integers between a and b. Moreover, for even integers a and b with $a < b$, let

$$E[a, b] = \{a, a + 2, a + 4, \ldots, b\}$$

be an *even consecutive set* of integers between a and b.

Thus for every graph G, $\text{spec}(G) \subseteq [\text{val}_{\min}(G),\ \text{val}_{\max}(G)]$. The γ-spectra of stars, paths, cycles and complete graphs are determined in [2, 7]. Next we recall the γ-spectrum of C_n for each integer $n \geq 3$.

Theorem 1.2 ([7]). *For each integer $n \geq 3$,*

$$\text{spec}(C_n) = E\,[\text{val}_{\min}(C_n),\ \text{val}_{\max}(C_n)].$$

Fonseca, Saenpholphat, and Zhang [5] suggest one to determine the γ-spectrum of $C_n^{\triangle}$. In this paper, we study not only the γ-spectrum of cycle with a triangle $C_n^{\triangle}$ but also the γ-spectrum of cycle with one chord $C_n + e$ in order to make broader generalization. Observe that in Theorem 1.2, the value of any γ-labeling of cycle is always even. We show that the value of any γ-labeling of cycle with one chord is not the case.

The reader is referred to Chartrand and Zhang [4] for basic definitions and terminology not mentioned here.

2. Main Result

In this section we show that the γ-spectrum of $C_n + e$ is a consecutive set of integers between $\text{val}_{\min}(C_n + e)$ and $\text{val}_{\max}(C_n + e)$, in the main theorem as follows.

Theorem 2.1. *For every integer $n \geq 4$,*

$$\text{spec}(C_n + e) = \Big[\text{val}_{\min}(C_n + e), \text{val}_{\max}(C_n + e)\Big].$$

Throughout this main section, let $C_n + e$ be a cycle with one chord of order n which is obtained from

$$\text{a cycle } C_n : v_1, v_2, \ldots, v_{r-1}, v_r, v_{r+1}, \ldots, v_n, v_1 \text{ and a chord } e = v_1v_r,$$

where $3 \leq r \leq n - 1$.

Since $C_n + v_1v_r$ is isomorphic to $C_n + v_1v_{n-r+2}$ for each r with $3 \leq r \leq n-1$, it is sufficient to determine that for each r with $3 \leq r \leq \left\lceil \frac{n+1}{2} \right\rceil$,

$$\text{spec}(C_n + v_1v_r) = \Big[\text{val}_{\min}(C_n + v_1v_r), \text{val}_{\max}(C_n + v_1v_r)\Big]. \tag{1}$$

Observe that Theorem 1.1 provides that

$$\begin{aligned} \mathrm{val}_{\min}(C_n + v_1v_r) &= (2n-2)+1 \\ \mathrm{val}_{\max}(C_n + v_1v_r) &= (2n-2)+\Big|\Big[\mathrm{val}_{\min}(C_n + v_1v_r), \mathrm{val}_{\max}(C_n + v_1v_r)\Big]\Big|. \end{aligned}$$

Therefore our goal is to find a γ-labeling f_l of $C_n + v_1v_r$ with $\mathrm{val}(f_l) = (2n-2)+l$ for each integer l with

$$1 \le l \le \Big|\Big[\mathrm{val}_{\min}(C_n + v_1v_r), \mathrm{val}_{\max}(C_n + v_1v_r)\Big]\Big|.$$

The following propositions will be used to deduce a γ-labeling f_l of $C_n+v_1v_r$ for achieving the main result.

Proposition 2.2. *For every even integer $n \ge 6$,*

$$\mathrm{spec}(C_n + e) = \Big[\mathrm{val}_{\min}(C_n + e), \mathrm{val}_{\max}(C_n + e)\Big] = \left[2n-1, \frac{n^2+6n+2}{2}\right]$$

where e is a chord joining two vertices with odd distance in even cycle C_n.

Proof. By (1), we may assume that $C_n+e = C_n+v_1v_r$ where $3 \le r \le \frac{n}{2}+1$ and r is even.

For each integer i with $0 \le i \le \frac{n}{2}-1$, let

$$\Delta_i = \begin{cases} 2n+2 & \text{if } i = 0 \\ 2n-4i-1 & \text{if } 1 \le i \le \frac{n}{2}-2 \\ \frac{n}{2}+3 & \text{if } i = \frac{n}{2}-1. \end{cases}$$

We will show that, for each integer l with

$$1 \le l \le \sum_{i=0}^{\frac{n}{2}-1} \Delta_i = \Big|\Big[\mathrm{val}_{\min}(C_n + v_1v_r), \mathrm{val}_{\max}(C_n + v_1v_r)\Big]\Big|,$$

there exists a γ-labeling f_l whose value is $(2n-2)+l$.

