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EXISTENCE AND UNIQUENESS OF WEAK SOLUTION FOR A NONLOCAL PROBLEM INVOLVING THE P-LAPLACIAN

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Abstract: In the present paper, we deal the existence results of solutions for a nonlocal elliptic Dirichlet boundary value problem involving p-Laplacian. The existence and uniqueness results are obtained by Browder Theorem.

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Key Words: weak solutions, p-Laplacian, nonlocal problem

1. Introduction

Consider the boundary value problem

$$\begin{cases} -M\left(\int\limits_{\Omega} |\nabla u|^p \, dx\right) \triangle_p u = f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \in C^{0,1}$be a bounded domain in $\mathbb{R}^N$.Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be a caratheodory function which is decreasing with respect to the second variable, i.e.,

$$f(x,t_1) \leq f(x,t_2) \tag{1.2}$$

for a.a $x \in \Omega$ and $t_1, t_2 \in \mathbb{R}$, $t_1 \geq t_2$.

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Assume, moreover, that there exists $f_0 \in L^q(\Omega), q = \frac{p}{p-1}$ and $c > 0$ such that

$$|f(x,s)| \leq f_0(x) + c|s|^{p-1} \tag{1.3}$$

and $M : \mathbb{R}^+ \to \mathbb{R}^+$,where

$$M_0 \leq M \leq M_\infty \tag{1.4}$$

is a continuous and increasing function.

Since the equation (1.1) contains an integral over Ω , it is no longer a pointwise identity; therefore it is often called nonlocal problem.This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [13]. Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.5}$$

presented by Kirchhoff in 1883, see [11]. This equation is an extension of the classical d'Alembert's wave equation by considering the effect of the changing in the length of the string during the vibrations. The parameters in (1.5) have the following meanings: L is the length of the string, h is the area of the cross-section,E is the Young modulus of the material, is the mass density, and P_0 is the initialtension.In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [10, 11, 14, 15, 16, 21, 23, 24, 25],in which the authors have used variational method and topological method to get the existence of solutions for (1.1) in the cases when f could satisfy p-superlinear, p-sublinear or p-linear growth condition at in nity. In this paper, motivated by the ideas introduced in [12] and the properties of Kirchhoff type operators in [17, 18, 19] we study problem (1.1) in the semipositone case; i.e., $f(0) < 0$.In this papers using Browder Theorem we obtain the existence and uniqueness of solutions for (1.1).

We define the Sobolev space $X = W_0^{1,p}(\Omega)$ as the clusures of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}} \tag{1.6}$$

for all $u \in C_0^\infty(\Omega)$.

Definition1.1. We say that $u \in X$ is a weak solution to (1.1) if

$$M\left(\int_{\Omega} |\nabla u|^{p}\, dx\right)\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla v dx = \int_{\Omega} f\left(x,u\right) v dx \tag{1.7}$$

for all $u, v \in X$

Lemma 1.2. The space $X = W_0^{1,p}(\Omega)$ is continuous imbedded into the space $L^{p^*}(\Omega)$,where $p^* = \frac{Np}{N-p}$,i.e.

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \tag{1.8}$$

which means that

$$\|u\|_{L^{p^*}(\Omega)} \leq c_{emb}\, \|u\|_{W_0^{1,p}(h,\Omega)} \tag{1.9}$$

where c_{emb} is the constant of the embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$. and the compact embedded into the space $L^q(\Omega)$ where $q \in [1, \frac{Np}{N-p})$,

2. Preliminaries and Space Setting

Definition 2.1. Let $A : V \to V$ be an operator on a real Banach space V. We say that the operator A is:

(i) bounded iff it maps bounded sets into bounded i.e. for each $r > 0$ there exists $\alpha > 0$ (α depending on r) such that

$$\|u\| \leq r \Rightarrow \|A(u)\| \leq \alpha, \forall u \in V$$

(ii) coercive: iff

$$\lim_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} = \infty$$

(iii) monotone iff $\langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq 0$ for all $u_1, u_2 \in V$.

(iv) strictly monotone iff $\langle A(u_1) - A(u_2), u_1 - u_2\rangle > 0$ for all $u_1, u_2 \in V, u_1 \neq u_2$.

(v) strongly monotone iff $\langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq k\|u_1 - u_2\|$ for all $u_1, u_2 \in V, u_1 \neq u_2$.

(vi) continuous iff $(u_n) \to^{w} u$ implies $Au_n \to A(u)$ for all $u_n, u \in V$.

(viii) demicontinuous iff $(u_n) \to u$ implies $A(u_n) \to^w A(u)$ for all $u_n, u \in V$.

