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SIGNED CLIQUE EDGE-DOMATIC NUMBER OF A GRAPH

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Abstract: In this paper, the problem of signed clique edge-domatic number is considered for any finite,nonempty and simple graph G with the maximal clique K_ω. At first,using induction method, we study the signed clique edge-domatic number for a class of graphs with maximal clique $K_\omega(\omega = 2, 3)$. By inducing and summarizing,we obtain that the signed clique edge-domatic number on this class of graphs and the signed clique edge-domatic number of their maximal clique K_ω have some relations. Inspired by ways and regulars of the proof of graphs with $\omega(G) = 2, 3$, we take seriously to study the signed clique edge-domatic number for any graph G with $K_\omega(\omega \geq 4)$ and have proved that the signed clique edge-domatic numbers on G and on $K_\omega(\omega \geq 4)$ are equal, so we achieve the signed clique edge-domatic number for any graph G. This conclusion is a practical significance for the promotion of signed edge-domatic number.

AMS Subject Classification: 05C69
Key Words: clique, signed clique edge domination, signed clique edge-domatic number, signed clique edge dominating family

1. Introduction

In this paper, all of the graphs that we consider are finite,nonempty, simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order n of G denotes the number of vertices of G and the size m of G denotes the number of edges of G. For any edge $e \in E(G)$,$N_G(e)$ will

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denote the open neighborhood of e in G and $N_G[e] = N_G(e)\bigcup\{e\}$ will denote the closed neighborhood.

Definition 1. (see [1]) A function $f' : E(G) \to \{-1, 1\}$ is called the signed edge dominating function (SEDF) of G if $\sum\limits_{e'\in N[e]} f'(e') \geq 1$,for every $e \in E(G)$. The signed edge domination number of G is defined as: $\gamma'_s(G) = \min\{\sum\limits_{e\in E(G)} f'(e) \mid f \text{ is an SEDF of G}\}$.

Every complete subgraph K_ω of G that is not included by other any complete subgraph of G is called a maximal clique of G. The order of maximal complete subgraph is called clique number of G,which is denoted by $\omega(G)$.

Definition 2. (see [2]) A function $f' : E(G) \to \{-1, 1\}$ is called a signed clique edge dominating function(SCEDF)of G if $\sum\limits_{e\in E(K_\omega)} f'(e) \geq 1$,for every maximal clique K_ω of G. Let $F_{sced}(G)$ is a set of all signed clique edge dominating functions on G.The signed clique edge domination number of G is defined as:$\gamma'_{scl}(G) = \min\{\sum\limits_{e\in E(G)} f'(e) \mid f' \in F_{sced}(G)\ \}$.

If $S \subseteq E = E(G)$, we denote

$$f'(S) = \sum_{e\in S} f'(e),$$

then $\gamma'_{scl}(G) = \min\{f'(E(G))|f' \in F_{sced}(G)\}$. We have concluded some properties of $\gamma'_{scl}(G)$ based on its definition:

(a) $\gamma'_{scl}(G) = |E(G)|$ if and only if G has no triangles.

(b) For arbitrary two disjoint graphs G_1,G_2,there is

$$\gamma'_{scl}(G_1 \bigcup G_2) = \gamma'_{scl}(G_1) + \gamma'_{scl}(G_2).$$

Definition 3. (see [3]) A set$\{f'_1, f'_2, \cdots, f'_d\}$ of signed edge dominating functions on G with the property that$\sum\limits_{i=1}^{d} f'_i(e) \leq 1$ for each edge $e \in E(G)$ is called a signed edge dominating family on G. The maximum number of functions in a signed edge dominating family on G is the signed edge-domatic number of G, denoted by $d'_s(G)$.

Now we similarly define a signed clique edge-domatic number of G.

Definition 4. Let a set$\{f_1', f_2', \cdots, f_d'\} \subseteq F_{sced}(G)$ on G with the property that $\sum_{i=1}^{d} f_i'(e) \leq 1$ for each edge $e \in E(G)$ be called a signed clique edge dominating family on G. Use $\Psi(G)$ to denote the collection of families of signed clique edge-dominations on G. The signed clique edge-domatic number of G is denoted by $d'_{scl}(G) = \max\{|F| : F \in \Psi(G)\}$. Obviously, $d'_{sccl}(G) \geq 1$.