First, we consider $1 \le l \le \Delta_0$ as follows:

I. For $1 \leq l \leq 8$, we can define a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} r-3 & \text{if } l = 2,8 \\ r-2 & \text{if } l = 1,4,7 \\ r-1 & \text{if } l = 3,6 \\ r & \text{if } l = 5 \end{cases}$$

$$f_l(v_i) = \begin{cases} r-i-1 & \text{if } 2 \leq i \leq r-2 \text{ and } l = 1,3,4,5,6,7 \\ r-i-2 & \text{if } 2 \leq i \leq r-2 \text{ and } l = 2,8 \end{cases}$$

$$f_l(v_{r-1}) = \begin{cases} 0 & \text{if } l = 1,3,4,5,6,7 \\ r-2 & \text{if } l = 2,8 \end{cases}$$

$$f_l(v_r) = \begin{cases} r-1 & \text{if } l = 1,2 \\ r & \text{if } l = 3,4 \\ r+1 & \text{if } l = 5,6,7,8 \end{cases}$$

$$f_l(v_i) = f_l(v_r) + i - r \quad \text{if } r+1 \leq i \leq n \text{ and } 1 \leq l \leq 8.$$

If $l = 1,3,4,5,6$ and 7, then

$$\text{val}(f_l) = -f_l(v_1) - 2f_l(v_{r-1}) + f_l(v_r) + 2f_l(v_n) = (2n-2) + l.$$

If $l = 2$ and 8, then

$$\text{val}(f_l) = -f_l(v_1) - 2f_l(v_{r-2}) + f_l(v_r) + 2f_l(v_n) = (2n-2) + l.$$

II. For $9 \leq l \leq 2r+2$, let f_l be a γ-labeling of $C_n + v_1v_r$ defined by

$$f_l(v_1) = \begin{cases} r-2 & \text{if } l \text{ is even} \\ r-1 & \text{if } l \text{ is odd} \end{cases}$$

$$f_l(v_i) = \begin{cases} i-2 & \text{if } 2 \leq i \leq r-1 \\ r+2 & \text{if } i = r \\ i+2 & \text{if } r+1 \leq i \leq \lceil \frac{l}{2} \rceil + r - 5 \\ r+1 & \text{if } i = \lceil \frac{l}{2} \rceil + r - 4 \\ i+1 & \text{if } \lceil \frac{l}{2} \rceil + r - 3 \leq i \leq n. \end{cases}$$

If $l = 9$ and 10, then

$$\mathrm{val}(f_l) = -f_l(v_1) - 2f_l(v_2) + 3f_l(v_r) - 2f_l(v_{r+1}) + 2f_l(v_n) = (2n-2) + l.$$

If $11 \le l \le 2r+2$, then

$$\begin{aligned}\mathrm{val}(f_l) &= -f_l(v_1) - 2f_l(v_2) + f_l(v_r) + 2f_l\left(v_{\lceil \frac{l}{2}\rceil + r - 5}\right) - 2f_l\left(v_{\lceil \frac{l}{2}\rceil + r - 4}\right) \\ &\quad + 2f_l(v_n) \\ &= (2n-2) + l.\end{aligned}$$

III. For $2r+3 \le l \le \Delta_0 = 2n+2$, define a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_i) = \begin{cases} \left\lceil \frac{l-5}{2} \right\rceil & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ \left\lceil \frac{l-6}{2} \right\rceil - r + i & \text{if } 3 \le i \le r-1 \\ \left\lceil \frac{l}{2} \right\rceil - r + i & \text{if } r \le i \le n + r - \left\lceil \frac{l+2}{2} \right\rceil \\ \left\lceil \frac{l-2}{2} \right\rceil - n - r + i & \text{if } n + r - \left\lceil \frac{l+2}{2} \right\rceil + 1 \le i \le n. \end{cases}$$

If $2r+3 \le l \le 2n$, then

$$\begin{aligned}\mathrm{val}(f_l) &= f_l(v_1) - 2f_l(v_2) + f_l(v_r) + 2f_l\left(v_{n+r-\lceil \frac{l}{2}\rceil + 1}\right) - 2f_l\left(v_{n+r-\lceil \frac{l}{2}\rceil + 2}\right) \\ &= (2n-2) + l.\end{aligned}$$

If $l = 2n+1$ and $2n+2$, then

$$\mathrm{val}(f_l) = f_l(v_1) - 2f_l(v_2) + 3f_l(v_r) - 2f_l(v_{r+1}) = (2n-2) + l.$$

Next, we consider $\Delta_0 + 1 \le l \le \sum_{i=0}^{\frac{n}{2}-1} \Delta_i$, by letting (t, l_t) be a pair of integers with $1 \le t \le \frac{n}{2} - 1$ and $l_t = l - \sum_{i=0}^{t-1} \Delta_i$. Certainly, $1 \le l_t \le \Delta_t$. We construct a γ-labeling f_l as the following procedure:

