ϕ-2-ABSORBING IDEALS

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Abstract: Let $R$ be a commutative ring with identity. 2-absorbing ideals have been studied by A. Badawi. A proper ideal $I$ of $R$ is 2-absorbing if $a, b, c \in R$ with $abc \in I$ implies $ab \in I$ or $ac \in I$ or $bc \in I$. Let $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ be a function where $I(R)$ is the set of ideals of $R$. We call a proper ideal $I$ of $R$ a $\varphi$-2-absorbing ideal if $a, b, c \in R$ with $abc \in I - \varphi(I)$ implies $ab \in I$ or $ac \in I$ or $bc \in I$. So taking $\varphi_\emptyset(J) = \emptyset$ (resp., $\varphi_0(J) = 0, \varphi_2(J) = J^2$), a $\varphi_\emptyset$-2-absorbing ideal (resp., $\varphi_0$-2-absorbing ideal, $\varphi_2$-2-absorbing ideal) is a 2-absorbing ideal (resp., weakly 2-absorbing ideal, almost 2-absorbing ideal). We show that $\varphi$-2-absorbing ideals enjoy analogs of many of the properties of 2-absorbing ideals.

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Key Words: almost 2-absorbing ideal, 2-absorbing ideal, $\varphi$-2-absorbing ideal, weakly 2-absorbing ideal

Throughout, $R$ will be a commutative ring with identity. We denote the set of ideals of $R$ by $I(R)$. By a proper ideal $I$ of $R$ we mean an ideal $I \in I(R)$ with $I \neq R$. We denote the set of proper ideals of $R$ by $I^*(R)$.  

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Badawi (2007) recently defined a proper ideal \( I \) of \( R \) to be 2-absorbing if for \( a, b, c \in R \) with \( abc \in I \) implies \( ab \in I \) or \( ac \in I \) or \( bc \in I \). With a 2-absorbing ideal, we make the following definitions. Let \( R \) be a commutative ring and \( \varphi : I(R) \to I(R) \cup \{ \emptyset \} \) be a function. We call a proper ideal \( I \) of \( R \) \( \varphi \)-2-absorbing if for \( a, b, c \in R, abc \in I - \varphi(I) \) implies \( ab \in I \) or \( ac \in I \) or \( bc \in I \). There is no loss of generality in assuming that \( \varphi(I) \subseteq I \), because \( I - \varphi(I) = I - (I \cap \varphi(I)) \). We henceforth make this assumption. Given two functions \( \psi_1, \psi_2 : I(R) \to I(R) \cup \{ \emptyset \} \), we define \( \psi_1 \leq \psi_2 \) if \( \psi_1(J) \subseteq \psi_2(J) \) for each \( J \in I(R) \).

**Definition 1.** Let \( I(R) \) be the set of ideals of \( R \), \( I^*(R) \) the set of proper ideals of \( R \) and \( \varphi : I(R) \to I(R) \cup \{ \emptyset \} \) be a map. A non-zero proper ideal \( I \) of \( R \) is called a \( \varphi \)-prime ideal if for all \( a, b \in R, ab \in I - \varphi(I) \) implies \( a \in I \) or \( b \in I \).

**Definition 2.** (1) A non-zero proper ideal \( I \) of \( R \) is called a \( \varphi \)-prime ideal if for all \( a, b \in R, ab \in I - \varphi(I) \) implies \( a \in I \) or \( b \in I \).

**Lemma 3.** (i) Every 2-absorbing ideal is \( \varphi \)-2-absorbing ideal.

(ii) Every \( \varphi \)-prime ideal is a \( \varphi \)-2-absorbing ideal.

**Definition 4.** Given two functions \( \psi_1, \psi_2 : I(R) \to I(R) \cup \{ \emptyset \} \), we define \( \psi_1 \leq \psi_2 \) if \( \psi_1(J) \subseteq \psi_2(J) \) for each \( J \in I(R) \).

We maintain notation and terminology used in following example for the remainder of the article.

**Definition 5.** A non-zero proper ideal \( I \) of \( R \) is called a weakly 2-absorbing ideal if for all \( a, b \in R, ab \in I - \{ 0 \} \) implies \( a \in I \) or \( b \in I \).

**Definition 6.** A non-zero proper ideal \( I \) of \( R \) is called an almost 2-absorbing ideal if for all \( a, b \in R, ab \in I - \{ I^2 \} \) implies \( a \in I \) or \( b \in I \).

