International Journal of Pure and Applied Mathematics
Volume 105 No. 4 2015, 599-625
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v105i4.3

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PERIODIC SOLUTIONS OF SYSTEMS OF AUTONOMOUS DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND IMPULSES

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Abstract: Autonomous systems of differential equations with variable structure and impulsive effects are the main objects of our study. The structure changes and the impulsive effects realize at the switching moments, which are specific to each different solution of the system. In these moments, the trajectory of the corresponding initial value problem meets successively the switching sets. For this class of equations, sufficient conditions for the existence of periodic solutions are found. The results are based on the Brouwer's fixed point theorem. We investigate the question of existence of periodic solutions of a generalized model of predator-prey community.

AMS Subject Classification: 34A34
Key Words: variable structure, impulses, switching functions, periodic solution

1. Introduction

The impulsive differential equations are convenient mathematical apparatus for modeling the dynamic processes, subjected to the discrete external influences over time. Frequently, the duration of the external effects are negligibly small

Received: May 2, 2015

compared with the period of investigation of the dynamic process. For this reason, it can be considered that these effects are in the form of impulses. The qualitative theory of impulsive differential equations is developing intensively due to the numerous applications: [2], [4], [5], [10], [11], [12], [20], [28], [29], [35], [37] and [38]. In particular, the existence of periodic solutions of this type of equation is the object of study in a number of articles: [1], [3], [8], [9], [19], [22], [24], [27], [30], [31], [32], [40] and [42], and several monographs, we will point out: [6], [7], [14], [21] and [36]. Should be noted that the above mentioned results refer only to the differential equations with permanent structure and fixed moments of impulsive effects. The authors' research in this paper aims partially fill this gap. Here, we study the periodic solutions of the equations with variable structure and variable impulsive effects. Recall that the periodic solutions of impulsive differential equations are used for describing the repetitive processes with any leaps of changes. This type of equations and their solutions are especially useful in modeling bioprocesses and eco-processes: [18], [23], [25], [26], [39] and [41].

2. Statement of the Problem and Preliminary Remarks

Denote the Euclidean norm and dot product in R^n by $\|.\|$ and $\langle ., . \rangle$, respectively. For the points $x(x_1, x_2, ..., x_n)$ and $y(y_1, y_2, ..., y_n)$ in R^n, we have

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + ... + x_n y_n;$$
$$\|x\| = \langle x, x \rangle^{\frac{1}{2}} = \left(x_1^2 + x_2^2 + ... + x_n^2\right)^{\frac{1}{2}}.$$

The Euclidean distance between the nonempty sets X and Y, where $X, Y \subset R^n$, is denoted by

$$\rho(X, Y) = inf\left\{\|x - y\|;\ x \in X,\ y \in Y\right\}.$$

An open ball with a center $x_0 \in R^n$ and radius $\delta = const > 0$ is denoted by

$$B_\delta(x_0) = \left\{x \in R^n;\ \|x - x_0\| < \delta\right\}.$$

$\overline{X}$ and ∂X are notations for the closure and contour of the set X, respectively.

The length of curve γ is denoted by $l[\gamma]$. The closed segment with endpoints x and y is denoted by

$$[x, y] = \left\{z_\lambda \in R^n;\ z_\lambda = (1 - \lambda)x + \lambda y,\ 0 \le \lambda \le 1\right\}.$$

Definition 1. *The curve* γ *is said to be* p***-linear***, *if*

$$\gamma = [x_0, x_1] \cup [x_1, x_2] \cup ... \cup [x_{p-1}, x_p].$$

That is to say, p-linear curve is composed by p sequentially connected line segments.

Definition 2. *[15] The domain* G *is said to be* p***-convex***, *where* p *is a natural number, if*

$$(\forall x, y \in G)\left(\exists \gamma = \bigcup_{i=1}^{p} [x_{i-1}, x_i] \subset G\right);\ x_0 = x,\ x_p = y.$$

In other words, any two points of G can be connected by p-linear curve from G. It is clear that each 1-convex domain is convex.

Definition 3. *[15] The domain* G *is said to be* ***bounded-connected***, *if*

$$(\exists l_0 = const > 0)(\forall x, y \in G)(\exists \gamma :\ x, y \in \gamma,\ \gamma \subset G) :\ l[\gamma] \leq l_0.$$

Remark 1. *From the definition above, it follows immediately that each bounded-connected domain is bounded.*

The main object of investigation in this paper is the following initial value problem for autonomous systems of differential equations with variable structure and impulses:

$$\frac{dx}{dt} = f_i(x),\ \varphi_i\big(x(t)\big) \neq 0,\ \text{i.e. } t_{i-1} < t < t_i; \tag{1}$$

$$x(t_i + 0) = J_i\big(x(t_i)\big),\ \varphi_i\big(x(t_i)\big) = 0,\ i = 1, 2, ...; \tag{2}$$

$$x(0) = x_0, \tag{3}$$

where:

- the functions $f_i : G_i \to R^n$ and the phase spaces G_i of the considered system (1), (2) are nonempty domains in R^n;
- the functions $\varphi_i : D_i \to R$, where the domains $D_i \subset G_i$ and $D_i \neq \emptyset$;
- the sets $\Phi_i = \big\{x \in D_i;\ \varphi_i(x) = 0\big\}$;
- the functions $J_i : \Phi_i \to G_{i+1} \backslash \overline{D_{i+1}}$;
- the initial point $x_0 \in G_1 \backslash \overline{D_1}$ (therefore $x_0 \notin \Phi_1$).

Should be noted that the sets $\Phi_i,\ i=1,2,...,$ are named switching. The solution of problem (1), (2), (3) is denoted by $x(t;x_0)$. It is fulfilled:

1.1. For $0=t_0\le t<t_1$, the solution of the studied problem coincides with the solution $x_1(t;t_0,x_0)$ of the problem

$$\frac{dx}{dt}=f_1(x),\quad x(t_0)=x_0=x_0^+; \tag{4}$$

1.2. For $t_0\le t<t_1$, it is satisfied $\varphi_1\big(x(t;x_0)\big)=\varphi_1\big(x_1(t;t_0,x_0)\big)\ne 0$;

1.3. For $t=t_1$, we have $\varphi_1\big(x(t_1;x_0)\big)=\varphi_1\big(x_1(t_1;t_0,x_0)\big)=0$;

1.4. The equality $x_1^+=x(t_1+0;x_0)=J_1\big(x(t_1;x_0)\big)=J_1\big(x_1(t_1;t_0,x_0)\big)$ is valid. It is fulfilled $x_1^+\in G_2\backslash\overline{D_2}$, i.e. $x_1^+\notin\Phi_2$;

2.1. For $t_1<t<t_2$, the solution of problem (1), (2), (3) coincides with the solution $x_2(t;t_1,x_1^+)$ of the problem

$$\frac{dx}{dt}=f_2(x),\quad x(t_1)=x_1^+; \tag{5}$$

2.2. For $t_1<t<t_2$, it is satisfied $\varphi_2\big(x(t;x_0)\big)=\varphi_2\big(x_2(t;t_1,x_1^+)\big)\ne 0$;

2.3. For $t=t_2$, we have $\varphi_2\big(x(t_2;x_0)\big)=\varphi_2\big(x_2(t_2;t_1,x_1^+)\big)=0$;

2.4. For $t=t_2$, the impulsive equality $x_2^+=x(t_2+0;x_0)=J_2\big(x(t_2;x_0)\big)=J_2\big(x_2(t_2;t_1,x_1^+)\big)$ is valid, etc.