IV. For $\Delta_0 + 1 \le l \le \sum_{i=0}^{\frac{n-r}{2}-1} \Delta_i$, define a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} n - (t-1) & \text{if } l_t = \Delta_t - 4 \\ n - t & \text{if } l_t \text{ is even and } l_t \ne \Delta_t - 3, \Delta_t - 1 \\ n - (t+1) & \text{if } l_t \text{ is odd and } l_t \ne \Delta_t - 4 \\ n - (t+2) & \text{if } l_t = \Delta_t - 3, \Delta_t - 1 \end{cases}$$

$$f_l(v_2) = 0$$

$$f_l(v_i) = \begin{cases} n+i-(t+r+1) & \text{if } 3 \le i \le r-1 \\ & \text{and } 1 \le l_t \le \Delta_t-(2r-1) \\ n+i-(t+r+2) & \text{if } 3 \le i \le f_l(v_{2t+r})+t+r-n+1 \\ & \text{and } \Delta_t-(2r-2) \le l_t \le \Delta_t-7 \\ n+i-(t+r+1) & \text{if } f_l(v_{2t+r})+t+r-n+2 \le i \le r-1 \\ & \text{and } \Delta_t-(2r-2) \le l_t \le \Delta_t-7 \\ n+i-(t+r+2) & \text{if } 3 \le i \le r-1 \text{ and } \Delta_t-6 \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} \frac{i-r+1}{2} & \text{if } i \text{ is odd and } r \le i \le 2t+r+1 \\ n+1-\left(\frac{i-r}{2}\right) & \text{if } i \text{ is even and } r \le i \le 2t+r-2 \end{cases}$$

$$f_l(v_{2t+r}) = \begin{cases} \left\lceil \frac{l_t+2}{2} \right\rceil + t & \text{if } 1 \le l_t \le \Delta_t-5 \\ \left\lceil \frac{l_t}{2} \right\rceil + t & \text{if } l_t = \Delta_t-4 \\ \left\lceil \frac{l_t+3}{2} \right\rceil + t & \text{if } \Delta_t-3 \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} i-(t+r-1) & \text{if } 2t+r+2 \le i \le n \text{ and } l_t = 1,2 \\ i-(t+r) & \text{if } 2t+r+2 \le i \le f_l(v_{2t+r})+t+r-1 \\ & \text{and } 3 \le l_t \le \Delta_t-(2r+1) \\ i-(t+r-1) & \text{if } f_l(v_{2t+r})+t+r \le i \le n \\ & \text{and } 3 \le l_t \le \Delta_t-(2r+1) \\ i-(t+r) & \text{if } 2t+r+2 \le i \le n \\ & \text{and } \Delta_t-2r \le l_t \le \Delta_t. \end{cases}$$

Then

$$\begin{aligned} \mathrm{val}(f_l) &= f_l(v_1)-2f_l(v_2)+3f_l(v_r) - 2\sum_{\substack{r+1\le i\le 2t+r+1 \\ i \text{ is odd}}} f_l(v_i) + 2\sum_{\substack{r+2\le i\le 2t+r-2 \\ i \text{ is even}}} f_l(v_i) \\ &\quad +2f_l(v_{2t+r}) \\ &= 2nt+n-2t^2-1+f_l(v_1)+2f_l(v_{2t+r}) = (2n-2)+l. \end{aligned}$$

V. For $\sum_{i=0}^{\frac{n-r}{2}-1}\Delta_i + 1 \le l \le \sum_{i=0}^{\frac{n}{2}-3}\Delta_i$, now we construct a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} n-(t-1) & \text{if } l_t = \Delta_t - 4 \\ n-t & \text{if } l_t \text{ is even and } l_t \ne \Delta_t - 3, \Delta_t - 1 \\ n-(t+1) & \text{if } l_t \text{ is odd and } l_t \ne \Delta_t - 4 \\ n-(t+2) & \text{if } l_t = \Delta_t - 3, \Delta_t - 1 \end{cases}$$

$$f_l(v_2) = 0$$

$$f_l(v_i) = \begin{cases} \frac{n-2+i-r}{2} & \text{if } i \text{ is even and } 3 \le i \le 2t+r-n+4 \\ \frac{n+5+r-i}{2} & \text{if } i \text{ is odd and } 3 \le i \le 2t+r-n+1 \end{cases}$$

$$f_l(v_{2t+r-n+3}) = \begin{cases} \left\lceil \frac{l_t+2}{2} \right\rceil + t & \text{if } 1 \le l_t \le \Delta_t - 5 \\ \left\lceil \frac{l_t}{2} \right\rceil + t & \text{if } l_t = \Delta_t - 4 \\ \left\lceil \frac{l_t+3}{2} \right\rceil + t & \text{if } \Delta_t - 3 \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} n+i-(t+r+1) & \text{if } 2t+r-n+5 \le i \le r-1 \\ & \text{and } 1 \le l_t \le 4 \\ n+i-(t+r+2) & \text{if } 2t+r-n+5 \le i \le f_l(v_{2t+r-n+3}) \\ & \qquad +t+r-n+1 \\ & \text{and } 5 \le l_t \le \Delta_t - 7 \\ n+i-(t+r+1) & \text{if } f_l(v_{2t+r-n+3}) + t + r - n + 2 \le i \le r-1 \\ & \text{and } 5 \le l_t \le \Delta_t - 7 \\ n+i-(t+r+2) & \text{if } 2t+r-n+5 \le i \le r-1 \\ & \text{and } \Delta_t - 6 \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} \frac{i-r+1}{2} & \text{if } i \text{ is odd and } r \le i \le n-1 \\ n+1-\left(\frac{i-r}{2}\right) & \text{if } i \text{ is even and } r \le i \le n-1 \end{cases}$$

$$f_l(v_n) = \begin{cases} t+3 & \text{if } l_t = 1, 2 \\ t+2 & \text{if } 3 \le l_t \le \Delta_t. \end{cases}$$