**Example.** Let \( R \) be a commutative ring. Define the following functions \( \varphi_\alpha : I(R) \to I(R) \cup \{ \emptyset \} \) and the corresponding \( \varphi_\alpha \)-2-absorbing ideals:

<table>
<thead>
<tr>
<th>( \varphi_\alpha )</th>
<th>( \varphi(J) )</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2-absorbing ideal</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>weakly 2-absorbing ideal</td>
</tr>
<tr>
<td>2</td>
<td>( J^2 )</td>
<td>almost 2-absorbing ideal</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>( J^\alpha )</td>
<td>( \alpha )-almost 2-absorbing ideal</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( \cap J^n )</td>
<td>( \omega )-2-absorbing ideal</td>
</tr>
<tr>
<td>1</td>
<td>( J )</td>
<td>any ideal</td>
</tr>
</tbody>
</table>
Observe that $\varphi_0 \leq \varphi_0 \leq \varphi \leq \cdots \leq \varphi_{n+1} \leq \varphi_n \leq \cdots \leq \varphi_2 \leq \varphi_1$. Let $A$ be an ideal of $R$. Define the function $\varphi_A$ by $\varphi_A(J) = AJ$. So if $(R, m)$ is quasilocal, $\varphi_2 \leq \varphi_m \leq \varphi_1$.

**Example.** Let $(R, m)$ be a quasilocal ring and let $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ be a function with $\text{Im}\varphi \subseteq I(R)$. If $m^3 = 0$, then every proper ideal of $R$ is weakly 2-absorbing and hence $\varphi$-2-absorbing. More generally, if $I$ is a proper ideal of $R$ with $I \cap m^3 \subseteq \varphi(I)$, then $I$ is $\varphi$-2-absorbing. For if $xyz \in I - \varphi(I)$, then $xyz \notin m^3$. So $x$ or $y$ or $z$ is a unit and hence $xy \in I$ or $xz \in I$ or $yz \in I$. As an example, let $I = (x)$ in $R = k[[x, y]]/(x)(x, y), k$ a field. Then $I \cap m^3 = 0$ for $m = (x, y)$, so $I$ is weakly 2-absorbing.

**Proposition 7.** (1) Let $R$ be a commutative ring and $J$ a proper ideal of $R$. Let $\psi_1, \psi_2 : I(R) \to I(R) \cup \{\emptyset\}$ be functions with $\psi_1 \leq \psi_2$. Then, if $J$ is a $\psi_1$-2-absorbing ideal, then $J$ is a $\psi_2$-2-absorbing ideal.

(2) $J$ 2-absorbing $\implies J$ weakly 2-absorbing $\implies J$ $\omega$-2-absorbing $\implies J$ $(n+1)$-almost 2-absorbing $\implies J$ $n$-almost 2-absorbing ($n \geq 2$) $\implies J$ almost 2-absorbing

(3) $J$ is $\omega$-2-absorbing if only if $J$ is $n$-almost 2-absorbing for all $n \geq 2$.

**Proof.** (1) Suppose that $abc \in J - \psi_2(J)$, where, $a, b, c \in R$. Since $\psi_1(J) \subseteq \psi_2(J)$, we have $abc \in J - \psi_1(J)$ and therefore $ab \in J$ or $ac \in J$ or $bc \in J$. Thus $J$ is $\psi_2$-2-absorbing. (2) This follows from (1) and the ordering of the $\phi_{\alpha}$'s given in Example. (3) Clear by (2).

**Theorem 8.** Let $R$ be a commutative ring with identity and $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ be a function, $J \in I^*(R)$ such that $J$ is $\varphi$-2-absorbing that is not 2-absorbing. Then $J^3 \subseteq \varphi(J)$.