V. Lutz, B. Zelinka study the signed domatic number and they give some new ways to solve those problems in [6] and [7], then X.J.Li and J.M.Xu research the signed edge domatic number in [3] and B.Xu proves the signed cycle domination number on graphs in [8]. So this article mainly studys the signed clique edge-domatic number.

2. Basic Properties of the Signed Clique Edge-domatic Number of a Graph

Theorem 1. *Let G be a graph of size $|E(G)| = m$ with the signed clique edge domination number $\gamma'_{scl}(G)$ and the signed clique edge-domatic number $d'_{scl}(G)$,then $\gamma'_{scl}(G)d'_{scl}(G) \leq m$.*

Proof. If $\{f_1', f_2', \cdots, f_d'\} \in \Psi(G)$ such that $d = d'_{scl}(G)$,then the definitions imply:

$$\begin{aligned}\gamma'_{scl}(G)d'_{scl}(G) = \gamma'_{scl}(G)d = \gamma'_{scl}(G)\sum_{i=1}^{d} 1 &\leq \sum_{i=1}^{d} \sum_{e\in E(G)} f_i'(e) \\ &= \sum_{e\in E(G)} \sum_{i=1}^{d} f_i'(e) \leq \sum_{e\in E(G)} 1 = m.\end{aligned}$$

Theorem 2. *For an arbitrary graph G, $d'_{scl}(G)$ is an odd integer.*

Proof. For an arbitrary graph G, suppose that $d = d'_{scl}(G)$ is an even number. Let $\{f_1', f_2', \cdots, f_d'\} \in \Psi(G)$ be the corresponding signed clique edge dominating family on G. We have $\sum_{i=1}^{d} f_i'(e) \leq 1$ for each $e \in E(G)$ based on definition. But on the left-hand side of this inequality a sum of an even number of odd summands occurs.Therefore,it is an even number and we obtain $\sum\limits_{i=1}^{d} f_i'(e) \leq 0$ for each $e \in E(G)$.This forces:

$$d = \sum_{i=1}^{d} 1 \leq \sum_{i=1}^{d} \sum_{e \in E(K\omega)} f_i'(e) = \sum_{e \in E(K\omega)} \sum_{i=1}^{d} f_i'(e) \leq 0,$$

which contradicts $d'_{sccl}(G) \geq 1$.

Theorem 3. *If* G *is a graph with maximal clique* $K_\omega, \omega \geq 2$*, then*

$$d'_{scl}(G) \leq \begin{cases} \frac{\omega(\omega-1)}{4} & if \quad \omega \equiv 0 \quad or \quad 1(mod4) \\ \frac{\omega(\omega-1)}{2} & if \quad \omega \equiv 2 \quad or \quad 3(mod4) \end{cases}$$

Proof. Let the size of a maximal clique K_ω on G be $|E(K_\omega)| = \frac{\omega(\omega-1)}{2}$. Let $d = d'_{scl}(G)$, $\{f_1', f_2', \cdots, f_d'\} \in \Psi(G)$ be the corresponding signed clique edge dominating family on G. Now we have two cases for the size of a maximal clique on G.

Case 1. $\omega \equiv 0 \quad or \quad 1(mod4)$.

If $\omega \equiv 0 \quad or \quad 1(mod4)$,$|E(K_\omega)| = \frac{\omega(\omega-1)}{2}$ is an even number. Let $f_i' \in \{f_1', f_2', \cdots, f_d'\}$, then $f_i'(E(K_\omega)) \geq 1$. Therefore,$f_i'(E(K_\omega)) \geq 2$. The imply:

$$2d = \sum_{i=1}^{d} 2 \leq \sum_{i=1}^{d} \sum_{e \in E(K_\omega)} f_i'(e) = \sum_{e \in E(K_\omega)} \sum_{i=1}^{d} f_i'(e) \leq \sum_{e \in E(K_\omega)} 1 = \frac{\omega(\omega-1)}{2},$$

so $d = d'_{scl}(G) \leq \frac{\omega(\omega-1)}{4}$.

Case 2. $\omega \equiv 2 \quad or \quad 3(mod4)$.