Then

$$\begin{aligned}\text{val}(f_l) &= f_l(v_1) + 3f_l(v_r) - 2\sum_{\substack{2\le i\le 2t+r-n+4\\ i \text{ is even}}} f_l(v_i) - 2\sum_{\substack{r+1\le i\le n-1\\ i \text{ is odd}}} f_l(v_i) + 2\sum_{\substack{3\le i\le 2t+r-n+1\\ i \text{ is odd}}} f_l(v_i)\\ &\quad +2f_l(v_{2t+r-n+3}) + 2\sum_{\substack{r+2\le i\le n-2\\ i \text{ is even}}} f_l(v_i)\\ &= 2nt + n - 2t^2 - 1 + f_l(v_1) + 2f_l(v_{2t+r-n+3}) = (2n-2) + l.\end{aligned}$$

VI. For $\sum_{i=0}^{\frac{n}{2}-3}\Delta_i + 1 \le l \le \sum_{i=0}^{\frac{n}{2}-2}\Delta_i$, observe that $\Delta_{\frac{n}{2}-2} = 7$. We define a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} \frac{n}{2} - l_t & \text{if } l_t = 1,2,3\\ \frac{n}{2} + 4 - l_t & \text{if } l_t = 4,5,6,7\end{cases}$$

$$f_l(v_2) = \begin{cases} 0 & \text{if } l_t = 2,5\\ \frac{n}{2} - 2 & \text{if } l_t = 1,3\\ \frac{n}{2} - 1 & \text{if } l_t = 4,6,7\end{cases}$$

$$f_l(v_3) = \begin{cases} 0 & \text{if } l_t = 1,4\\ \frac{n}{2} - 1 & \text{if } l_t = 2,3\\ \frac{n}{2} & \text{if } l_t = 5,6,7\end{cases}$$

$$f_l(v_i) = \tfrac{n+4+r-i}{2} \quad \text{if } i \text{ is even and } 4 \le i \le r-2$$

$$f_l(v_i) = n + 1 + \left(\tfrac{i-r}{2}\right) \quad \text{if } i \text{ is even and } r \le i \le n$$

$$f_l(v_i) = \begin{cases} \frac{n}{2} & \text{if } i = 5 \text{ and } l_t = 1,2,3\\ \frac{i-5}{2} & \text{if } i \text{ is odd, } 7 \le i \le n-3 \text{ and } l_t = 1,2,3\\ \frac{i-3}{2} & \text{if } i \text{ is odd, } 5 \le i \le n-3 \text{ and } l_t = 4,5,6\\ \frac{i-3}{2} & \text{if } i \text{ is odd, } 5 \le i \le n-5 \text{ and } l_t = 7\\ 0 & \text{if } i = n-3 \text{ and } l_t = 7\end{cases}$$

$$f_l(v_{n-1}) = \begin{cases} 0 & \text{if } l_t = 3, 6 \\ \frac{n}{2} - 3 & \text{if } l_t = 1, 2 \\ \frac{n}{2} - 2 & \text{if } l_t = 4, 5, 7. \end{cases}$$

Since $l_t = l - \sum_{i=0}^{\frac{n}{2}-3} \Delta_i = l - \frac{n^2}{2} - \frac{n}{2} + 7$, it follows that for each l with $\sum_{i=0}^{\frac{n}{2}-3} \Delta_i + 1 \leq l \leq \sum_{i=0}^{\frac{n}{2}-2} \Delta_i$, $\text{val}(f_l) = \frac{n^2}{2} + \frac{5n}{2} - 9 + l_t = (2n-2) + l$.