**Proof.** Suppose that $J^3 \not\subseteq \varphi(J)$, We show that $J$ is 2-absorbing. Let $a, b, c \in R$ with $abc \in J$. If $abc \notin \varphi(J)$, then $J, \varphi$-2-absorbing gives $ab \in J$ or $ac \in J$ or $bc \in J$. So assume that $abc \in \varphi(J)$. First, suppose that $abJ \not\subseteq \varphi(J)$; say $abj_0 \notin \varphi(J)$ where $j_0 \in J$. Then $ab(c + j_0) \in J - \varphi(J)$. So $ab \in J$ or $a(c + j_0) \in J$ or $b(c + j_0) \in J$; and hence $ab \in J$ or $ac \in J$ or $bc \in J$. So we can assume that $abJ \subseteq \varphi(J)$. Likewise, we can assume that $acJ \subseteq \varphi(J)$, $bcJ \subseteq \varphi(J)$. Since $J^3 \not\subseteq \varphi(J)$, there exist $j_1, j_2, j_3 \in J$ with $j_1j_2j_3 \notin \varphi(J)$. Then $(a + j_1)(b + j_2)(c + j_3) \in J - \varphi(J)$. So $J, \varphi$-2-absorbing gives $(a + j_1)(b + j_2) \in J$ or $(a + j_1)(c + j_3) \in J$ or $(b + j_2)(c + j_3) \in J$; hence $ab \in J$ or $ac \in J$ or $bc \in J$. So $J$ is 2-absorbing.
Corollary 9. Let $J$ be $\varphi$- 2-absorbing and $J^3 \not\subseteq \varphi(J)$. Then $J$ is a 2-absorbing ideal.

Corollary 10. Let $J$ be a $\varphi$- 2-absorbing ideal, where $\varphi \leq \varphi_4$. Then $J$ is $\omega$- 2-absorbing.

Proof. If $J$ is 2-absorbing, then $J$ is $\varphi$- 2-absorbing for each $\varphi$. Thus $J$ is $\omega$- 2-absorbing. Suppose that $J$ is not 2-absorbing. By Theorem 5, $J^3 \subseteq \varphi(J) \subseteq J^4$. Hence $\varphi(J) = J^n$ for each $n \geq 3$, so $J$ is $n$-almost 2-absorbing for each $n \geq 3$ and thus $\omega$- 2-absorbing.

In following we give a way to construct $\varphi$-2-absorbing ideals $J$ where, $\varphi_\omega \leq \varphi$.  

Remark. If $I$ is a weakly 2-absorbing ideal of $T$, then $J = I \times S$ need not be a weakly 2-absorbing ideal of $R = T \times S$, where $T$ and $S$ are commutative rings. Indeed, $J$ is weakly 2-absorbing if and only if $J$ is actually 2-absorbing. However, $J$ is $\varphi$- 2-absorbing for each $\varphi$ with $\varphi_\omega \leq \varphi$. If $I$ is actually 2-absorbing, then $J$ is 2-absorbing and hence is $\varphi$- 2-absorbing for all $\varphi$.

Theorem 11. Suppose that $T$ and $S$ are commutative rings with identity, $I$ an ideal of $T$ and $J$ an ideal of $S$. Then:

(1) $I$ is a 2-absorbing ideal of $T$ if and only if $I \times S$ is a 2-absorbing ideal of $T \times S$.

(2) $J$ is a 2-absorbing ideal of $S$ if and only if $T \times J$ is a 2-absorbing ideal of $T \times S$.

Proof. Because the proof of (1) and (2) are similar, we only prove (1). Hence suppose that $I$ is a 2-absorbing ideal of $T$ and $(a, b)(c, d)(e, f) \in I \times S$. Thus $ace \in I$, and hence $ac \in I$ or $ae \in I$ or $ce \in I$. Therefore $(a, b)(c, d) \in I \times S$ or $(a, b)(e, f) \in I \times S$ or $(c, d)(e, f) \in I \times S$. Conversely, suppose that $abc \in I$. Hence $(abc, 1) \in I \times S$ and hence $(a, 1)(b, 1)(c, 1) \in I \times S$. Since $I \times S$ is 2-absorbing $(a, 1)(b, 1) \in I \times S$ or $(a, 1)(c, 1) \in I \times S$ or $(b, 1)(c, 1) \in I \times S$, and hence $ab \in I$ or $ac \in I$ or $bc \in I$. Hence $I$ is a 2-absorbing ideal.

Theorem 12. Let $R$ and $S$ be commutative rings and let $\psi_1 : I(R) \to I(R) \cup \{\emptyset\}, \psi_2 : I(S) \to I(S) \cup \{\emptyset\}$ be functions. Let $\varphi = \psi_1 \times \psi_2$. Then the following are $\varphi$- 2-absorbing ideals of $R \times S$. Also if $I \times J$ is $\varphi$-2-absorbing, then $I$ is $\psi_1$- 2-absorbing and $J$ is $\psi_2$-2-absorbing.
(1) $I \times J$, where $I$ is a proper ideal of $R$ and $J$ is a proper ideal of $S$ with $\psi_1(I) = I$ and $\psi_2(J) = J$.

(2) $I \times S$, where $I$ is $\psi_1$-2-absorbing in $R$ which must be 2-absorbing if $\psi_2(S) \neq S$.