If $\omega \equiv 2 \quad or \quad 3(mod4)$, then $|E(K_\omega)| = \frac{\omega(\omega-1)}{2}$ is an odd number. Let $f_i' \in \{f_1', f_2', \cdots, f_d'\}$,then $f_i'(E(K_\omega)) \geq 1$. This imply:

$$d = \sum_{i=1}^{d} 1 \leq \sum_{i=1}^{d} \sum_{e \in E(K_\omega)} f_i'(e) = \sum_{e \in E(K_\omega)} \sum_{i=1}^{d} f_i'(e) \leq \sum_{e \in E(K_\omega)} 1 = \frac{\omega(\omega-1)}{2},$$

so $d = d'_{scl}(G) \leq \frac{\omega(\omega-1)}{2}$.

Theorem 4. *Let* G *be a graph of size* m *with no triangles. Then* $d'_{scl}(G) = 1$.

Proof. We can know the value of the signed clique edge domination number on G from its properties(a). We have $\gamma'_{scl}(G) = |E(G)| = m$.

Following from theorem 1, $\gamma'_{scl}(G)d'_{scl}(G) \leq m$, we can obtain $d'_{scl}(G) \leq \frac{m}{|E(G)|} = 1$. Since $d'_{scl}(G) \geq 1$, the equality can be achieved: $d'_{scl}(G) = 1$.

Corollary 1. *For any tree T of order $n \geq 3$, then $d'_{scl}(T) = 1$.*

A plane graph G is called a maximal plane graph if its every face is a triangle.

Lemma 1. *For any maximal plane graph G of order $n(n \geq 3)$, then it has $2n-4$ faces (including its outside boundary), see [2].*

Lemma 2. *For any maximal planar graph G of order $n(n \geq 3)$, then $\gamma'_{scl}(G) \geq n-2$.*

Proof. For any maximal planar graph G of order $n(n \geq 3)$ has $2n-4$ faces(including its outside boundary) from Lemma 1.

Let $F_1, F_2, \cdots, F_{2n-4}$ be all faces of G and $E(F_i)$ be a set of all edges of the face F_i.It is not hard to know that every face F_i is a maximal clique of G. Let $f' \in F_{sced}(G)$ satisfy $\gamma'_{scl}(G) = f'(E(G))$, then $f'(E(F_i)) \geq 1$ for any face F_i, based on definition of the function f',so

$$2\gamma'_{scl}(G) = \sum_{i=1}^{2n-4} \sum_{e \in E(F_i)} f'(e) \geq 2n-4, \text{ so } \gamma'_{scl}(G) \geq n-2.$$

Next we will introduce some class of graphs with some properties such that equality can be achieved at above.

Let $G_1 = K_3$,G_m is the graph obtained from G_{m-1} by adding a vertex u and three edges uv_1, uv_2, uv_3,where v_1, v_2, v_3 are three vertices on the same inner face(triangle) of $G_{m-1}(m = 2, 3, \cdots)$. Obviously,$G_m$is a maximal planar graph,and $|V(G_m)| = m + 2$.

Lemma 3. *Let $M = \{G_1, G_2, \cdots, G_m, \cdots\}$,then $\gamma'_{scl}(G) = n-2$ for any graph $G \in M$ with order n.*

Proof. Obviously, for any graph $G \in M$ with order $n(n \geq 3)$,by Lemma 2,$\gamma'_{scl}(G) \geq n-2$. Next we can obtain $\gamma'_{scl}(G) \leq n-2$.

Using induction on order n of G,we prove that $f'(E(G)) \leq n-2$ holds for all $n(n = 3, 4, 5, \cdots)$.

Clearly,when $n = 3$, we let $G = K_3$,then G is the maximal clique for itself.Let $E(G) = \{e_1, e_2, e_3\}$ and $f'(E(G)) = \gamma'_{scl}(G)$,there exist exactly values of two edges 1, so we can suppose that $f'(e_1) = 1, f'(e_2) = 1, f'(e_3) = -1$. Obviously,$f'(E(G)) \leq n - 2$ is true.