VII. For $\sum_{i=0}^{\frac{n}{2}-2} \Delta_i + 1 \leq l \leq \sum_{i=0}^{\frac{n}{2}-1} \Delta_i$, define a γ-labeling f_l of $C_n + v_1 v_r$ by

$$f_l(v_1) = \begin{cases} \frac{n}{2} - 1 - l_t & \text{if } 1 \leq l_t \leq 3 \\ \frac{n}{2} + 3 - l_t & \text{if } 4 \leq l_t \leq \Delta_{\frac{n}{2}-1} \end{cases}$$

$$f_l(v_2) = \begin{cases} \frac{n}{2} & \text{if } 1 \leq l_t \leq 3 \\ \frac{n}{2} + 2 & \text{if } 4 \leq l_t \leq \Delta_{\frac{n}{2}-1} \end{cases}$$

$$f_l(v_i) = \frac{i-1}{2} \qquad \text{if } i \text{ is odd, } 3 \leq i \leq n-1 \text{ and } i \neq 2f_l(v_1) + 1$$

$$f_l(v_{2f_l(v_1)+1}) = 0$$

$$f_l(v_i) = \begin{cases} \frac{n+4+r-i}{2} & \text{if } i \text{ is even, } 4 \leq i \leq r-2 \text{ and } 1 \leq l_t \leq 3 \\ \frac{n+2+r-i}{2} & \text{if } i \text{ is even, } 4 \leq i \leq r-2 \text{ and } 4 \leq l_t \leq \Delta_{\frac{n}{2}} - 1 \end{cases}$$

$$f_l(v_i) = n + 1 - \left(\frac{r-i}{2}\right) \quad \text{if } i \text{ is even and } r \leq i \leq n.$$

Then

$$\begin{aligned} \text{val}(f_l) &= -3f_l(v_1) + 2f_l(v_2) + 3f_l(v_r) - 2\sum_{\substack{3 \leq i \leq n-1 \\ i \text{ is odd}}} f_l(v_i) + 2\sum_{\substack{4 \leq i \leq r-2 \\ i \text{ is even}}} f_l(v_i) \\ &\quad + 2\sum_{\substack{r+2 \leq i \leq n \\ i \text{ is even}}} f_l(v_i) \\ &= -\frac{n^2}{4} + \frac{3n}{2} + 1 - f_l(v_1) + 2f_l(v_2) + 2\sum_{\substack{4 \leq i \leq n \\ i \text{ is even}}} f(v_i) \\ &= (2n-2) + l. \end{aligned}$$

□

We now illustrate the proof of Proposition 2.2. The table 1 shows all variables in the proof of Proposition 2.2 that we use to find $\text{spec}(C_8 + v_1 v_4)$.

val$(f_l) = (2n-2) + l$ **of** $C_8 + v_1v_4 \in \left[2n-1, \frac{n^2+6n+2}{2}\right] = [15, 57]$										
$\Delta_0 = 2n+2 = 18, 1 \leq l \leq 18$										
l		γ-*labeling* f_l : v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	val(f_l)
1		2	1	0	3	4	5	6	7	15
2		1	0	2	3	4	5	6	7	16
3		3	1	0	4	5	6	7	8	17
4		2	1	0	4	5	6	7	8	18
5		4	1	0	5	6	7	8	9	19
6		3	1	0	5	6	7	8	9	20
7		2	1	0	5	6	7	8	9	21
8		1	0	2	5	6	7	8	9	22
9		3	0	1	6	5	7	8	9	23
10		2	0	1	6	5	7	8	9	24
11		3	0	2	6	7	8	9	1	25
12		4	0	2	6	7	8	9	1	26
13		4	0	3	7	8	9	1	2	27
14		5	0	3	7	8	9	1	2	28
15		5	0	4	8	9	1	2	3	29
16		6	0	4	8	9	1	2	3	30
17		6	0	5	9	1	2	3	4	31
18		7	0	5	9	1	2	3	4	32
$\Delta_1 = 2n - 4(1) - 1 = 11, 19 \leq l \leq 29$										
l	l_1	γ-*labeling* f_l : v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	val(f_l)
19	1	6	0	5	9	1	3	2	4	33
20	2	7	0	5	9	1	3	2	4	34
21	3	6	0	5	9	1	4	2	3	35
22	4	7	0	5	9	1	4	2	3	36
23	5	6	0	4	9	1	5	2	3	37
24	6	7	0	4	9	1	5	2	3	38
25	7	8	0	4	9	1	5	2	3	39
26	8	5	0	4	9	1	7	2	3	40
27	9	6	0	4	9	1	7	2	3	41
28	10	5	0	4	9	1	8	2	3	42
29	11	6	0	4	9	1	8	2	3	43
$\Delta_2 = 2n - 4(2) - 1 = 7, 30 \leq l \leq 36$										
l	l_2	γ-*labeling* f_l : v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	val(f_l)
30	1	3	2	0	9	4	8	1	7	44
31	2	2	0	3	9	4	8	1	7	45
32	3	1	2	3	9	4	8	0	7	46
33	4	4	3	0	9	1	8	2	7	47
34	5	3	0	4	9	1	8	2	7	48
35	6	2	3	4	9	1	8	0	7	49
36	7	1	3	4	9	0	8	2	7	50
$\Delta_3 = \frac{n}{2} + 3 = 7, 37 \leq l \leq 43$										
l	l_3	γ-*labeling* f_l : v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	val(f_l)
37	1	2	4	1	9	0	8	3	7	51
38	2	1	4	0	9	2	8	3	7	52
39	3	0	4	1	9	2	8	3	7	53
40	4	3	6	1	9	2	8	0	7	54
41	5	2	6	1	9	0	8	3	7	55
42	6	1	6	0	9	2	8	3	7	56
43	7	0	6	1	9	2	8	3	7	57