(3) $R \times J$, where $J$ is $\psi_2$-2-absorbing in $S$ which must be 2-absorbing if $\psi_1(R) \neq R$.

Proof. Case (1) is clear since $I \times J - \varphi(I \times J) = I \times J - \psi_1(I) \times \psi_2(J) = I \times J - I \times J = \emptyset$. If $I$ is 2-absorbing, certainly, $I \times S$ is 2-absorbing and hence $\varphi$-2-absorbing. So suppose that $I$ is $\psi_1$-2-absorbing and $\psi_2(S) = S$. Suppose that $(a_1,a_2,a_3,b_1,b_2,b_3) = (a_1,b_1)(a_2,b_2)(a_3,b_3) \in I \times S - \psi_1(I) \times \psi_2(S) = I \times S - \psi_1(I) \times S = (I - \psi_1(I)) \times S$. Then $a_1a_2a_3 \in I - \psi_1(I)$ if $a_1a_2 \in I$ or $a_1a_3 \in I$ or $a_2a_3 \in I$, so $(a_1a_2,b_1b_2) \in I \times S$ or $(a_1a_3,b_1b_3) \in I \times S$ or $(a_2a_3,b_2b_3) \in I \times S$. So $I \times S$ is $\varphi$-2-absorbing. The proof for case (3) is similar. Next suppose that $I \times J$ is $\varphi$-2-absorbing. Let $abc \in I - \psi_1(I)$. Then $(a,0)(b,0)(c,0) = (abc,0) \in I \times J - \varphi(I \times J)$, so $(ab,0) \in I \times J$ or $(ac,0) \in I \times J$ or $(bc,0) \in I \times J$, i.e., $ab \in I$ or $ac \in I$ or $bc \in I$. So $I$ is $\psi_1$-2-absorbing. Likewise, $J$ is $\psi_2$-2-absorbing.

Theorem 13. Suppose that $R$ and $S$ are commutative rings, and $I$ is an ideal in $R$. Then $I \times S$ is a weakly 2-absorbing ideal of $R \times S$ if and only if $I$ is a 2-absorbing ideal in $R$.

Proof. Suppose that $I$ is a 2-absorbing ideal of $R$, then $I \times S$ is a 2-absorbing ideal in $R \times S$, hence $I \times S$ is a weakly 2-absorbing ideal of $R \times S$. Conversely, suppose that $I \times S$ is a weakly 2-absorbing ideal in $R \times S$ and $abc \in I$ for some $a,b,c \in R$, then $(abc,1) \in I \times S$, hence $(abc,1) \in I \times S - 0$. Since $I \times S$ is weakly 2-absorbing and we have $(a,1)(b,1)(c,1) = (abc,1) \in I \times S - 0$, hence $ab \in I$ or $ac \in I$ or $bc \in I$.

Note. If $I$ is a weakly 2-absorbing ideal of $T$, then $I \times S$ in $R \times S$ is $\varphi$-2-absorbing for each $\varphi$ with $\varphi_w \leq \varphi$. If $I$ is actually 2-absorbing, then $I \times S$ is 2-absorbing and hence is $\varphi$-2-absorbing for all $\varphi$. Suppose that $I$ is not 2-absorbing. Then $I^3 = 0$. So $(I \times S)^3 = I^3 \times S = 0 \times S$ and hence $\varphi_w(I \times S) = 0 \times S$. Hence $I \times S - \varphi_w(I \times S) = I \times S - 0 \times S = (I - \{0\}) \times S$. Then $(x_1,x_2)(y_1,y_2)(z_1,z_2) = (x_1y_1z_1,x_2y_2z_2) \in I \times S - \varphi_w(I \times S)$ and hence $x_1y_1z_1 \in I - 0 \Rightarrow x_1y_1 \in I$ or $x_1z_1 \in I$ or $y_1z_1 \in I \Rightarrow (x_1,x_2)(y_1,y_2) \in I \times S$ or $(x_1,x_2)(z_1,z_2) \in I \times S$ or $(y_1,y_2)(z_1,z_2) \in I \times S$. So $I \times S$ is $\varphi_w$-2-absorbing and hence $\varphi$-2-absorbing.
Theorem 14. (1) Let $T$ and $S$ be commutative rings and let $I$ be a weakly 2-absorbing ideal of $T$. Then $J = I \times S$ is a $\varphi$-2-absorbing ideal of $R = T \times S$ for each $\varphi$ with $\varphi_w \leq \varphi \leq \varphi_1$.