The time constants $t_1,t_2,...$ are named switching moments. We introduce the notations $x_i=x(t_i;x_0)$ and $x_i^+=J_i(x_i)$. Note that $x_i^+\notin\Phi_{i+1},\ i=1,2,....$

For $i=1,2,...,$ we consider the following initial value problems for autonomous systems of differential equations with fixed structure and without impulses:

$$\frac{dx}{dt}=f_i(x),\quad x(0)=x_{0i}. \tag{6}$$

The functions $f_i:G_i\to R^n,\ i=1,2,...,$ belong to the right hand side of the system with variable structure (1). The initial points $x_{0i}\in G_i$. The solution of any one of initial value problems (6) is denoted by $x_i(t;0,x_{0i}),\ t\in R,\ i=1,2,....$

Definition 4. *[33] Let for each $i=1,2,...,$ the following conditions be valid:*

1. The sets $X_i^+ \subset G_i,\ \Phi_i \subset G_i,\ X_i^+ \neq \emptyset$ *and* $\Phi_i \neq \emptyset$.
2. For each point $x_{0i} \in X_i^+$*, the solution* $x_i(t;0,x_{0i})$ *is defined and unique in the interval* $[0,\infty)$.
3. It is fulfilled

$$\big(\forall x_{0i} \in X_i^+\big)\big(\exists \theta = \theta(x_{0i}) \geq 0\big):$$

$$x_i(\theta;0,x_{0i}) \in \Phi_i \text{ and } x_i(t;0,x_{0i}) \notin \Phi_i, \quad \text{for } 0 \leq t < \theta.$$

Then:
- The set Φ_i *is said to be* ***positive reachable*** *from the set* X_i^+ *via system (6);*
- The set Φ_i *is said to be* ***totally positive reachable*** *via system (6), if* $X_i^+ = G_i$.

Definition 5. *[13] The solutions of system (6),* $i = 1, 2, ...,$ *is said to be* ***uniformly Lipschitz stable****, if*

$$(\exists L_i = const > 0)(\exists \delta_{L_i} = const > 0):$$

$$\big(\forall x_{0i}^*, x_{0i}^{**} \in G_i,\ \|x_{0i}^* - x_{0i}^{**}\| < \delta_{L_i}\big)$$
$$\Rightarrow \big\|x(t;0,x_{0i}^*) - x(t;0,x_{0i}^{**})\big\| < L_i\|x_{0i}^* - x_{0i}^{**}\|,\ t \geq 0.$$

The validity of the assumptions below will be required during the basic consideration:

Assumption A1. *Each of the sets* Φ_i *is totally positive reachable via system (6),* $i = 1, 2, ...$

Assumption A2. *The moments of switching* $t_1, t_2, ...$ $(0 < t_1 < t_2 < ...)$ *do not possess a compression point, i.e.* $lim_{i\to\infty} t_i = \infty$.

Definition 6. *Let Assumption A1 be valid.*
Then the functions $\Theta_i^+ : G_i \to R^+,\ i = 1, 2, ...,$ *are called* **a function of** ***positive reachability*** *of system (6), if*

$$\big(\forall x_{0i} \in G_i\big)\big(\exists \theta = \Theta_i^+(x_{0i}) \geq 0\big):$$

- $x_i(\theta;0,x_{0i}) \in \Phi_i$,
- $\big(\forall t,\ 0 \leq t < \theta = \Theta_i^+(x_{0i})\big) \Rightarrow x_i(t;0,x_{0i}) \notin \Phi_i$.

Assumption A3. *The functions* $\Theta_i^+ \in C[G_i, R^+],\ i = 1, 2, ...$
The following conditions are introduced:

C1. *The constants* $C_{Lipf_i} > 0$ *satisfy the inequalities*

$$(\forall x', x'' \in G_i) \Rightarrow \big\| f_i(x') - f_i(x'') \big\| \le C_{Lip_i} \|x' - x''\|, \ i = 1, 2, ...$$

C2. *The constants* $C_{f_i} > 0$ *satisfy the inequalities*

$$(\forall x \in G_i) \Rightarrow \|f_i(x)\| \le C_{f_i}, \ i = 1, 2, ...$$

C3. *The solutions of systems (6),* $i = 1, 2, ...$, *exist and are unique in* R.
C4. *The solutions of systems (6),* $i = 1, 2, ...$, *are uniformly Lipschitz stable.*
C5. *The functions* $\varphi_i \in C[\overline{D_i}, R]$ *and furthermore* $\varphi_i \in C^1[\Phi_i, R]$, *where the sets* $\Phi_i = \big\{ x \in D_i; \ \varphi_i(x) = 0 \big\} \ne \emptyset$.
C6. *The constants* $C_{\langle grad\varphi_i, f_i \rangle} > 0$ *satisfy the inequalities*

$$(\forall x \in \Phi_i) \Rightarrow \big\langle grad\varphi_i(x), f_i(x) \big\rangle \ge C_{\langle grad\varphi_i, f_i \rangle}, \ i = 1, 2, ...$$

C7. *The sets* $\Phi_i, \ i = 1, 2, ...$, *are connected.*
C8. *The following inclusions* $\overline{\Phi_i} \backslash \Phi_i \subset \partial G_i, \ i = 1, 2, ...$, *are satisfied.*
C9. *The constants* $C_{\varphi_i} > 0$ *satisfy the inequalities*

$$(\forall x \in D_i) \Rightarrow |\varphi_i(x)| \le C_{\varphi_i} . \rho(x, \Phi_i), \ i = 1, 2, ...$$

C10. *The constants* $C_{Lip\varphi_i} > 0$ *satisfy the inequalities*

$$(\forall x', x'' \in D_i) \Rightarrow \big| \varphi_i(x') - \varphi_i(x'') \big| \le C_{Lip\varphi_i} \|x' - x''\|, \ i = 1, 2, ...$$

C11. *The constants* $C_{\partial D_i} > 0$ *satisfy the inequalities*

$$(\forall x \in \partial D_i \cap G_i) \Rightarrow \big| \varphi_i(x) \big| \ge C_{\partial D_i}, \ i = 1, 2, ...$$

C12. *The functions* $J_i \in C[\Phi_i, G_{i+1} \backslash \overline{D_{i+1}}], \ i = 1, 2, ...$
C13. *The domains* $G_i, \ i = 1, 2, ...$, *are p-convex and bounded or bounded-connected.*
C14. *The series*

$$\sum_{i=1}^{\infty} \frac{C_{\partial D_i}}{C_{f_i} . C_{Lip\varphi_i}}$$

is divergent.

As a result of [16] we will formulate the following theorem, in which the sufficient conditions are given under which Assumption A1 is valid.

Theorem 1. *[16] Let Conditions C1-C9 and C13 hold.*

Then each of the sets Φ_i, $i = 1, 2, ...$, is positively totally reachable by the corresponding system (6).

Conditions, under which Assumption A2 is valid, are given in the paper [17]. As a consequence we get the following theorem.

Theorem 2. *[17] Let Conditions C1-C3, C5, C6, C10-C12 and C14 hold.*
Then the switching moments of initial value problem (1), (2), (3), regardless of the initial point $x_0 \in G_1 \backslash \overline{D_1}$, do not have a compression point.

The next theorem is equivalent to Assumption A3. We obtain it as a result of article [15].

Theorem 3. *[15] Let Conditions C1-C3 and C5-C9 hold.*
Then $X_i^+ = G_i$ and the functions $\Theta_i^+ \in C[G_i, R^+]$, $i = 1, 2, ...$.

3. Continuous Dependence

We will recall the following definition.