Table 1: spec$(C_8 + v_1v_4)$

Proposition 2.3. *For every even integer $n \geq 4$,*

$$\begin{aligned}\mathrm{spec}(C_n+e) &= \Big[\mathrm{val}_{\min}(C_n+e), \mathrm{val}_{\max}(C_n+e)\Big] \\ &= \begin{cases} \Big[2n-1, \frac{n^2+5n-2}{2}\Big] & \text{if } n = 4, 6 \\ \Big[2n-1, \frac{n^2+6n-10}{2}\Big] & \text{if } n \geq 8 \end{cases}\end{aligned}$$

where e is a chord joining two vertices with even distance in even cycle C_n.

Proof. Assume by (1) that $C_n+e = C_n + v_1v_r$ where $3 \leq r \leq \frac{n}{2}+1$ and r is odd.

First we consider $C_n + v_1v_r$ when $n \geq 8$. For each integer i with $0 \leq i \leq \frac{n}{2}-1$, let

$$\Delta_i = \begin{cases} 2n+2 & \text{if } i = 0 \\ 2n-4i-1 & \text{if } 1 \leq i \leq \frac{n}{2}-2 \\ \frac{n}{2}-3 & \text{if } i = \frac{n}{2}-1. \end{cases}$$

For each integer l with

$$1 \leq l \leq \sum_{i=0}^{\frac{n}{2}-1} \Delta_i = \left|\Big[\mathrm{val}_{\min}(C_n+v_1v_r), \mathrm{val}_{\max}(C_n+v_1v_r)\Big]\right|,$$

we show that there is a γ-labeling f_l whose value is $(2n-2)+l$.

1. For $1 \leq l \leq \Delta_0 = 2n+2$, a γ-labeling f_l is defined in similar way as described in **I.**, **II.**, and **III.** of Proposition 2.2.

Now we consider $\Delta_0 + 1 \leq l \leq \sum_{i=0}^{\frac{n}{2}-1} \Delta_i$, by letting (t, l_t) be a pair of integers with $1 \leq t \leq \frac{n}{2}-1$ and $l_t = l - \sum_{i=0}^{t-1} \Delta_i$, that is, $1 \leq l_t \leq \Delta_t$. A γ-labeling f_l can be constructed as the following procedure:

2. For $\Delta_0 + 1 \leq l \leq \sum_{i=0}^{\frac{n-r-1}{2}} \Delta_i$ of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} n-(t-1) & \text{if } l_t = \Delta_t - 4 \\ n-t & \text{if } l_t \text{ is even and } l_t \neq \Delta_t-3, \Delta_t-1 \\ n-(t+1) & \text{if } l_t \text{ is odd and } l_t \neq \Delta_t - 4 \\ n-(t+2) & \text{if } l_t = \Delta_t-3, \Delta_t-1 \end{cases}$$

$$f_l(v_2) = 0$$

$$f_l(v_i) = \begin{cases} n+i-(t+r+1) & \text{if } 3 \le i \le r-1 \text{ and } 1 \le l_t \le \Delta_t-(2r-1) \\ n+i-(t+r+2) & \text{if } 3 \le i \le f_l(v_{2t+r})+t+r-n+1 \\ & \text{and } \Delta_t-(2r-2) \le l_t \le \Delta_t-7 \\ n+i-(t+r+1) & \text{if } f_l(v_{2t+r})+t+r-n+2 \le i \le r-1 \\ & \text{and } \Delta_t-(2r-2) \le l_t \le \Delta_t-7 \\ n+i-(t+r+2) & \text{if } 3 \le i \le r-1 \text{ and } \Delta_t-6 \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} \frac{i-r+1}{2} & \text{if } i \text{ is even and } r \le i \le 2t+r+1 \\ n+1-\left(\frac{i-r}{2}\right) & \text{if } i \text{ is odd and } r \le i \le 2t+r-2 \end{cases}$$

$$f_l(v_{2t+r}) = \begin{cases} \left\lceil \frac{l_t+2}{2} \right\rceil + t & \text{if } 1 \le l_t \le \Delta_t-5 \\ \left\lceil \frac{l_t}{2} \right\rceil + t & \text{if } l_t = \Delta_t-4 \\ \left\lceil \frac{l_t+3}{2} \right\rceil + t & \text{if } \Delta_t-3 \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} i-(t+r-1) & \text{if } 2t+r+2 \le i \le n \text{ and } l_t = 1,2 \\ i-(t+r) & \text{if } 2t+r+2 \le i \le f_l(v_{2t+r})+t+r-1 \\ & \text{and } 3 \le l_t \le \Delta_t-(2r+1) \\ i-(t+r-1) & \text{if } f_l(v_{2t+r})+t+r \le i \le n \\ & \text{and } 3 \le l_t \le \Delta_t-(2r+1) \\ i-(t+r) & \text{if } 2t+r+2 \le i \le n \text{ and } \Delta_t-2r \le l_t \le \Delta_t. \end{cases}$$