Suppose that there exists an SCEDF, say g' on $G = G_{n-3} \in M$ with order $n - 1$ such that $g'(E(G)) \leq (n - 1) - 2$ is true. Then we consider $G = G_{n-2} \in M$. Based on the definition of graph G, we can know that there exists 3-degree vertex v in some inner face of $G = G_{n-2}$. Let $H = G - \{v\} \in M$, then by the induction hypothesis, there is $g'(E(H)) \leq (n - 1) - 2$. Let $S = N_G(v) = \{u_1, u_2, u_3\}$, then $H[S] = K_3$ and the values of three edges of $H[S]$ exactly two are 1. Without loss of generality, let $g'(u_1u_2) = -1, g'(u_2u_3) = 1$ and $g'(u_1u_3) = 1$. When the vertex v adds to $H[S] = K_3$, it forms three sets $S_1 = \{u_1, u_2, v\}$,$S_1 = \{u_1, u_3, v\}$,$S_1 = \{u_3, u_2, v\}$, which constitute three maximal inner faces of $G = G_{n-2}$. Define a function f' on $G = G_{n-2}$ as follows:

$$f'(e) = \begin{cases} g'(e) & if \quad e \in E(G - \{v\}) \\ 1 & if \quad e \in \{u_1v, u_2v\} \\ -1 & if \quad e = u_3v \end{cases}$$

It is not hard to verify that f' is an SCEDF of $G = G_{n-2}$ such that $f'(E(G)) = 2 - 1 + f'(E(H)) \leq n - 2$ is true. So $f'(E(G)) = n - 2$.

Theorem 5. *Let $M = \{G_1, G_2, \cdots, G_m, \cdots\}$, for any graph G in M,then $d'_{scl}(G) = 3$.*

Proof. For any graph G with the order n in M, the number of the maximal cliques on G is $2n - 4$ by Lemma 1. Let $F_1, F_2, \cdots, F_{2n-4}$ denotes all maximal cliques of G. By theorem1,we determine the signed clique edge-domatic number $d'_{scl}(G) \leq 3$, since the size of G is $|E(G)| = 3n - 6$. Therefore, we can get $d'_{scl}(G) = 1$ or 3 by theorem 1 and theorem 2. Now we prove that $d'_{scl}(G) = 3$ is right.

Using induction on order n of G, we prove that $d'_{scl}(G) = 3$ holds for all $n(n = 3, 4, 5, \cdots)$.

When $n = 3$, then $G = K_3 = G_1$, so it is easy to verify that $d'_{scl}(G) = 3$ is true.

When $|V(G)| = n-1$, then $G = G_{n-3}$, so we suppose that $d'_{scl}(G) = 3$ is also true. Let $F_1, F_2, \cdots, F_{2(n-1)-4}$ denote all the faces of G_{n-3} and $\{g'_1, g'_2, g'_3\} \in \Psi(G_{n-3})$. Based on the definition of graph $G = G_{n-2}$, we can add the vertex u to any inner face F_i of G_{n-3} such that u is adjacent to three vertices of F_i. Let $V(F_i) = \{v_{1i}, v_{2i}, v_{3i}\}$, so we can get additional three inner faces F_{1i}, F_{2i}, F_{3i}

which are formed by $\{v_{1i}, v_{2i}, u\}$,$\{v_{2i}, v_{3i}, u\}$,$\{v_{3i}, v_{1i}, u\}$.
Therefore,F_{1i}, F_{2i}, F_{3i} are inner faces of G_{n-2}. Define f_1', f_2', f_3' on G_{n-2} as following:

When $g_1'(v_{1i}v_{2i}) = -1$,there must be $g_1'(v_{1i}v_{3i}) = g_1'(v_{2i}v_{3i}) = 1$. Let

$$f_1'(e) = \begin{cases} g_1'(e) & if \quad e \in E(G_{n-2} - \{u\}) \\ 1 & if \quad e \in \{uv_{1i}, uv_{2i}\} \\ -1 & if \quad e = uv_{3i} \end{cases}$$

When $g_2'(v_{2i}v_{3i}) = -1$,there must be $g_2'(v_{1i}v_{3i}) = g_2'(v_{1i}v_{2i}) = 1$. Let

$$f_2'(e) = \begin{cases} g_2'(e) & if \quad e \in E(G_{n-2} - \{u\}) \\ 1 & if \quad e \in \{uv_{2i}, uv_{3i}\} \\ -1 & if \quad e = uv_{1i} \end{cases}$$