Definition 7. *[14] We say that the solution $x(t; x_0)$ of problem (1), (2), (3)* ***depends continuously on the initial point****, if*

$$(\forall x_0 \in G_1 \backslash \overline{D_1})(\forall \varepsilon = const > 0)(\forall \eta = const > 0)(\forall T = const > 0)$$
$$\big(\exists \delta = \delta(x_0, \varepsilon, \eta, T) > 0\big): \ \big(\forall x_0^* \in G_1 \backslash \overline{D_1}, \ \|x_0^* - x_0\| < \delta\big)$$
$$\Rightarrow \big\|x(t; x_0^*) - x(t; x_0)\big\| < \varepsilon, \ 0 \le t \le T, \ |t - t_i| > \eta, \ i = 1, 2, ...$$

Remark 2.*Pay attention to the specific feature of the upper definition of continuous dependence. More precisely, the different solutions have different sets of switching (impulsive) moments. This means that in the general case, the inequalities $t_1^* \neq t_1$, $t_2^* \neq t_2, ...$ are valid.*

Consider the solutions $x(t; x_0^*)$ and $x(t; x_0)$ in the arbitrary interval with the switching moments t_i^* and t_i, $i = 1, 2, ...$, as its ends. Both solutions are subjected to the impulsive effects in different ends of this interval. Consequently (in the general case), they are not "close" inside in the interval. For example, it is possible that their difference to be greater than the impulsive effect at the left end of the interval considered. Furthermore, assuming that the corresponding switching moments vary "significantly", then a proximity between the solutions may exists only in a negligible part of the interval $0 \le t \le T$. This assumption negates the practical value of concept of continuous dependence. Therefore, it is natural to suppose that

$$(\forall \eta = const > 0)\big(\forall x_0^* \in G_1 \backslash \overline{D_1},\ \|x_0^* - x_0\| < \delta\big)$$

$$\Rightarrow |t_i^* - t_i| < \eta,\ i = 1, 2, ...$$

Then in the surroundings of moments $t_1, t_2, ...$ with a radius η, both solutions are not close. Exactly, this is recognized in Definition 7.

Theorem 4. *Assume that:*
1. Conditions C1-C3 and C12 hold.
2. Assumptions A1-A3 are valid.

Then the solution of problem (1), (2), (3) depends continuously on the initial point.

Proof. Let the point $x_0 \in G_1 \backslash \overline{D_1}$. Let ε, η and T be the arbitrary positive constants.

Subsequently, we conclude:
1. Under Assumption A1, the solution $x_1(t; t_0, x_0^+)$ of problem (4) meets switching set Φ_1 at the moment t_1. Taking into account Condition C12, we find $x_1 \in G_2 \backslash \overline{\Phi_2}$;
2. We obtain that the solution $x_2(t; t_1, x_1^+)$ of problem (5) meets the switching set Φ_2 at the moment t_2, etc.

In this way, we find out that the solution

$$x(t; x_0) = \begin{cases} x_1(t; t_0, x_0^+),\ 0 = t_0 \le t \le t_1; \\ x_2(t; t_1, x_1^+),\ t_1 < t \le t_2; \\ .... \\ x_i(t; t_{i-1}, x_{i-1}^+),\ t_{i-1} < t \le t_i; \\ .... \end{cases}$$

of initial value problem (1), (2), (3) meets the switching sets $\Phi_1, \Phi_2, ...$ at the moments $t_1, t_2, ...$, respectively. The inequalities $t_0 < t_1 < t_2 < ...$ are valid. Furthermore, by Assumption A2, it follows that $\lim_{i \to \infty} t_i = \infty$. It means that there exists a number k such that

$$0 = t_0 < t_1 < ... < t_{k-1} < T \le t_k. \tag{7}$$

Consider a perturbed solution $x(t;x_0^*)$ of system (1), (2) with initial condition $x(0)=x_0^*$, where $x_0^*=x_0^{*+}\in G_1\backslash\overline{\Phi_1}$. Analogously to the formula for $x(t;x_0)$, we obtain

$$x(t;x_0^*)=\begin{cases} x_1(t;t_0^*,x_0^{*+}),\ 0=t_0^*\le t\le t_1^*;\\ x_2(t;t_1^*,x_1^{*+}),\ t_1^*<t\le t_2^*;\\ \dots\dots\dots\dots\dots\dots\\ x_i(t;t_{i-1}^*,x_{i-1}^{*+}),\ t_{i-1}^*<t\le t_i^*;\\ \dots\dots\dots\dots\dots\dots, \end{cases}$$

where:
- $t_1^*,t_2^*,...$ $(0=t_0^*<t_1^*<t_2^*<...)$ are moments at which the solutions $x_1(t;t_0^*,x_0^{*+}),\ x_2(t;t_1^*,x_1^{*+}),...$ reach the switching sets $\Phi_1,\Phi_2,...$, respectively;
- $x_i^*=x_i(t_i^*;t_{i-1}^*,x_{i-1}^{*+})=x(t_i^*;,x_0^*),\ x_i^{*+}=J_i(x_i^*),\ x_i^{*+}\in G_{i+1}\backslash\overline{\Phi_{i+1}},\ i=1,2,...$.

Introduce the notations

$$t_i^{min}=min\{t_i^*,t_i\},\quad t_i^{max}=max\{t_i^*,t_i\},\quad i=1,2,.... \tag{8}$$

We have:
1.1. Using Condition C1 and Gronwall′s inequality for $0=t_0^{max}\le t\le t_1^{min}$, we obtain

$$\begin{aligned}&\big\|x(t;x_0^*)-x(t;x_0^*)\big\|\\ &\quad=\big\|x_1(t;t_0^*,x_0^*)-x_1(t;t_0,x_0)\big\|\\ &\quad=\left\|x_0^*+\int_{t_0^{max}}^t f_1\big(x_1(\tau;t_0^*,x_0^*)\big)d\tau-x_0-\int_{t_0^{max}}^t f_1\big(x_1(\tau;t_0,x_0)\big)d\tau\right\|\\ &\quad\le\|x_0^*-x_0\|+\int_{t_0^{max}}^t\Big\|f_1\big(x_1(\tau;t_0^*,x_0^*)\big)-f_1\big(x_1(\tau;t_0,x_0)\big)\Big\|d\tau\\ &\quad\le\|x_0^*-x_0\|+\int_{t_0^{max}}^t C_{Lipf_1}\big\|x_1(\tau;t_0^*,x_0^*)-x_1(\tau;t_0,x_0)\big\|d\tau\\ &\quad\le\|x_0^*-x_0\|exp\big(C_{Lipf_1}(t-t_0^{max})\big)\\ &\quad\le\|x_0^*-x_0\|exp\big(C_{Lipf_1}(t_1^{min}-t_0^{max})\big),\end{aligned}$$

whence we get

$$\lim_{x_0^*\to x_0}\big\|x(t;x_0^*)-x(t;x_0)\big\|=0,\quad t_0^{max}\le t\le t_1^{min}.$$

1.2. From Assumption A3, it follows that

$$\lim_{x_0^*\to x_0}(t_1^{max}-t_1^{min})$$

$$= \lim_{x_0^* \to x_0} |t_1^* - t_1|$$
$$= \lim_{x_0^* \to x_0} \left|\Theta_1^+(x_0^*) - \Theta_1^+(x_0)\right| = 0.$$