Theorem 2.2. (Browder [4]) Let A be a reflexive real Banach space. Moreover let $A : V \to V$ be an operator which is: bounded, demicontinuous, coercive, and monotone on the space V. Then, the system $A(u) = F$ has at least one solution $u \in V$ for each $F \in V'$: If moreover, A is strictly monotone operator, then the system (1.1) has precisely one solution $u \in V$ for every $F \in V'$.

Proof. We consider the Sobolev space $X = W_0^{1,p}(\Omega)$ with the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}$$

We define operators $J, : X \to X^*$ by

$$\langle J(u), v \rangle = M\left(\int_\Omega |\nabla u|^p \, dx \right) \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx$$

and $F : \Omega \times \mathbb{R} \to \mathbb{R}$,by

$$\langle F(u), v \rangle = \int_\Omega f(x, u) \, v dx$$

for all $u, v \in W_0^{1,p}(\Omega)$.

We say that u is a weak solution of (1.1) if

$$\langle A(u), v \rangle = \langle J(u), v \rangle - \langle F(u), v \rangle = 0$$

holds for any $v \in W_0^{1,p}(\Omega)$.Thus, to find a weak solution of (1.1) is equivalent to finding $u \in W_0^{1,p}(\Omega)$ which satisfies the operator equation $A(u) = 0$.

Now, we have the following properties of the operators J and F:

a) J and F are well defined. Using Holder's inequality, we have

$$\begin{aligned} |\langle J(u), v \rangle| &= \left| M\left(\int_\Omega |\nabla u|^p \, dx \right) \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx \right| \\ &\leq M_\infty \left(\int_\Omega |\nabla u|^{p-1} |\nabla v| \, dx \right) \end{aligned}$$

$$\leq \quad M_\infty \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{q}} \left(\int_\Omega |\nabla v|^p \, dx \right)^{\frac{1}{p}} < \infty$$

$$\begin{aligned} |\langle F(u), v \rangle| &= \left| \int_\Omega f(x, u) \, v dx \right| \\ &\leq \int_\Omega \left(f_0(x) + c |u|^{p-1} \right) |v| \, dx \\ &\leq \left(\int_\Omega |f_0(x)|^q \, dx \right)^{\frac{1}{q}} \left(\int_\Omega |v|^p \, dx \right)^{\frac{1}{p}} \\ &\quad + c \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{q}} \left(\int_\Omega |v|^p \, dx \right)^{\frac{1}{p}} \\ &< \infty \end{aligned}$$

and hence J and F are well defined.

b) J, and F are bounded operators. Indeed, for every u such that

$$\|u\|_{W_0^{1,p}} \leq \alpha$$

we have

$$\begin{aligned} \|J(u)\|_{X^*} &= \sup_{\|v\|_{X^* \leq 1}} |\langle J(u), v \rangle| \\ &\leq \sup_{\|v\|_{X^* \leq 1}} M \left(\int_\Omega |\nabla u|^p \, dx \right) \left| \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx \right| \\ &\leq M_\infty \sup_{\|v\|_{X^* \leq 1}} \left[\int_\Omega |\nabla u|^{p-1} |\nabla v| \, dx \right] \end{aligned}$$

Using Holder's inequality, we obtain

$$\|J(u)\|_{X^*} \leq M_\infty \sup_{\|v\|_{X^* \leq 1}} \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{q}} \left(\int_\Omega |\nabla v|^p \, dx \right)^{\frac{1}{p}} \leq \alpha^{\frac{p}{q}} M_\infty$$

Also, we get

$$\begin{aligned}
\|F(u)\|_{X^*} &= \sup_{\|v\|_{X^*\le 1}} |\langle f(x,u), v\rangle| \\
&\le \sup_{\|v\|_{X^*\le 1}} \int_\Omega \left(f_0(x) + c|u|^{p-1}\right)|v|\,dx \\
&\le \sup_{\|v\|_{X^*\le 1}} \left[\left(\int_\Omega |f_0(x)|^q\,dx\right)^{\frac{1}{q}} + \left(\int_\Omega |u|^{(p-1)q}\,dx\right)^{\frac{1}{q}}\right] \\
&\quad \times \left(\int_\Omega |v|^p\,dx\right)^{\frac{1}{p}} \\
&\le c_{emb}\left(\|f_0\|_{L^q(\Omega)} + c_{emb}\|u\|_X^{\frac{p}{q}}\right) \\
&\le c_{emb}\left(\|f_0\|_{L^q(\Omega)} + c_{emb}\alpha^{\frac{p}{q}}\right)
\end{aligned}$$

c) J and F are continuous operators. If $u_n \to u$ in X: Then, we have $\|u_n - u\|_X \to 0$,so that

$$\|u_n - u\|_{L^p(\Omega)} \to 0.$$

Applying Dominated Convergence Theorem, we obtain

$$\left\|\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\right\|_{L^p(\Omega)} \to 0$$