When $g_3'(v_{1i}v_{3i}) = -1$,there must be $g_3'(v_{1i}v_{2i}) = g_3'(v_{2i}v_{3i}) = 1$. Let

$$f_3'(e) = \begin{cases} g_3'(e) & if \quad e \in E(G_{n-2} - \{u\}) \\ 1 & if \quad e \in \{uv_{1i}, uv_{3i}\} \\ -1 & if \quad e = uv_{2i} \end{cases}$$

It is not hard to verify that $\{f_1', f_2', f_3'\} \in \Psi(G_{n-2})$, so $d_{scl}'(G) = d_{scl}'(G_{n-2}) \geq 3$.Therefore, $d_{scl}'(G) = 3$.

Lemma 4. *For any graph G with size m, let $\omega(G) = \omega(\omega \geq 4)$ and the number of maximal cliques of G be $\alpha(\alpha \geq 1)$, then $d_{scl}'(G) = d_{scl}'(K_\omega)$.*

Proof. Let $K_\omega \in \{K_{\omega_1}, K_{\omega_2}, \cdots, K_{\omega_\alpha}\}$ be any maximal clique of G, then we can know $|V(K_{\omega_i})| = \omega, i = 1, 2, \cdots, \alpha$.

Let $E_1 = E(K_{\omega_1}) \bigcup E(K_{\omega_2}) \bigcup \cdots \bigcup E(K_{\omega_\alpha})$, $E_2 = E(G) \setminus E_1$. Let $G_1 = G[E_1]$ be an induced subgraph of edges E_1 in G.

We can prove this lemma from two aspects next.

Let $\{g_{1j}', g_{2j}', \cdots, g_{d_j j}'\} \in \Psi(K_{\omega_j}), j = 1, 2, 3, \cdots, \alpha$, then we can get $d_1 = d_2 = \cdots = d_j = d = d_{scl}'(K_\omega)$, since $K_{\omega_1}, K_{\omega_2}, \cdots, K_{\omega_\alpha}$ are maximal cliques of G. Now we consider two cases as following.

Case 1. Let $E(K_{\omega_i}) \bigcap E(K_{\omega_j}) = \emptyset$, if $i \neq j$ for any $i, j = 1, 2, \cdots, \alpha$.

Let $f_{1j}', f_{2j}', \cdots, f_{dj}', j = 1, 2, \cdots, \alpha$ be functions on G such that satisfying :

$$f_{ij}'(e) = \begin{cases} g_{ij}'(e) & if \quad e \in E(K_{\omega_j}), i = 1, 2, \cdots, d \\ 0 & if \quad e \in E(G) \setminus E(K\omega_j) \end{cases}$$

Let $F'_1, F'_2, \cdots, F'_d$ be functions on G such that

$$F'_i(e) = \begin{cases} \sum\limits_{j=1}^{\alpha} f'_{ij}(e) & if \quad e \in E_1, i = 1, 2, \cdots, d \\ -1 & if \quad e \in E(G) \setminus E_1 \end{cases}$$

It is not hard to verify that $\{F'_1, F'_2, \cdots, F'_d\} \in \Psi(G)$, then $d'_{scl}(G) \geq d = d'_{scl}(K_\omega)$.

Case 2. Let $E(K_{\omega_i}) \bigcap E(K_{\omega_j}) \neq \emptyset$, if $i \neq j$ for some $i, j = 1, 2, \cdots, \alpha$.

Let $E_1 = \{e_{1\beta_1}, e_{2\beta_2}, \cdots, e_{|E_1|\beta_{|E_1|}}\}$, where $e_{h\beta_h} \in E_1 (h = 1, 2, \cdots, |E_1|, \beta_h = 1, 2, \cdots, \alpha)$ is an edge that appears in β_h cliques of G. For some $k, j, k \neq j$, let $E(K_{\omega_j}) \bigcap E(K_{\omega_k}) = \{e_1, e_2, \cdots, e_s\}, s \geq 1$. Since $K_{\omega_1}, K_{\omega_2}, \cdots, K_{\omega_\alpha}$ are maximal cliques, there must exist functions g'_{ij}, g'_{ik} such that $g'_{ij}(e) = g'_{ik}(e)$ for every $e \in E(K_{\omega_j}) \bigcap E(K_{\omega_k})$.