1.3. Using Condition C2 and the previous points 1.1 and 1.2, we establish

$$\begin{aligned}
&\lim_{x_0^* \to x_0} \|x_1^* - x_1\||\\
&= \lim_{x_0^* \to x_0} \left\|x(t_1^*; x_0^*) - x(t_1; x_0))\right\|\\
&= \lim_{x_0^* \to x_0} \left\|x_1(t_1^*; t_0^*, x_0^*) - x_1(t_1^{min}; t_0^*, x_0^*)\right\|\\
&\quad + \lim_{x_0^* \to x_0} \left\|x_1(t_1; t_0, x_0) - x_1(t_1^{min}; t_0, x_0)\right\|\\
&\quad + \lim_{x_0^* \to x_0} \left\|x_1(t_1^{min}; t_0^*, x_0^*) - x_1(t_1^{min}; t_0, x_0)\right\|\\
&\le \lim_{x_0^* \to x_0} \int_{t_1^{min}}^{t_1^*} \left\|f_1\big(x_1(\tau; t_0^*, x_0^*)\big)\right\| d\tau\\
&\quad + \lim_{x_0^* \to x_0} \int_{t_1^{min}}^{t_1} \left\|f_1\big(x_1(\tau; t_0, x_0)\big)\right\| d\tau\\
&\quad + \lim_{x_0^* \to x_0} \left\|x(t_1^{min}; x_0^*) - x(t_1^{min}; x_0)\right\|\\
&\le C_{f_1} \lim_{x_0^* \to x_0} (t_1^* - t_1^{min}) + C_{f_1} \lim_{x_0^* \to x_0} (t_1 - t_1^{min})\\
&= C_{f_1} \lim_{x_0^* \to x_0} (t_1^{max} - t_1^{min}) = 0.
\end{aligned}$$

1.4. By Condition C12 and point 1.3, we obtain

$$\begin{aligned}
&\lim_{x_0^* \to x_0} \|x_1^{*+} - x_1^+\|\\
&= \lim_{x_0^* \to x_0} \left\|J_1(x_1^*) - J_1(x_1)\right\|\\
&= \lim_{x_1^* \to x_1} \left\|J_1(x_1^*) - J_1(x_1)\right\| = 0.
\end{aligned}$$

1.5. Taking into account Condition C2, the previous point 1.4 and point 1.2, we reach the conclusion

$$\begin{aligned}
&\lim_{x_0^* \to x_0} \left\|x(t_1^{max} + 0; x_0^*) - x(t_1^{max} + 0; x_0)\right\|\\
&= \lim_{x_0^* \to x_0} \left\|x_2(t_1^{max} + 0; t_1^*, x_1^{*+}) - x_2(t_1^{max} + 0; t_1, x_1^+)\right\|
\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{x_0^*\to x_0} \left\|x_2(t_1^{max}+0;t_1^*,x_1^{*+})-x_1^{*+}\right\| \\
&\quad + \lim_{x_0^*\to x_0} \left\|x_2(t_1^{max}+0;t_1,x_1^{+})-x_1^{+}\right\| \\
&\quad + \lim_{x_0^*\to x_0} \|x_1^{*+}-x_1^{+}\| \\
&\le \lim_{x_0^*\to x_0} \int_{t_1^*}^{t_1^{max}} \left\|f_2\big(x_2(\tau;t_1^*,x_1^{*+})\big)\right\|d\tau \\
&\quad + \lim_{x_0^*\to x_0} \int_{t_1}^{t_1^{max}} \left\|f_2\big(x_2(\tau;t_1,x_1^{+})\big)\right\|d\tau \\
&\le C_{f_2} \lim_{x_0^*\to x_0} (t_1^{max}-t_1^{min}) = 0.
\end{aligned}
$$

Further, we will use the following equalities:

$$x(t;x_0^*) = x\big(t;x(t_1^{max}+0;x_0^*)\big),\ t_1^{max} < t \le t_2^{min}; \tag{9}$$

$$x(t;x_0) = x\big(t;x(t_1^{max}+0;x_0)\big),\ t_1^{max} < t \le t_2^{min}; \tag{10}$$

$$\lim_{x_0^*\to x_0} x(t_1^{max}+0;x_0^*) = x(t_1^{max}+0;x_0) \text{ (see 1.5)}. \tag{11}$$

We have:

2.1. For $t_1^{max} < t \le t_2^{min}$, as we get into consideration the equalities (9) and (10), analogously to 1.1, we obtain

$$
\begin{aligned}
&\left\|x(t;x_0^*)-x(t;x_0^*)\right\| \\
&\quad = \left\|x\big(t;x(t_1^{max}+0;x_0^*)\big)-x\big(t;x(t_1^{max}+0;x_0)\big)\right\| \\
&\quad = \left\|x_2\big(t;t_1^{max},x(t_1^{max}+0;x_0^*)\big)-x_2\big(t;t_1^{max},x(t_1^{max}+0;x_0)\big)\right\| \\
&\quad \le \left\|x(t_1^{max}+0;x_0^*)-x(t_1^{max}+0;x_0)\right\| \\
&\qquad + \int_{t_1^{max}}^{t} \left\|f_2\Big(x_2\big(\tau;t_1^{max},x(t_1^{max}+0;x_0^*)\big)\Big)\right. \\
&\qquad\qquad \left. -f_2\Big(x_2\big(\tau;t_1^{max},x(t_1^{max}+0;x_0)\big)\Big)\right\|d\tau \\
&\quad \le \left\|x(t_1^{max}+0;x_0^*)-x(t_1^{max}+0;x_0)\right\| \\
&\qquad + \int_{t_1^{max}}^{t} C_{Lipf_2}\left\|x_2\big(\tau;t_1^{max},x(t_1^{max}+0;x_0^*)\big)\right. \\
&\qquad\qquad \left. -x_2\big(\tau;t_1^{max},x(t_1^{max}+0;x_0)\big)\right\|d\tau \\
&\quad \le \left\|x(t_1^{max}+0;x_0^*)-x(t_1^{max}+0;x_0)\right\| exp\big(C_{Lipf_2}(t_2^{min}-t_1^{max})\big).
\end{aligned}
$$

By (11), it follows that

$$\begin{aligned}&\lim_{x_0^*\to x_0}\left\|x(t;x_0^*)-x(t;x_0)\right\|\\&\quad\le \lim_{x_0^*\to x_0}\left\|x(t_1^{max}+0;x_0^*)-x(t_1^{max}+0;x_0)\right\|exp\big(C_{Lipf_2}(t_2^{min}-t_1^{max})\big)\\&\quad=0,\ \ t_1^{max}<t\le t_2^{max}.\end{aligned}$$

2.2. From Assumption A3 and point 1.4, it follows

$$\begin{aligned}&\lim_{x_0^*\to x_0}(t_2^{max}-t_2^{min})\\&\quad=\lim_{x_0^*\to x_0}|t_2^*-t_2|\\&\quad=\lim_{x_0^*\to x_0}\left|\Theta_2^+(x_1^{*+})-\Theta_2^+(x_1^+)\right|\\&\quad=\lim_{x_1^{*+}\to x_1^+}\left|\Theta_2^+(x_1^{*+})-\Theta_2^+(x_1^+)\right|=0.\end{aligned}$$

2.3. Similarly to 1.3, using Condition C2 and the results of points 2.1 and 2.2, we obtain

$$\begin{aligned}&\lim_{x_0^*\to x_0}\|x_2^*-x_2\|\\&\quad=\lim_{x_0^*\to x_0}\left\|x(t_2^*;x_0^*)-x(t_2;x_0)\right\|\\&\quad=\lim_{x_0^*\to x_0}\left\|x_2(t_2^*;t_1^*,x_1^{*+})-x_2(t_2;t_1,x_1^+)\right\|\\&\quad\le\lim_{x_0^*\to x_0}\left\|x_2(t_2^*;t_1^*,x_1^{*+})-x_2(t_2^{min};t_1^*,x_1^{*+})\right\|\\&\qquad+\lim_{x_0^*\to x_0}\left\|x_2(t_2;t_1,x_1^+)-x_2(t_2^{min};t_1,x_1^+)\right\|\\&\qquad+\lim_{x_0^*\to x_0}\left\|x_2(t_2^{min};t_1^*,x_1^{*+})-x_2(t_2^{min};t_1,x_1^+)\right\|\\&\quad\le\lim_{x_0^*\to x_0}\int_{t_2^{min}}^{t_2^*}\left\|f_2\big(x_2(\tau;t_1^*,x_1^{*+})\big)\right\|d\tau\\&\qquad+\lim_{x_0^*\to x_0}\int_{t_2^{min}}^{t_2}\left\|f_2\big(x_2(\tau;t_1,x_1^+)\big)\right\|d\tau\\&\qquad+\lim_{x_0^*\to x_0}\left\|x(t_2^{min};x_0^*)-x(t_2^{min};x_0)\right\|\\&\quad\le C_{f_2}\lim_{x_0^*\to x_0}(t_2^*-t_2^{min})+C_{f_2}\lim_{x_0^*\to x_0}(t_2-t_2^{min})\\&\quad=C_{f_2}\lim_{x_0^*\to x_0}(t_2^{max}-t_2^{min})=0.\end{aligned}$$