Then

$$\begin{aligned} \text{val}(f_l) &= f_l(v_1)-2f_l(v_2)+3f_l(v_r) - 2\sum_{\substack{r+1\le i\le 2t+r+1 \\ i \text{ is even}}} f_l(v_i) + 2\sum_{\substack{r+2\le i\le 2t+r-2 \\ i \text{ is odd}}} f_l(v_i) + 2f_l(v_{2t+r}) \\ &= 2nt+n-2t^2-1+f_l(v_1)+2f_l(v_{2t+r}) = (2n-2)+l. \end{aligned}$$

3. For $\sum_{i=0}^{\frac{n-r-1}{2}} \Delta_i+1 \le l \le \sum_{i=0}^{\frac{n}{2}-2} \Delta_i$, define a γ-labeling f_l of $C_n+v_1v_r$ by

$$f_l(v_1) = \begin{cases} n-(t-1) & \text{if } l_t = \Delta_t - 4 \\ n-t & \text{if } l_t \text{ is even and } l_t \neq \Delta_t - 3, \Delta_t - 1 \\ n-(t+1) & \text{if } l_t \text{ is odd and } l_t \neq \Delta_t - 4 \\ n-(t+2) & \text{if } l_t = \Delta_t - 3, \Delta_t - 1 \end{cases}$$

$$f_l(v_2) = 0$$

$$f_l(v_i) = \begin{cases} \frac{n-1+i-r}{2} & \text{if } i \text{ is even and } 3 \leq i \leq 2t+r-n+3 \\ \frac{n+4+r-i}{2} & \text{if } i \text{ is odd and } 3 \leq i \leq 2t+r-n \end{cases}$$

$$f_l(v_{2t+r-n+2}) = \begin{cases} \left\lceil \frac{l_t+2}{2} \right\rceil + t & \text{if } 1 \leq l_t \leq \Delta_t - 5 \\ \left\lceil \frac{l_t}{2} \right\rceil + t & \text{if } l_t = \Delta_t - 4 \\ \left\lceil \frac{l_t+3}{2} \right\rceil + t & \text{if } \Delta_t - 3 \leq l_t \leq \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} n+i-(t+r+1) & \text{if } 2t+r-n+4 \leq i \leq r-1 \text{ and } l_t = 1, 2 \\ n+i-(t+r+2) & \text{if } 2t+r-n+4 \leq i \leq f_l(v_{2t+r-n+2}) \\ & \qquad + t + r - n + 1 \\ & \text{and } 3 \leq l_t \leq \Delta_t - r \\ n+i-(t+r+1) & \text{if } f_l(v_{2t+r-n+2}) + t + r - n + 2 \leq i \leq r-1 \\ & \text{and } 3 \leq l_t \leq \Delta_t - r \\ n+i-(t+r+2) & \text{if } 2t+r-n+4 \leq i \leq r-1 \\ & \text{and } \Delta_t - (r-1) \leq l_t \leq \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} \frac{i-r+1}{2} & \text{if } i \text{ is even and } r \leq i \leq n \\ n+1-\left(\frac{i-r}{2}\right) & \text{if } i \text{ is odd and } r \leq i \leq n. \end{cases}$$

Then

$$\begin{aligned} \mathrm{val}(f_l) &= f_l(v_1) + 3f_l(v_r) - 2\sum_{\substack{2 \leq i \leq 2t+r-n+3 \\ i \text{ is even}}} f_l(v_i) - 2\sum_{\substack{r+1 \leq i \leq n \\ i \text{ is even}}} f_l(v_i) + 2\sum_{\substack{3 \leq i \leq 2t+r-n \\ i \text{ is odd}}} f_l(v_i) \\ &= 2nt + n - 2t^2 - 1 + f_l(v_1) + 2f_l(v_{2t+r-n+2}) = (2n-2) + l. \end{aligned}$$

4. For $\sum_{i=0}^{\frac{n}{2}-2} \Delta_i + 1 \leq l \leq \sum_{i=0}^{\frac{n}{2}-1} \Delta_i$, define a γ-labeling f_l of $C_n + v_1 v_r$

by

$$\begin{aligned} f_l(v_1) &= \tfrac{n}{2} - 3 - l_t \\ f_l(v_2) &= \tfrac{n}{2} \\ f_l(v_i) &= \tfrac{n+4+r-i}{2} && \text{if } i \text{ is odd, } 3 \le i \le r-2 \\ f_l(v_i) &= \tfrac{i}{2} - 1 && \text{if } i \text{ is even, } 4 \le i \le n \\ & && \text{and } i \ne 2f_l(v_1)+2 \\ f_l(v_{2f_l(v_1)+2}) &= 0 \\ f_l(v_i) &= n+1+\tfrac{r-i}{2} && \text{if } i \text{ is odd, } r \le i \le n-1. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{val}(f_l) &= -3f_l(v_1) + 3f_l(v_r) + 2\sum_{\substack{3\le i\le r-2 \\ i \text{ is odd}}} f_l(v_i) + 2\sum_{\substack{r+2\le i\le n-1 \\ i \text{ is odd}}} f_l(v_i) - 2\sum_{\substack{4\le i\le n-2 \\ i \text{ is even}}} f_l(v_i) \\ &= \tfrac{n^2}{2} + \tfrac{5n}{2} - 2 + l - \tfrac{n^2}{2} - \tfrac{n}{2} = (2n-2)+l. \end{aligned}$$