Let $f'_{1j}, f'_{2j}, \cdots, f'_{dj}, j = 1, 2, \cdots, \alpha$ be functions on G such that satisfying :

$$f'_{ij}(e) = \begin{cases} g'_{ij}(e_{h\beta_h})/\beta_h & if \quad e = e_{h\beta_h} \in E(K_{\omega_j}), i = 1, 2, \cdots, d \\ 0 & if \quad e \in E(G) \setminus E(K_{\omega_j}) \end{cases}$$

Let $F'_1, F'_2, \cdots, F'_d$ be functions on G such that

$$F'_i(e) = \begin{cases} \sum\limits_{j=1}^{\alpha} f'_{ij}(e) & if \quad e \in E_1, i = 1, 2, \cdots, d \\ -1 & if \quad e \in E(G) \setminus E_1 \end{cases}$$

It is not hard to verify that $\{F'_1, F'_2, \cdots, F'_d\} \in \Psi(G)$, then $d'_{scl}(G) \geq d = d'_{scl}(K_\omega)$.

On the other hand, let $\{f'_1, f'_2, \cdots, f'_d\} \in \Psi(G)$ and $d = d'_{scl}(G)$. Therefore, there exist $f'_i(E(K_{\omega_j})) \geq 1$ for every $K_{\omega_j}, j = 1, 2, \cdots, \alpha$ and $\sum\limits_{i=1}^{d} f'_i(e) \leq 1$ for each $e \in E(G)$. Let $g'_k : E_1 \to \{-1, 1\}, k = 1, 2, \cdots, d$ be a function of G_1 and satisfy: $g'_i(e) = f'_i(e)$, if $e \in E_1, i = 1, 2, \cdots, d$.It is easy to verify that $\{g'_1, g'_2, \cdots, g'_d\}$ is a signed clique edge dominating family of K_ω, so $d'_{scl}(K_\omega) \geq d = d'_{scl}(G)$.

From two aspects above, we obtain that $d'_{scl}(K_\omega) = d'_{scl}(G)$.

Theorem 6. *For any graph G with size m, let $\omega(G) = \omega(\omega \geq 4)$ and the number of maximal cliques of G be $\alpha(\alpha \geq 1)$,then*

$$d'_{scl}(G) = \tfrac{\omega(\omega-1)}{2}, if \quad \omega \equiv 2 or 3 (mod 4) \tag{2.1}$$

$$d'_{scl}(G) = \begin{cases} \frac{\omega(\omega-1)}{4} - 1 & if \quad \omega \equiv 0(mod4), \frac{\omega}{4} \quad is \quad even \\ \frac{\omega(\omega-1)}{4} & if \quad \omega \equiv 0(mod4), \frac{\omega}{4} \quad is \quad odd \end{cases} \tag{2.2}$$

$$d'_{scl}(G) = \begin{cases} \frac{\omega(\omega+1)}{4} - 1 & if \quad \omega \equiv 1(mod4), \frac{\omega}{4} \quad is \quad even \\ \frac{\omega(\omega+1)}{4} & if \quad \omega \equiv 1(mod4), \frac{\omega}{4} \quad is \quad odd \end{cases} \tag{2.3}$$

Proof. By the lemma4, we only consider its maximal clique of G. Let $K_\omega \in \{K_{\omega_1}, K_{\omega_2}, \cdots, K_{\omega_\alpha}\}$ be any maximal clique of G and

$$E(K_\omega) = \{e_1, e_2, \cdots, e_{\frac{\omega(\omega-1)}{2}}\}$$

be an edge set of K_ω. Therefore, a function $f' \in F_{sced}(G)$ if only and if $f'(E(K_\omega)) \geq 1$.We consider three cases as following.