2.4. It is satisfied (given the previous point and Condition C12)

$$\begin{aligned}
&\lim_{x_0^*\to x_0} \|x_2^{*+} - x_2^+\| \\
&\quad = \lim_{x_0^*\to x_0} \big\|J_2(x_2^*) - J_2(x_2)\big\| \\
&\quad = \lim_{x_2^*\to x_2} \big\|J_2(x_2^*) - J_2(x_2)\big\| = 0.
\end{aligned}$$

2.5. We find out

$$\begin{aligned}
&\lim_{x_0^*\to x_0} \big\|x(t_2^{max}+0;x_0^*) - x(t_2^{max}+0;x_0)\big\| \\
&\quad = \lim_{x_0^*\to x_0} \big\|x_3(t_2^{max}+0;t_2^*,x_2^{*+}) - x_3(t_2^{max}+0;t_2,x_2^+)\big\| \\
&\quad = \lim_{x_0^*\to x_0} \big\|x_3(t_2^{max}+0;t_2^*,x_2^{*+}) - x_2^{*+}\big\| \\
&\qquad + \lim_{x_0^*\to x_0} \big\|x_3(t_2^{max}+0;t_2^*,x_2^{*+}) - x_2^{+}\big\| \\
&\qquad + \lim_{x_0^*\to x_0} \|x_2^{*+} - x_2^+\| \\
&\quad \le \lim_{x_0^*\to x_0} \int_{t_2^*}^{t_2^{max}} \Big\|f_3\big(x_3(\tau;t_2^*,x_2^{*+})\big)\Big\| d\tau \\
&\qquad + \lim_{x_0^*\to x_0} \int_{t_2}^{t_2^{max}} \Big\|f_3\big(x_3(\tau;t_2,x_2^{+})\big)\Big\| d\tau \\
&\quad \le C_{f_3} \lim_{x_0^*\to x_0} (t_2^{max} - t_2^{min}) = 0.
\end{aligned}$$

By induction for $i = 1, 2, ..., k$, we have:

$$\begin{aligned}
&i.1.\ \lim_{x_0^*\to x_0} \big\|x(t;x_0^*) - x(t;x_0)\big\| = 0,\ t_{i-1}^{max} < t \le t_i^{min}; \\
&i.2.\ \lim_{x_0^*\to x_0} (t_i^{max} - t_i^{min}) = 0; \\
&i.3.\ \lim_{x_0^*\to x_0} \|x_i^* - x_i\| = 0; \\
&i.4.\ \lim_{x_0^*\to x_0} \|x_i^{*+} - x_i^+\| = 0; \\
&i.5.\ \lim_{x_0^*\to x_0} \big\|x(t_i^{max}+0;x_0^*) - x(t_i^{max}+0;x_0)\big\| = 0.
\end{aligned}$$

By the equalities i.1, where $i = 1, 2, ..., k$, we obtain

$$(\exists \delta_1 = const > 0):\ \big(\forall x_0^* \in G_1,\ \|x_0^* - x_0\| < \delta_1\big) \tag{12}$$

$$\Rightarrow \left\|x(t;x_0^*) - x(t;x_0)\right\| < \varepsilon,\ t \in \bigcup_{i=1}^{k}(t_{i-1}^{max}, t_i^{min}).$$

From the equalities i.2, where $i = 1, 2, ..., k$, we get

$$(\exists\delta_2 = const > 0) : \left(\forall x_0^* \in G_1,\ \|x_0^* - x_0\| < \delta_2\right)$$
$$\Rightarrow (t_i^{max} - t_i^{min}) < \eta,\ i = 1, 2, ..., k,$$

whence, taking into consideration the inequalities (7), we reach the inclusion

$$\begin{aligned}
&\bigcup_{i=1}^{k}(t_{i-1}^{max}, t_i^{min}) \qquad (13)\\
&= (t_0, t_k)\backslash\bigcup_{i=1}^{k}[t_i^{min}, t_i^{max}]\\
&\supset (t_0, T)\backslash\bigcup_{i=1}^{k} B_\eta(t_i)\\
&= \left\{t;\ t_0 < t < T,\ |t - t_i| > \eta,\ i = 1, 2, ..., k\right\}\\
&= \left\{t;\ t_0 < t < T,\ |t - t_i| > \eta,\ i = 1, 2, ...\right\}.
\end{aligned}$$

Finally, take into account the arbitrary choice of constants $x_0, \varepsilon, \eta, T$, from (12) and (13), it follows that

$$(\forall x_0 \in G_1)(\forall\varepsilon = const > 0)(\forall\eta = const > 0)(\forall T = const > 0)$$
$$\left(\exists\delta = min\{\delta_1, \delta_2\} > 0\right) : \left(\forall x_0^* \in G_1,\ \|x_0^* - x_0\| < \delta\right)$$
$$\Rightarrow \left\|x(t;x_0^*) - x(t;x_0)\right\| < \varepsilon,\ t_0 < t < T,\ |t - t_i| > \eta,\ i = 1, 2, ...$$

The theorem is proved.

From Theorem 1 - Theorem 4 we get the following statements.

Corollary 1. *Let Conditions C1-C14 hold.*

Then the solution of problem (1), (2), (3) depends continuously on the initial point.

In the next corollary, we derive the sufficient conditions for continuous dependence of the solutions of original system (1), (2) without requiring for the solutions of initial values problems (6) to be uniformly Lipschitz stable (i.e. without Condition C4).

Corollary 2. *Assume that:*
1. Conditions C1-C3 and C5-C14 hold.
2. Assumption A1 is valid.

Then the solution of problem (1), (2), (3) depends continuously on the initial point.

4. Periodicity

We introduce the next condition.
C15. There exists $k_0 \in N$ such that $\overline{J_{k_0}\big(\Phi_{k_0}\big)} \subset G_1 \backslash \overline{D_1}$ and $\overline{J_{k_0}\big(\Phi_{k_0}\big)}$ is convex.

Theorem 5. *Assume that:*
1. Conditions C1-C3, C7, C8, C12, C13 and C15 hold.
2. Assumptions A1-A3 are valid

Then

$$\Big(\exists x_0 \in \overline{J_{k_0}\big(\Phi_{k_0}\big)}\Big): \; J_{k_0}\big(x(t_{k_0}; x_0)\big) = J_{k_0}(x_{k_0}) = x_{k_0}^+ = x_0.$$

Proof. Let x_0 be an arbitrary point from $\overline{J_{k_0}\big(\Phi_{k_0}\big)} \subset G_1 \backslash \overline{D_1}$. The solution $x(t; x_0)$ of system (1), (2), (3) consistently meets the switching sets $\Phi_1, \Phi_2, ..., \Phi_{k_0}$. The meetings take place in the moments $t_1, t_2, ..., t_{k_0}$ $(0 < t_1 < t_2 < ... < t_{k_0})$, respectively. Given Condition C15 and Assumption A1, we conclude that $x(t_{k_0}; x_0) = x_{k_0} \in \Phi_{k_0}$ and

$$J_{k_0}\big(x(t_{k_0}; x_0)\big) = x_{k_0}^+ \in J_{k_0}\big(\Phi_{k_0}\big) \subset \overline{J_{k_0}\big(\Phi_{k_0}\big)}.$$