(2) Let $R$ be a commutative ring and let $J$ be a finitely generated proper ideal of $R$. Suppose that $J$ is $\varphi$-2-absorbing where $\varphi \leq \varphi_4$. Then either $J$ is weakly 2-absorbing or $J^3 \neq 0$ is idempotent and $R$ decomposed as $T \times S$ where $S = J^3$ and $J = I \times S$ where $I$ is weakly 2-absorbing. Hence $J$ is $\varphi$-2-absorbing for each $\varphi$ with $\varphi_w \leq \varphi \leq \varphi_1$.

Proof. (1) is proved in the previous paragraph. (2) If $J$ is 2-absorbing, then $J$ is weakly 2-absorbing. So we can assume that $J$ is not 2-absorbing. Then $J^3 \subseteq \varphi(J)$; and hence $J^3 \subseteq \varphi(J) \subseteq \varphi_4(J) = J^4$. So $J^3 = J^4$. Hence $J^3$ is idempotent. Since $J^3$ is finitely generated, $J^3 = (e)$ for some idempotent $e \in R$. Suppose $J^3 = 0$. Then $\varphi(J) \subseteq J^4 = 0$. So $\varphi(J) = 0$ and hence $J$ is weakly 2-absorbing. So assume $J^3 \neq 0$. Put $S = J^3 = Re$ and $T = R(1 - e)$; So $R$ decomposes as $T \times S$ where $S = J^3$. Let $I = J(1 - e)$, so $J = I \times S$ where $I^3 = (J(1 - e))^3 = J^3(1 - e)^3 = (e)(1 - e) = 0$. We show that $I$ is weakly 2-absorbing. Let $xyz \in I^3 - \{0\}$; so $(x, 1)(y, 1)(z, 1) = (xyz, 1) \in I \times S - (I \times S)^3 = I \times S - 0 \times S \subseteq J - \varphi(J)$ since $\varphi \leq \varphi_4$ implies $\varphi(J) \subseteq J^4 = (I \times S)^4 = 0 \times S$. Hence $(xy, 1) \in J$ or $(xz, 1) \in J$ or $(yz, 1) \in J$ so $xy \in I$ or $xz \in I$ or $yz \in I$. Hence $I$ is weakly 2-absorbing.

Corollary 15. Let $R$ be an indecomposable commutative ring and $J$ a finitely generated $\varphi$-2-absorbing ideal of $R$, where $\varphi \leq \varphi_4$. Then $J$ is weakly 2-absorbing. If further $R$ is an integral, $J$ is actually 2-absorbing.

Corollary 16. Let $R$ be a Noetherian integral domain. A proper ideal $J$ of $R$ is 2-absorbing if and only if $xyz \in J - J^4$ implies $xy \in J$ or $xz \in J$ or $yz \in J$.

Definition 17. A non-zero nonunit element $a$ in a commutative ring $R$ is irreducible if $a = bc$ implies $b \in (a)$ or $c \in (a)$.

Theorem 18. Let $R$ be a commutative ring and let $a \in R$ be a nonunit.

(1) Suppose that $(0 : a) \subseteq (a)$. Then $(a)$ is $\varphi$- 2-absorbing for some $\varphi$ with $\varphi \leq \varphi_2$ if and only if $(a)$ is 2-absorbing.

(2) Suppose that $(R, M)$ is quasi-local.

$(C_1)$ The ideal $(a)$ is $\varphi$- 2-absorbing for some $\varphi$ with $\varphi \leq \varphi_4$ if and only if $(a)$ is weakly 2-absorbing.
(C2) The ideal \( a \) is \( \varphi_{M} \)-2-absorbing if and only if \( a \) is irreducible.

Proof. (1) \((\Leftarrow)\) A 2-absorbing ideal is \( \varphi \)-2-absorbing for every \( \varphi \).
\((\Rightarrow)\) we may assume that \( (a) \) is \( \varphi_{2} \)-2-absorbing. Let \( xyz \in (a) \). If \( xyz \notin (a)^{2} \), then \( xyz \in (a) \) or \( xz \in (a) \) or \( yz \in (a) \). So suppose that \( xyz \in (a)^{2} \).
Now \( (x + a)yz \in (a) \). If \( (x + a)yz \notin (a)^{2} \), then \( (x + a)y \in (a) \) or \( (x + a)z \in (a) \) or \( yz \in (a) \) and hence \( xy \in (a) \) or \( xz \in (a) \) or \( xz \in (a) \). So assume that \( (x + a)yz \in (a)^{2} \). Then \( xyz \in (a)^{2} \) gives \( yz \in (a)^{2} \) and hence \( yz \in (a) \) or \( (0 : a) \subseteq (a) \).