Case 1: $\omega \equiv 2 \quad or \quad 3(mod4)$. When $\omega \equiv 2 \quad or \quad 3(mod4)$,$|E(K_\omega)| = \frac{\omega(\omega-1)}{2}$ is an odd number. Let $\frac{\omega(\omega-1)}{2} = 2p+1$ and p be nonnegative integer.Define the signed clique edge dominating functions $f'_1, f'_2, \cdots, f'_{\frac{\omega(\omega-1)}{2}}$ by $f'_i(e_i) = f'_i(e_{i+1}) = \cdots = f'_i(e_{i+p}) = 1$ and $f'_i(e_j) = -1$ for every $e_j \in E(K_\omega) \setminus \{e_i, e_{i+1}, \cdots, e_{i+p}\}, i = 1, 2, \cdots, \frac{\omega(\omega-1)}{2}, j = 1, 2, \cdots, \frac{\omega(\omega-1)}{2}$, where all subscripts are taken modulo $\frac{\omega(\omega-1)}{2}$. It is easy to see that $f'_i(E(K_\omega)) = 1$ for $i = 1, 2, \cdots, \frac{\omega(\omega-1)}{2}$ and $\sum_{i=1}^{\frac{\omega(\omega-1)}{2}} f'_i(e) = 1$ for each $e \in E(K_\omega)$. Hence

$$\{f'_1, f'_2, \cdots, f'_{\frac{\omega(\omega-1)}{2}}\} \in \Psi(K_\omega)$$

and conclude that $d'_{scl}(K_\omega) \geq \frac{\omega(\omega-1)}{2}$. In view of theorem3, we obtain (2.1).

Case 2: $\omega \equiv 0(mod4)$. Let $\omega = 4k$, (k is a positive integer), then $|E(K_\omega)| = \frac{\omega(\omega-1)}{2} = 2k(4k-1)$ is an even number. We show that $d'_{scl}(G) \leq \frac{\omega(\omega-1)}{4}$ by theoreom3. We have two cases discuss the parity about $k(4k-1)$.

Case 2.1 $k(4k-1)$ is odd. Define the family of signed clique edge dominating functions $f'_1, f'_2, \cdots, f'_{k(4k-1)}$ by $f'_1(e_1) = f'_1(e_2) = \cdots = f'_1(e_{k(4k-1)+1}) = 1$ and $f'_1(e_j) = -1$ for each $e_j \in E(K_\omega) \setminus \{e_1, e_2, \cdots, e_{k(4k-1)+1}\}$,
$f'_2(e_3) = f'_2(e_4) = \cdots = f'_2(e_{k(4k-1)+3}) = 1$ and $f'_2(e_j) = -1$ for each $e_j \in E(K_\omega) \setminus \{e_3, e_4, \cdots, e_{k(4k-1)+3}\}$,

$\vdots \qquad \vdots \qquad \vdots$

$f'_{\frac{k(4k-1)-1}{2}}(e_{k(4k-1)-2}) = f'_{\frac{k(4k-1)-1}{2}}(e_{k(4k-1)-1}) = \cdots = f'_{\frac{k(4k-1)-1}{2}}(e_{2k(4k-1)-2})$

$= 1$ and $f'_{\frac{k(4k-1)-1}{2}}(e_j) = -1$ for each $e_j \in E(K_\omega) \setminus \{e_{k(4k-1)-2}, e_{k(4k-1)-1}, \cdots, e_{2k(4k-1)-2}\}$,

$\vdots \qquad \vdots \qquad \vdots$

$f'_{k(4k-1)}(e_{2k(4k-1)-1}) = f'_{k(4k-1)}(e_{2k(4k-1)}) = \cdots = f'_{k(4k-1)}(e_{3k(4k-1)-1}) = 1$ and $f'_{k(4k-1)}(e_j) = -1$ for each $e_j \in E(K_\omega) \setminus \{e_{2k(4k-1)-1}, e_{2k(4k-1)}, \cdots, e_{3k(4k-1)-1}\}$, where all subscripts are taken modulo $\frac{\omega(\omega-1)}{2}$.

It is not hard to obtain $f'_i(E(K_\omega)) = 2$ for $i = 1, 2, \cdots, k(4k-1)$ and $\sum_{i=1}^{k(4k-1)} f'_i(e) = 1$ for each $e \in E(K_\omega)$,Hence, $\{f'_1, f'_2, \cdots, f'_{k(4k-1)}\} \in \Psi(K_\omega)$ and conclude that $d'_{scl}(K_\omega) \geq k(4k-1) = \frac{\omega(\omega-1)}{4}$, so we obtain (2.2).