We define the function $F_{k_0} : \overline{J_{k_0}\big(\Phi_{k_0}\big)} \to \overline{J_{k_0}\big(\Phi_{k_0}\big)}$ as follows

$$\Big(\forall x_0 \in \overline{J_{k_0}\big(\Phi_{k_0}\big)}\Big) \Rightarrow F_{k_0}(x_0) = J_{k_0}\big(x(t_{k_0}; x_0)\big) = J_{k_0}(x_{k_0}).$$

Similar to Theorem 4 (see i.3), we have

$$\begin{aligned} &\lim_{x_0^* \to x_0, \; x_0^* \in \overline{J_{k_0}\left(\Phi_{k_0}\right)}} F_{k_0}(x_0^*) \\ &= \lim_{x_0^* \to x_0, \; x_0^* \in \overline{J_{k_0}\left(\Phi_{k_0}\right)}} J_{k_0}\big(x(t_{k_0}^*; x_0^*)\big) \end{aligned}$$

$$= \lim_{x_0^* \to x_0,\ x_0^* \in \overline{J_{k_0}\left(\Phi_{k_0}\right)}} J_{k_0}(x_{k_0}^*)$$
$$= \lim_{x_0^* \to x_0,\ x_0^* \in G_1} J_{k_0}(x_{k_0}^*)$$
$$= J_{k_0}(x_{k_0}) = F_{k_0}(x_0).$$

Therefore, the function F_{k_0} is continuous, i.e. $F_{k_0} \in C[\overline{J_{k_0}\left(\Phi_{k_0}\right)}, \overline{J_{k_0}\left(\Phi_{k_0}\right)}]$.

In accordance with Condition C15, we have $\overline{J_{k_0}\left(\Phi_{k_0}\right)} \subset G_1 \backslash \overline{D_{k_0}} \subset G_1$. On the other hand, by Condition C13 (and in particular from Remark 1), it follows that G_1 is a bounded set. Thus, the conclusion is that the set $\overline{J_{k_0}\left(\Phi_{k_0}\right)}$ is bounded. Again from Condition C15, we have that this set is convex.

We obtain that the set $\overline{J_{k_0}\left(\Phi_{k_0}\right)}$ is bounded, closed and convex. The function F_{k_0} displayed continuously $\overline{J_{k_0}\left(\Phi_{k_0}\right)}$ in $\overline{J_{k_0}\left(\Phi_{k_0}\right)}$. Then from the Brouwer's fixed point theorem, it follows

$$\left(\exists x_0 \in \overline{J_{k_0}\left(\Phi_{k_0}\right)}\right) : \ F_{k_0}(x_0) = x_0 \quad \Leftrightarrow \quad x_{k_0}^+ = x_0.$$

The theorem is proved.

From Theorem 1 - Theorem 3 and Theorem 5, we get next statement.

Corollary 3. *Let Conditions C1-C15 hold.*

Then $\left(\exists x_0 \in \overline{J_{k_0}\left(\Phi_{k_0}\right)}\right) : \ x_{k_0}^+ = x_0$.

We introduce the conditions.

C16. For $i = 1, 2, ...$, the following equalities are satisfied:

- $G_{i+k_0} = G_i, \quad D_{i+k_0} = D_i;$
- $f_{i+k_0}(x) = f_i(x),\ x \in G_i;$
- $\varphi_{i+k_0}(x) = \varphi_i(x),\ x \in D_i;$
- $J_{i+k_0}(x) = J_i(x),\ x \in \Phi_i.$

Remark 3. *Condition C14 follows from Condition C16. Indeed, under Condition C16, we obtain*

$$\frac{C_{\partial D_{i+k_0}}}{C_{f_{i+k_0}}.C_{Lip\varphi_{i+k_0}}} = \frac{C_{\partial D_i}}{C_{f_i}.C_{Lip\varphi_i}},\ i = 1, 2, ...$$

Then

$$lim_{i \to \infty} \frac{C_{\partial D_i}}{C_{f_i}.C_{Lip\varphi_i}} \neq 0.$$

Thus, we find that the series $\sum \frac{C_{\partial D_i}}{C_{f_i}.C_{Lip\varphi_i}}$ *is divergent.*

Condition C14 is ignored in the following statements.

As a consequence of Theorem 5, we will formulate several conditions for existence of a periodic solution of basic problem (1), (2), (3).

Theorem 6. *Assume that:*
1. Conditions C1-C3, C7, C8, C12, C13, C15 and *C16 hold.*
2. Assumptions A1-A3 are valid.

Then there exists an initial point $x_0 \in \overline{J_{k_0}(\Phi_{k_0})}$ *such that the solution* $x(t;x_0)$ *of problem (1), (2), (3) is periodic with a period* t_{k_0}.

Corollary 4. *Assume that:*
1. Conditions C1-C3, C5-C13, C15 and C16 hold.
2. Assumption A1 is true.

Then there exists an initial point $x_0 \in \overline{J_{k_0}(\Phi_{k_0})}$ *such that the solution* $x(t;x_0)$ *of problem (1), (2), (3) is periodic with a period* t_{k_0}.

Corollary 5. *Conditions C1-C13, C15 and C16 hold.*

Then there exists an initial point $x_0 \in \overline{J_{k_0}(\Phi_{k_0})}$ *such that the solution* $x(t;x_0)$ *of problem (1), (2), (3) is periodic with a period* t_{k_0}.

5. Application

The classical Lotka-Volterra model describes fairly adequately the dynamics of an isolated community of predator-prey type without external influences. Assume that the community could modify its parameters of development. We suppose that these changes are performed instantaneously. The parameters change reflects on the sharp variation of the growth speed of the biomass of both species. Further, we will specify the moments in which these changes are made. The corresponding initial value problem has the form:

$$\frac{dm}{dt} = f_i^1(m, M) = m(r_i^1 - q_i^1 M); \tag{14}$$

$$\frac{dM}{dt} = f_i^2(m, M) = -M(r_i^2 - q_i^2 m); \tag{15}$$

$$m(0) = m_0; \quad M(0) = M_0, \tag{16}$$

where:

- $m = m(t) > 0$ and $M = M(t) > 0$ are the prey and predator biomasses at the

moment $t \geq 0$;
- $r_i^1 = const > 0$ and $r_i^2 = const > 0,\ i = 1, 2, ...,$ are specific growth factors, relevant to the first species (prey) and the second (predator), respectively;
- $q_i^1 = const > 0$ and $q_i^2 = const > 0,\ i = 1, 2, ...,$ are the coefficients indicating interspecies competition. In the common case, they are different for the prey and predator;
- $m_0 > 0$ and $M_0 > 0$ are the prey and predator biomasses at the initial moment $t = 0$.

It is known that for $i = 1, 2, ...,$ the system (14), (15) possesses:
1. Unstable (saddle) stationary point $(0, 0)$;
2. Stable stationary point $(m_i^{00}, M_i^{00}) = (r_i^2/q_i^2, r_i^1/q_i^1)$;
3. First integral of the form

$$V_i(m, M) = q_i^1 M + q_i^2 m - r_i^1 lnM - r_i^2 lnm + r_i^1\Big(ln\frac{r_i^1}{q_i^1} - 1\Big) + r_i^2\Big(ln\frac{r_i^2}{q_i^2} - 1\Big);$$

4. For each constant $c \geq 0$, implicitly given curve

$$\gamma_i^c = \big\{(m, M);\ V_i(m, M) = c\big\}$$

is a trajectory of system (14), (15). This trajectory is closed. The initial point (m_0, M_0) of this trajectory is suitably chosen. It is sufficient $V_i(m_0, M_0) = c$;
5. For each constant $c > 0$, the set

$$D_i^c = \big\{(m, M);\ 0 < V_i(m, M) < c\big\}$$

is a connected domain, situated in $R^+ \times R^+$, which possesses contour $\partial D_i^c = \gamma_i^c \cup \{(m_i^{00}, M_i^{00})\}$;
6. If $0 < c_1 < c_2$, then $\gamma_i^{c_1} \subset D_i^{c_2}$.