(2)(C1) If \( (a) \) is weakly 2-absorbing, then \( (a) \) is \( \varphi \)-2-absorbing for each \( \varphi \) with \( \varphi_{0} \leq \varphi \leq \varphi_{1} \). Conversely, suppose that \( (a) \) is \( \varphi \)-2-absorbing for some \( \varphi \) with \( \varphi \leq \varphi_{4} \). Since a quasilocal ring has no nontrivial idempotents, \( (a) \) is weakly 2-absorbing by Theorem 11(2).

(C2) Note that \( a \) is irreducible if and only if \( a = bcd \).

Implies one of the following types:
1) \( b \in (a) \) or \( cd \in (a) \)
2) \( c \in (a) \) or \( bd \in (a) \)
3) \( d \in (a) \) or \( bc \in (a) \)

But \( (a) \) is \( \varphi_{M} \)-2-absorbing if and only if \( bcd \in (a) - M(a) \) implies \( bc \in (a) \) or \( bd \in (a) \) or \( cd \in (a) \). But \( bcd \in (a) - M(a) \) if and only if \( bcd = ua \) for some unit \( u \in R \). Thus \( (a) \) \( \varphi_{M} \)-2-absorbing if and only if \( a = bcd \) implies \( bc \in (a) \) or \( bd \in (a) \) or \( cd \in (a) \).

Theorem 19. Let \( I \) be a proper ideal of a commutative ring \( R \) and let \( \varphi : I(R) \to I(R) \cup \{ \emptyset \} \) be a function. Then the following are equivalent:

1) \( I \) is \( \varphi \)-2-absorbing;
2) For \( x, y \in R \) such that \( xy \in R - I, (I : xy) = (I : x) \cup (I : y) \cup (\varphi(I) : xy) \);

3) For \( x, y \in R \) such that \( xy \in R - I, (I : xy) = (I : x) \) or \( (I : xy) = (I : y) \) or \( (I : xy) = (\varphi(I) : xy) \);

4) For ideals \( A, B, C \) of \( R \), \( ABC \subseteq I, ABC \not\subseteq \varphi(I) \) implies \( AB \subseteq I \) or \( AC \subseteq I \) or \( BC \subseteq I \).

Proof. (1 \( \Rightarrow \) 2) let \( xy \in R - I \). Let \( z \in (I : xy) \); so \( xyz \in I \). If \( xyz \notin \varphi(I) \) then \( yz \in I \) or \( xz \in I \); so \( z \in (I : y) \) or \( z \in (I : x) \). If \( xyz \in \varphi(I) \) then \( z \in (\varphi(I) : xy) \). So \( (I : xy) \subseteq (I : x) \cup (I : y) \cup (\varphi(I) : xy) \), the other containment always holds(remember we are assuming \( \varphi(I) \subseteq I \)).

(2 \( \Rightarrow \) 3) If an ideal is a union of two ideals, it is equal to one of them.
(3 → 4) Let $A, B, C$ be ideals of $R$ with $ABC \subseteq I$. Suppose that $AB \nsubseteq I$ and $AC \nsubseteq I$ and $BC \nsubseteq I$, we show that $ABC \subseteq \varphi(I)$. Let $ab \in AB$. First, suppose that $ab \notin I$. Then $abC \subseteq I$ gives $C \subseteq (I : ab)$. Now $AC \nsubseteq I$, $BC \nsubseteq I$; so $(I : ab) = (\varphi(I) : ab)$. Hence $abC \subseteq \varphi(I)$. Next, let $ab \in I \cap AB$. Choose $a'b' \in AB - I$. Then $(ab + a'b')C \subseteq \varphi(I)$. Let $c \in C$ then $abc = (ab + a'b')c - a'b'c \in \varphi(I)$. Thus $ABC \subseteq \varphi(I)$.

(4 → 1) Let $abc \in I - \varphi(I)$. Then $(a)(b)(c) \subseteq I$, $(a)(b)(c) \nsubseteq \varphi(I)$. So $(a)(b) \subseteq I$ or $(b)(c) \subseteq I$ or $(a)(c) \subseteq I$; i.e., $ab \in I$ or $bc \in I$ or $ac \in I$.