Next, we consider $C_4 + v_1v_3$, by letting $\Delta_0 = 10$ and $\Delta_1 = 1$. Therefore $\Delta_0 + \Delta_1 = |[\mathrm{val}_{\min}(C_4+v_1v_3), \mathrm{val}_{\max}(C_4+v_1v_3)]|$.

If $1 \le l \le \Delta_0 = 10$, then we define a γ-labeling f_l similar to γ-labeling f_l in **1.** of $C_n + v_1v_r$ when $n \ge 8$.

Otherwise, if $l = \Delta_0 + \Delta_1 = 11$, then a γ-labeling f_l of $C_4 + v_1v_3$ defined by $f_l(v_1) = 4, f_l(v_2) = 0, f_l(v_3) = 5,$ and $f_l(v_4) = 1$ has $\mathrm{val}(f_l) = (2n-2)+l$.

Last, we consider $C_6 + v_1v_3$, by letting $\Delta_0 = 14, \Delta_1 = 7$ and $\Delta_2 = 1$. Then $\Delta_0 + \Delta_1 + \Delta_2 = |[\mathrm{val}_{\min}(C_6+v_1v_3), \mathrm{val}_{\max}(C_6+v_1v_3)]|$.

If $1 \le l \le \Delta_0 + \Delta_1 = 21$, then we define a γ-labeling f_l similar to γ-labeling f_l in **1.** of $C_n + v_1v_r$ when $n \ge 8$.

Otherwise, if $l = \Delta_0+\Delta_1+\Delta_2 = 22$, then define a γ-labeling f_l of $C_6+v_1v_3$ by $f_l(v_1) = 5, f_l(v_2) = 0, f_l(v_3) = 7, f_l(v_4) = 1, f_l(v_5) = 6,$ and $f_l(v_6) = 2$ having $\mathrm{val}(f_l) = (2n-2)+l$. □

Proposition 2.4. *For every odd integer $n \ge 5$,*

$$\mathrm{spec}(C_n+e) = \left[\mathrm{val}_{\min}(C_n+e), \mathrm{val}_{\max}(C_n+e)\right] = \left[2n-1, \frac{n^2+6n-3}{2}\right].$$

The proof of Proposition 2.4 is similar to Propositions 2.2 and 2.3 and is therefore omitted.

3. Final Remarks

In this paper, we have established the γ-spectrum of cycle with one arbitrary chord. A natural question arises how to determine the extremal values and then γ-spectrum of cycle with two or more chords.

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References

[1] G. D. Bullington, L. L. Eroh, and S. J. Winters, γ-labelings of complete bipartite graphs, *Discuss. Math. Graph Theory*, **30** (2010), 45-54.

[2] G. Chartrand, D. Erwin, D.W. VanderJagt, P. Zhang, γ-labelings of graphs, *Bull. Inst. Combin. Appl.*, **44** (2005), 51-68.

[3] G. Chartrand, D. Erwin, D.W. VanderJagt, P. Zhang, On γ-labelings of trees, *Discuss. Math. Graph Theory*, **25**, No. 3 (2005), 363-383.

[4] G. Chartrand, P. Zhang, *Introduction to Graph Theory*, The Walter Rudin student series in advanced mathematics, McGraw-Hill Higher Education, Boston (2005).

[5] C.M. da Fonseca, V. Saenpholphat, and P. Zhang, Extremal values for a γ-labeling of a cycle with a triangle, *Utilitas Math.*, **92** (2013), 167-185.

[6] C.M. da Fonseca, V. Saenpholphat, P. Zhang, On γ-labelings of graphs, *Utilitas Math.*, to appear.

[7] C.M. da Fonseca, V. Saenpholphat, P. Zhang, The γ-spectrum of a graph, *Ars Combin.*, **101** (2011), 109-127.

[8] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin. DS6*, **16** (2009), 1-219.

[9] S.W. Golomb, How to number a graph, In: *Graph Theory and Computing*, Academic Press, New York (1972).

[10] V. Khemmani, S. Saduakdee, γ-labeling of cycle with one chord, *Lecture Notes in Computer Science*, submitted.

[11] A. Rosa, On certain valuations of the vertices of a graph, In: *Theory of Graphs (Internat. Sympos., Rome, 1966)*, Gordon and Breach, New York; Dunod, Paris (1966).