Let c and C be arbitrary constants such that $0 < c < C$. For $i = 1, 2, ...,$ the phase space of system (14), (15) is defined as $G_i^{c,C} = D_i^C \backslash \overline{D_i^c}$, i.e.

$$G_i^{c,C} = \big\{(m, M);\ c < V_i(m, M) < C\big\}.$$

Assume that the biomass of the prey is useful. For this reason, predetermined quantity of this biomass is produced for a "relatively long period of time". For this purpose, in the form of impulsive effects, certain quantities of biomass (for example $m_i,\ i = 1, 2, ...$) are withdrawn repeatedly from the prey. The moments $t_1, t_2, ...,$ at which the impulsive effects are realized upon the modeled community, coincide with the moments when the trajectory of problem (14), (15), (16) meets the so-called reachable (or switching) sets $\Phi_1, \Phi_2, ...$ These sets are situated in $D_i^{c,C} \subset G_i^{c,C}$ and are defined by the switching functions $\varphi_1, \varphi_2, ...$ Specifically, $\varphi_1 : D_1^{c,C} \to R,\ \varphi_2 : D_2^{c,C} \to R, ...$, where the switching functions are defined as follows:

$$\varphi_i(m, M) = M - M_i^{00} = M - \frac{r_i^1}{q_i^1},\ i = 1, 2, ... \tag{17}$$

and the domains

$$D_i^{c,C} = \left(\left\{m;\ m_i^{00} = \tfrac{r_i^2}{q_i^2} < m < \infty\right\} \times R^+\right) \cap G_i^{c,C},\ i = 1, 2, ... \tag{18}$$

We will specify that changes of parameters in the studied system are realized in moments $t_1, t_2, ...$

Further, we denote by $m_i^{c,max}$ and $m_i^{C,max}$, respectively the bigger solutions of the equations

$$V_i(m, M_i^{00}) = V_i\left(m, \frac{r_i^1}{q_i^1}\right) = c \Leftrightarrow m - \frac{r_i^2}{q_i^2} lnm = \frac{r_i^2}{q_i^2}\left(1 - ln\frac{r_i^2}{q_i^2}\right) + \frac{c}{q_i^2}$$

and

$$V_i(m, M_i^{00}) = V_i\left(m, \frac{r_i^1}{q_i^1}\right) = C \Leftrightarrow m - \frac{r_i^2}{q_i^2} lnm = \frac{r_i^2}{q_i^2}\left(1 - ln\frac{r_i^2}{q_i^2}\right) + \frac{C}{q_i^2}.$$

The constants $m_i^{c,min}$ and $m_i^{C,min}$ are smaller solutions of the above two equations.

Then the reachable sets have the form

$$\Phi_i^{c,C} = \left\{(m, M);\ m_i^{c,max} < m < m_i^{C,max},\ M = M_i^{00}\right\},\ i = 1, 2, ... \tag{19}$$

The moments of impulsive effects $t_1, t_2, ...$ satisfy the equalities:

$$\begin{aligned} &\varphi_i\big(m(t_i), M(t_i)\big) = 0,\ m(t_i) > m_i^{00} \\ \Leftrightarrow\quad &M(t_i) = \frac{r_i^1}{q_i^1},\ m(t_i) > \frac{r_i^2}{q_i^2},\ i = 1, 2, ... \end{aligned}$$

Remark 4. *Pay attention that for*

$$M(t) = M_i^{00} = \frac{r_i^1}{q_i^1} \quad and \quad m(t) > m_i^{00} = \frac{r_i^2}{q_i^2},$$

the right hand side of (14) becomes zero. Therefore, we have $(dm(t))/(dt) = 0$. It can be shown that just then the victim's biomass is maximum. Consequently, the withdrawal of biomass from the victim in these moments $(t_1, t_2, ...)$ is justified.

Remark 5. *It is natural to assume that the quantity of victim's biomass taken away at the moment t_i depends on its volume at this moment. Usually, the quantity taken away is $m_i = p_i.m(t_i)$, where the constant p_i satisfies $0 \leq p_i \leq 1,\ i = 1, 2, ...$*

Let the impulsive functions are defined by equalities

$$J_i(m, M) = \big(J_i^1(m, M), J_i^2(m, M)\big) = \left((1 - p_i).m, \frac{r_{i+1}^1}{q_{i+1}^1}\right),\ i = 1, 2, ...$$

Then:

$$\begin{aligned} m(t_i + 0) &= J_i^1\big(m(t_i), M(t_i)\big) \\ &= (1 - p_i).m(t_i) = q_i.m(t_i),\ q_i = 1 - p_i,\ 0 \leq q_i \leq 1; \\ M(t_i + 0) &= J_i^2\big(m(t_i), M(t_i)\big) = \frac{r_{i+1}^1}{q_{i+1}^1},\ i = 1, 2, ... \end{aligned} \tag{20}$$

In addition, we suppose that

$$\begin{aligned} & m_{i+1}^{C,min} < m(t_i + 0) < m_{i+1}^{c,min} \\ \Leftrightarrow\ & \frac{m_{i+1}^{C,min}}{m(t_i)} < q_i < \frac{m_{i+1}^{c,min}}{m(t_i)},\ i = 1, 2, ... \end{aligned} \tag{21}$$

Since $m_i^{c,max} < m(t_i) < m_i^{C,max}$, the inequalities (21) are valid, if

$$\frac{m_{i+1}^{C,min}}{m_i^{c,max}} < q_i < \frac{m_{i+1}^{c,min}}{m_i^{C,max}},\ i = 1, 2, ... \tag{22}$$

The inequalities (22) will be valid, if

$$m_{i+1}^{C,min}.m_i^{C,max} \leq m_{i+1}^{c,min}.m_i^{c,max},\ i = 1, 2, ... \tag{23}$$

Note that, in case $r_1^1 = r_2^1 = ...,\ r_1^2 = r_2^2 = ...,\ q_1^1 = q_2^1 = ...$ and $q_1^2 = q_2^2 = ...$ the inequalities (23) are fulfilled.

Remark 6. *We consider that the inequalities (21) are natural and useful of the following three reasons:*
- Quantity of biomass m_i (taken away from the prey at the moment t_i) is

$$m_i = m(t_i) - m(t_i + 0),\ i = 1, 2, ...$$

Since $M(t_i+0) = M_{i+1}^{00}$, we conclude that $m(t_i+0)$ satisfies (21) or the next inequalities

$$m_{i+1}^{c,max} < m(t_i+0) < m_{i+1}^{C,max}, \ i = 1, 2, ... \tag{24}$$

Obviously, the quantity of biomass m_i is greater, if the inequalities (21) are true;

- After the moment t_i, the solution of problem starts from the initial point $\big(m(t_i+0), M_{i+1}^{00}\big)$. It can be shown that, if (21) is satisfied, then from this moment on (until to the next switching moment t_{i+1}), the victim's biomass increases, which is favorable for the user. On the contrary, if the inequalities (24) are valid, immediately after the switching moment, the victim's biomass decreases;

- The time interval between two adjacent moments of removal of victim's biomass $\Delta_i = t_{i+1} - t_i$ is shorter in the case when $m(t_i+0)$ satisfies (21) in comparison with the case when $m(t_i+0)$ satisfies (24). In other words, the time for reproduction of withdrawn biomass in case (21) is shorter than in case (24).