**Lemma 20.** Suppose that $I$ is a 2-absorbing ideal of a ring $R$ and let $S$ be a multiplicatively closed subset of $R$. Then $IR_S$ is a 2-absorbing ideal of $R_S$.

**Proof.** Suppose that $xyz \in IR_S$ for some $x, y, z \in R_S$. Then there are elements $s \in S_1$ and $x_1, y_2, z_3 \in R$ such that $xyz = \left(\frac{x_1}{s}\right)\left(\frac{y_2}{s}\right)\left(\frac{z_3}{s}\right) \in IR_S$. Thus, $x_1x_2x_3 \in I$. Since $I$ is a 2-absorbing ideal of $R$. We have $x_1x_2 \in I$ or $x_1x_3 \in I$ or $x_2x_3 \in I$, and thus $xy \in IR_S$ or $xz \in IR_S$ or $yz \in IR_S$.

**Note.** Given a function $\varphi : I(R) \to I(R) \cup \{0\}$ we define $\varphi_S : I(R_S) \to I(R_S) \cup \{0\}$ by $\varphi_S(J) = (\varphi(J \cap R))_S$ (and $\varphi_S(J) = 0$ if $\varphi(J \cap R) = 0$). Note that $\varphi_S(J) \subseteq J$ and $(\varphi_S)_{\alpha} = \varphi_{\alpha}$ for $\alpha \in \{0\} \cup \{0\} \cup \mathbb{N}$. We show that if $(\varphi(I))_S \subseteq \varphi_S(IS)(\text{which is the case for } \varphi_{\alpha} \text{ for } \alpha \in \{0\} \cup \{0\} \cup \mathbb{N})$, then $I$ $\varphi_S$-2-absorbing. Given an ideal $J$ of $R$, define $\varphi_J : I(J) \to I(J) \cup \{0\}$ by $\varphi_J(J) = \varphi(J) + J$ for $I \supseteq J$ (and $\varphi(J) = 0$ if $\varphi(I) = 0$). Note that $\varphi_J(J) \subseteq J$ and $(\varphi_J)_{\alpha} = \varphi_{\alpha}$ for $\alpha \in \{0\} \cup \{0\} \cup \mathbb{N}$. If $I \supseteq J$ are ideals of $R$ and $I$ is 2-absorbing (resp., weakly 2-absorbing, $n$-almost 2-absorbing), then so is $\frac{J}{I}$. We show that if $I$ is $\varphi$-2-absorbing, then $\frac{J}{I}$ is $\varphi_J$-2-absorbing in $\frac{R}{J}$.

**Proposition 21.** Let $R$ be a commutative ring and let $\varphi : I(R) \to I(R) \cup \{0\}$ be a function. Let $I$ be a $\varphi$- 2-absorbing ideal of $R$.

(1) If $I$ is an ideal of $R$ with $J \subseteq I$. Then $\frac{J}{I}$ is a $\varphi_J$- 2-absorbing ideal of $\frac{R}{J}$.

(2) Suppose that $S$ is a multiplicatively closed subset of $R$ with $I \cap S = \emptyset$ and $\varphi(S) \subseteq \varphi_S(IS)$. Then $IS$ is a $\varphi_S$- 2-absorbing ideal of $R_S$.

**Proof.** (1) Let $a, b, c \in R$. Suppose that $\overline{abc} \in \frac{J}{I} - \varphi_J(J)$, $\frac{I}{J} - \frac{(\varphi(I) + J)}{J}$. Hence $abc \in I - \varphi(I) + J$. Thus $abc \in I - \varphi(I)$; so $ab \in I$ or $ac \in I$ or $bc \in I$. Therefore $\overline{ab} \in \frac{J}{I}$ or $\overline{ac} \in \frac{J}{I}$ or $\overline{bc} \in \frac{J}{I}$; so $\frac{J}{I}$ is $\varphi_J$-2-absorbing.

(2) Let $\frac{x}{s}, \frac{y}{q}, \frac{z}{q} \in IS - \varphi_S(IS)$. So $xyzu \in I$ for some $u \in S$; but $xyzw \notin \varphi_S(IS) \cap R$ for every $w \in S$. Now if $xyzw \in \varphi(I)$, then $\frac{x}{s} \frac{y}{q} \frac{z}{q} \in (\varphi(I)S) \subseteq \varphi_S(IS)$;
a contradiction. So $xy(zu) \in I - \varphi(I)$ and hence $I$ is $\varphi$-2-absorbing gives $xy \in I$ or $x(zu) \in I$ or $y(zu) \in I$. Hence $\frac{xz}{t} \in I_S$ or $\frac{xz}{q} \in I_S$ or $\frac{yz}{t} \in I_S$. Thus $I_S$ is $\varphi$-2-absorbing.