Case 2.2: $k(4k-1)$ is even. Define the family of signed clique edge dominating functions $f'_1, f'_2, \cdots,$
$f'_{k(4k-1)-1}$ by:

$f'_1(e_1) = f'_1(e_2) = \cdots = f'_1(e_{k(4k-1)+1}) = 1$ and $f'_1(e_j) = -1$ for each $e_j \in E(K_\omega) \setminus \{e_1, e_2, \cdots, e_{k(4k-1)+1}\}$,

$f'_2(e_4) = f'_2(e_5) = \cdots = f'_2(e_{k(4k-1)+4}) = 1$ and $f'_2(e_j) = -1$ for each $e_j \in E(K_\omega) \setminus \{e_4, e_5, \cdots, e_{k(4k-1)+4}\}$,

$f'_3(e_6) = f'_3(e_7) = \cdots = f'_3(e_{k(4k-1)+6}) = 1$ and $f'_3(e_j) = -1$ for each $e_j \in E(K_\omega) \setminus \{e_6, e_7, \cdots, e_{k(4k-1)+6}\}$,

$\vdots \qquad \vdots \qquad \vdots$

$f'_{\frac{k(4k-1)}{2}}(e_{k(4k-1)}) = f'_{\frac{k(4k-1)}{2}}(e_{k(4k-1)+1}) = \cdots = f'_{\frac{k(4k-1)}{2}}(e_{2k(4k-1)}) = 1$ and $f'_{\frac{k(4k-1)}{2}}(e_j) = -1$ for each $e_j \in E(K_\omega) \setminus \{e_{k(4k-1)}, e_{k(4k-1)+1}, \cdots, e_{2k(4k-1)}\}$, $f'_{\frac{k(4k-1)}{2}+1}(e_{k(4k-1)+3}) = f'_{\frac{k(4k-1)}{2}+1}(e_{k(4k-1)+4}) = \cdots$
$= f'_{\frac{k(4k-1)}{2}+1}(e_{2k(4k-1)+3}) = 1$ and $f'_{\frac{k(4k-1)}{2}+1}(e_j) = -1$ for each $e_j \in E(K_\omega) \setminus \{e_{k(4k-1)+3}, e_{k(4k-1)+4}, \cdots, e_{2k(4k-1)+3}\}$.

$\vdots \qquad \vdots \qquad \vdots$

$f'_{k(4k-1)-1}(e_{2k(4k-1)-1}) = f'_{k(4k-1)-1}(e_{2k(4k-1)}) = \cdots =$
$f'_{k(4k-1)-1}(e_{3k(4k-1)-1}) = 1$ and $f'_{k(4k-1)-1}(e_j) = -1$ for each $e_j \in E(K_\omega) \setminus \{e_{2k(4k-1)-1}, e_{2k(4k-1)}, \cdots, e_{3k(4k-1)-1}\}$,where all subscripts are taken modulo $\frac{\omega(\omega-1)}{2}$.Note that we have a jump of three in the arguments from f'_1 to f'_2 and from $f'_{\frac{k(4k-1)}{2}}$ to$f'_{\frac{k(4k-1)}{2}+1}$ and a jump of two in the subscripts.

It is a simple matter to obtain $f'_i(E(K_\omega)) = 2$ for $i = 1, 2, \cdots, k(4k-1) -$

1.Furthermore,we have verified that $\sum_{i=1}^{k(4k-1)-1} f'_i(e_{k(4k-1)+2}) = -1$ and

$$\sum_{i=1}^{k(4k-1)-1} f'_i(e) = 1$$

for each

$$e \in E(K_\omega) \setminus \{e_{k(4k-1)+2}\}.$$

Hence

$$\{f'_1, f'_2, \cdots, f'_{k(4k-1)-1}\} \in \Psi(K_\omega)$$

and conclude that $d'_{scl}(K_\omega) \geq k(4k-1)-1, sod'_{scl}(K_\omega) \geq \frac{\omega(\omega-1)}{4} - 1$, so we obtain (2.2) by the theoreom3.

Case 3: $\omega \equiv 1(mod4)$. We can prove this case is true in same way of proof of Case 2.

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