Consider the following problem (model of predator-prey community with the impulsive extractions of biomass from the victim and impulsive change of the parameters):

$$\frac{dm}{dt} = m(r_i^1 - q_i^1 M), \ M(t) \neq M_i^{00}, \ \text{i.e. } t_{i-1} < t < t_i; \tag{25}$$

$$\frac{dM}{dt} = -M(r_i^2 - q_i^2 m), \ M(t) \neq M_i^{00}, \ \text{i.e. } t_{i-1} < t < t_i; \tag{26}$$

$$m(t_i+0) = q_i.m(t_i), \ 0 \leq q_i \leq 1; \tag{27}$$

$$M(t_i+0) = M_{i+1}^{00}, \ i = 1, 2, ...; \tag{28}$$

$$m(0) = m_0; \quad M(0) = M_0. \tag{29}$$

We will apply Corollary 4 for system (25)-(28). Therefore, we will check the validity of Conditions C1-C3, C5-C13, C15, C16 and Assumption A1. Sequentially for $i = 1, 2, ...$, we find:

- Condition C1: The functions $f_i \in C^1[\overline{G_i^{c,C}}, R^2]$, where

  $$\begin{aligned} f_i(m, M) &= \big(f_i^1(m, M), f_i^2(m, M)\big) \\ &= \big(m(r_i^1 - q_i^1.M), -M(r_i^2 - q_i^2.m)\big) \end{aligned}$$

  are the right hand sides of system (25), (26). Therefore,

  $$\frac{\partial f_i^1}{\partial m}, \ \frac{\partial f_i^1}{\partial M}, \ \frac{\partial f_i^2}{\partial m} \text{ and } \frac{\partial f_i^2}{\partial M}$$

are bounded in $G_i^{c,C} \subset \overline{G_i^{c,C}}$. This means that the functions f_i satisfy the Lipschitz conditions.

- Condition C2: The functions $f_i \in C[\overline{G_i^{c,C}}, R^2]$. Therefore, f_i are bounded in $G_i^{c,C} \subset \overline{G_i^{c,C}}$.

- Condition C3: We have $\big(\forall (m_0, M_0) \in G_i^{c,C}\big)(\forall t \in R)$ it follows that the solutions of problem (25)-(29) $\big(m_i(t;0,m_0,M_0), M_i(t;0,m_0,M_0)\big) \in \gamma_i^c$, where $c = V_i(m_0, M_0)$. Therefore, the solutions of systems (25)-(28) exist and are unique in R.

- Condition C5: It is clear that the functions $\varphi_i(m,M) = M - M_i^{00}$ are continuously differentiable in $\overline{G_i^{c,C}}$.

- Condition C6: We obtain

$$\begin{aligned} &\Big(\forall (m,M) \in \Phi_i^{c,C} \\ &\qquad = \big\{(m,M);\ m_i^{c,max} < m < m_i^{C,max},\ M = M_i^{00}\big\}\Big) \\ \Rightarrow\ & \big\langle grad\varphi_i(m, M_i^{00}), f_i(m, M_i^{00})\big\rangle \\ =\ & \Big\langle (0,1).\big(m(r_i^1 - q_i^1.M_i^{00}), -M_i^{00}(r_i^2 - q_i^2.m)\big)\Big\rangle \\ =\ & -\frac{r_i^1}{q_i^1}(r_i^2 - q_i^2.m) = \frac{r_i^1.q_i^2}{q_i^1}\big(m - m_i^{00}\big) \\ \geq\ & \frac{r_i^1.q_i^2}{q_i^1}\big(m_i^{c,max} - m_i^{00}\big) = const = C_{\langle grad\varphi_i, f_i\rangle}. \end{aligned}$$

- Condition C7: The sets $\Phi_i^{c,C}$ are open intervals. Therefore, they are connected sets.

- Condition C8: It is fulfilled

$$\overline{\Phi_i^{c,C}} \backslash \Phi_i^{c,C} = \Big\{\big(m_i^{c,max}, M_i^{00}\big),\ \big(m_i^{C,max}, M_i^{00}\big)\Big\} \subset \partial G_i^{c,C}.$$

- Condition C9: For every point $(m,M) \in D_i^{c,C}$, it is true

$$\rho\big((m,M), \Phi_i^{c,C}\big) \geq \big|M - M_i^{00}\big| = \big|\varphi_i(m,M)\big|.$$

Then the constants $C_{\varphi_i} = 1$.

- Condition C10: For $(m', M'), (m'', M'') \in G_i^{c,C}$, we have
$$|\varphi_i(m', M') - \varphi_i(m'', M'')| = |M' - M''| \leq \big\|(m', M') - (m'', M'')\big\|.$$
The constants $C_{Lip\varphi_i} = 1$.

- Condition C11: We establish
$$\partial D_i^{c,C} \cap G_i^{c,C} = \{m_i^{00}\}\times$$
$$\times\Big(\{M : M_i^{C,min} < M < M_i^{c,min}\} \cup \{M : M_i^{c,max} < M < M_i^{C,max}\}\Big),$$
where $M_i^{c,min}$ and $M_i^{c,max}$ are small and large solutions of the equation $V_i(m_i^{00}, M) = c$, respectively. Similarly, $M_i^{C,min}$ and $M_i^{C,max}$ are small and large solutions of the equation $V_i(m_i^{00}, M) = C$. Then for $(m, M) \in \partial D_i^{c,C} \cap G_i^{c,C}$, we have
$$|\varphi_i(m, M)| = |M - M_i^{00}|$$
$$\geq \max\Big\{M_i^{00} - M_i^{c,min}, M_i^{c,max} - M_i^{00}\Big\} = C_{D_i}.$$

- Condition C12: The functions $J_i(m, M) = \big(q_i.m, M_{i+1}^{00}\big)$ are continuous for $(m, M) \in \Phi_i^{c,C}$. Furthermore, from (21), it follows
$$m(t_i + 0) = J_i^1\big(m(t_i), M(t_i)\big) < m_{i+1}^{c,min} < m_{i+1}^{00},$$
$$M(t_i + 0) = J_i^2\big(m(t_i), M(t_i)\big) = M_{i+1}^{00}.$$
The above inequalities indicate that $J_i(m, M) \in G_{i+1}^{c,C} \backslash \overline{D_{i+1}^{c,C}}$.

- Condition C13: The domains $G_i^{c,C}$ are bounded-connected.

- Condition C15: It is enough to assume that $(\exists k_0 \in N)$ such that the inequalities
$$c < V_1\big(m_{k_0}^{C,min}, M_{k_0}^{00}\big), \quad V_1\big(m_{k_0}^{c,min}, M_{k_0}^{00}\big) < C$$
are valid.

- Condition C16: We will assume that:
$$r_{i+k_0}^1 = r_i^1, \quad r_{i+k_0}^2 = r_i^2, \quad q_{i+k_0}^1 = q_i^1, \quad q_{i+k_0}^2 = q_i^2, \quad q_{i+k_0} = q_i,$$
from where, it follows this condition.

- Assumption 1: The next statements are valid:
  - If $\gamma_i^{const} \cap G_i^{c,C} \neq \emptyset \;\Rightarrow\; \gamma_i^{const} \subset G_i^{c,C}$, i.e. each trajectory passing through the point of phase space is contained in $G_i^{c,C}$;
  - If $\gamma_i^{const} \subset G_i^{c,C} \;\Rightarrow\; \gamma_i^{const} \cap \Phi_i^{c,C} \neq \emptyset$, i.e. each trajectory from $G_i^{c,C}$ crosses switching set $\Phi_i^{c,C}$.

  From the above two statements, it follows that the sets $\Phi_i^{c,C}$ are totally reachable.

Remark 7. *Through Corollary 4, we find that system (25)-(28) with variable structure and impulses has a periodic solution.*

Acknowledgments

The authors thank to University of Chemical Technology and Metallurgy, Sofia for the financial support.

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