**Theorem 22.** A commutative ring $R$ has every proper (principal) ideal almost 2-absorbing if and only if $R$ is von Neumann regular or $(R, M)$ is quasilocal with $M^3 = 0$.

**Proof.** ($\Leftarrow$) Suppose that $R$ is von Neumann regular. Then each proper ideal of $R$ is idempotent and hence almost 2-absorbing. If $(R, M)$ is quasilocal with $M^3 = 0$, then every proper ideal of $R$ is weakly 2-absorbing and hence almost 2-absorbing.

($\Rightarrow$) Suppose that every proper principal ideal of $R$ is almost 2-absorbing. Let $M$ be a maximal ideal of $R$. Then every proper principal ideal of $R_M$ is almost 2-absorbing. Let $a, b, c \in M_M$. Then $(abc)$ is almost 2-absorbing. Now $abc \in (abc)$. So either $ab \in (abc), ac \in (abc), bc \in (abc)$ or $abc \in (abc)^2$. Hence $(ab) = (ab)(c), (ac) = (ac)(b), (bc) = (bc)(a)$ or $(abc) = (abc)(abc)$. By Nakayama’s lemma $(abc) = 0$. Hence $M^3_M = 0$. Let $x \in R$. Then in $R_M, x^3$ is either a unit or 0. So $(x^3) = (x^4)$ locally and hence globally. Thus $(x^3) = (e)$ for some idempotent $e$. First suppose that $R$ is indecomposable. Then $x^3 = 0$ for each nonunit $x \in R$. So $R$ is quasilocal with unique maximal ideal $M$. Now suppose that $R$ is not indecomposable; say $R = R_1 \times R_2$. Suppose that $R_1$ is not von Neumann regular. Then $R_1$ has a nonidempotent ideal $I$. By hypothesis $I \times 0$ is almost 2-absorbing. But this contradicts Theorem 9. So $R_1$ is von Neumann regular. Likewise, $R_2$ is von Neumann regular. Hence their product $R = R_1 \times R_2$ is also von Neumann regular.

**Corollary 23.** Let $R$ be a commutative ring and $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ a function with $\varphi_w \leq \varphi \leq \varphi_2$. Then every proper (principal) ideal of $R$ is almost 2-absorbing if and only if $R$ is von Neumann regular.

**Proof.** ($\Rightarrow$) Since a $\varphi$-2-absorbing ideal with $\varphi \leq \varphi_2$ is almost 2-absorbing.

($\Leftarrow$) Suppose that $R$ is von Neumann regular. Let $I$ be an ideal of $R$. Since $I$ is idempotent, $I = \bigcap_{n=1}^{\infty} I^n \subseteq \varphi(I) \subseteq I^2 = I$; so $\varphi(I) = I$. Hence every proper ideal of $R$ is $\varphi$-2-absorbing.

**Lemma 24.** Let $R$ be a commutative local ring with maximal ideal $M$ such that $M^3 = 0$ (in particular $M^2 = 0$). Then every proper ideal is weakly 2-absorbing.
Proof. Suppose that $I$ is a proper ideal and $abc \in I - 0 = I - M^3$. Hence $abc \notin M^3$; so at least one of $a$ or $b$ or $c$ does not belong to $M$. So $a$ or $b$ or $c$ is a unit and hence $ab$ or $ac$ or $bc$ is in $I$. So $I$ is weakly 2-absorbing.

Lemma 25. Let $R$ be a commutative local ring with maximal ideal $M$ and $I$ is a proper ideal in $R$ such that $I^3 = M^3$ then $I$ is $\varphi$-2-absorbing for some $\varphi$ with $\varphi \geq \varphi_3$.

Proof. Let $abc \in I - \varphi(I)$ for some $a, b, c \in R$. Since $\varphi \geq \varphi_3$, we have $abc \in I - \varphi_3(I)$. Then $abc \in I - I^3 = I - M^3$. So $a$ or $b$ or $c$ is a unit and hence $ab$ or $ac$ or $bc$ is in $I$. So $I$ is $\varphi$-2-absorbing.

Corollary 26. If $(R, M)$ is a commutative local ring and $I$ is a proper ideal such that $M^3 \subseteq I \subseteq M$, then $I$ is almost 2-absorbing if and only if $M^3 = I^3$.

References