Hence

$$\begin{aligned}
\|J(u_n) - J(u)\|_{X^*} &= \sup_{\|v\|_{X^*\le 1}} |J(u_n) - J(u), v\rangle| \\
&\le M_\infty \sup_{\|v\|_{X^*\le 1}} \left(\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)^q dx\right)^{\frac{1}{q}} \\
&\quad \times \left(\int_\Omega |v|^p\,dx\right)^{\frac{1}{p}} \\
&\le M_\infty c_{emb}\left(\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)^q dx\right)^{\frac{1}{q}}
\end{aligned}$$

$$\rightarrow \quad 0 \text{ for } n \rightarrow \infty$$

Also, we get

$$\begin{aligned} \|F(u_n) - F(u)\|_{X^*} &= \sup_{\|v\|_{X^*} \leq 1} |F(u_n) - F(u), v\rangle| \\ &\leq c_{emb} \left(\int_\Omega |f(x, u_n) - f(x, u)|^q \, dx \right)^{\frac{1}{q}} \rightarrow 0 \text{ for } n \rightarrow \infty \end{aligned}$$

d) Let $p \geq 2, \forall x_1, x_2 \in \mathbb{R}^N$ we have the following inequality (see [6])

$$|x_2|^p \geq |x_1|^p + p\,|x_1|^{p-2}\, x_1 (x_2 - x_1) + \frac{|x_2 - x_1|^p}{2^{p-1} - 1}. \tag{2.1}$$

Now,

$$\begin{aligned} \langle J(u) - J(v), u - v\rangle &= M\left(\int_\Omega |\nabla u|^p \, dx \right) \\ &\quad \times \int_\Omega \left[|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right] (\nabla u - \nabla v) \, dx \\ &= M\left(\int_\Omega |\nabla u|^p \, dx \right) \int_\Omega |\nabla u|^{p-2} \nabla u (\nabla u - \nabla v) \, dx \\ &\quad - M\left(\int_\Omega |\nabla u|^p \, dx \right) \int_\Omega |\nabla v|^{p-2} \nabla v (\nabla u - \nabla v) \, dx \\ &= I_1 + I_2. \end{aligned}$$

$$I_1 + I_2 \geq M_0 \int_\Omega |\nabla u - \nabla v|^p = C \|u - v\|_X^p$$

So

$$\langle J(u) - J(v), u - v\rangle \geq C \|u - v\|_X^p \tag{2.2}$$

Also, we get

$$\langle F(u) - F(v), u - v\rangle = \int_\Omega [f(x, u) - f(x, v)]\,(u - v)\, dx$$

Since f is decreasing with respect to the second variable, we have

$$[f(x,u) - f(x,v)](u-v) \leq 0$$

consequently

$$\langle F(u) - F(v), u - v\rangle = \int_{\Omega} [f(x,u) - f(x,v)](u-v)\,dx \leq 0 \tag{2.3}$$

Equations (2.2) and (2.3) imply that

$$\langle A(u) - A(v), u - v\rangle \geq C\,\|u - v\|_X^p \tag{2.4}$$

So A is strongly monotone.

Now, to apply Browder Theorem, it remains to prove that A is a coercive operator.

From (2.4), we have

$$\langle A(u), u\rangle \geq \langle A(0), u\rangle + C\,\|u\|_X^p$$

On the other hand

$$\begin{aligned}
\langle A(0), u\rangle &= \langle J(0), u\rangle - \langle F(0), u\rangle \\
&= -\int_{\Omega} [f(x,0)\,u]\,dx \geq -\int_{\Omega} f_0 u dx \\
&\geq -\left[\int_{\Omega} (f_0(x))^q\right]^{\frac{1}{q}} \left(\int_{\Omega} |u|^p\,dx\right)^{\frac{1}{p}} \\
&\geq -c_{emb}\,\|f_0\|_{L^q(\Omega)}\,\|u\|_X .
\end{aligned}$$

then

$$\langle A(u), u\rangle \geq C\,\|u\|_X^p - c_{emb}\,\|f_0\|_{L^q(\Omega)}\,\|u\|_X$$

So,

$$\lim_{\|u\|_X \to \infty} \frac{\langle A(u), u\rangle}{\|u\|_X} = \infty.$$

This proves the coercivity condition and so, the existence of weak solution for (1.1).

The uniqueness of weak solution of (1.1), is a direct consequence of (2.4). Suppose that u, v be a weak solutions of (1.1) such that $u \neq v$.

Now, from (2.4), we have

$$0 = \langle A(u) - A(v), u - v\rangle \geq C\,\|u - v\|_X^2 \geq 0$$

therefore $u = v$. This completes the proof. □

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