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CHARACTERIZATIONS OF OPERATOR ORDER FOR THREE POSITIVE DEFINITE OPERATORS VIA OPERATOR MEAN

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Abstract: Motivated by Lin and Cho's characterizations of $A \geq B \geq C$ via extended grand Furuta inequality, we present two characterizations of $A \geq B \geq C$ via operator mean.

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1. Introduction

A capital letter (such as T) stands for a bounded linear operator on a Hilbert space. $T > 0$ and $T \geq 0$ mean T is a positive definite operator and T is a positive semidefinite operator, respectively.

As an important and historic extension of Löwner-Heinz inequality ($A \geq B \geq 0 \Rightarrow A^\alpha \geq B^\alpha$, $\alpha \in [0, 1]$), T. Furuta proved the following theorem in 1987.

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Theorem 1.1. (see Furuta Inequality, [2, 7]) *If* $A \geq B \geq 0$, *then* $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$ *holds for* $p \geq 0$, $r \geq 0$, $q \geq 1$ *with* $(1+r)q \geq p+r$.

In 1995, T. Furuta obtained the following grand form of Theorem 1.1.

Thoerem 1.2. (see Grand Furuta Inequality, [3, 8]) *If* $A \geq B \geq 0$ *with* $A > 0$, *then* $A^{1-t+r} \geq [A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}}$ *holds for* p, $s \geq 1$, $t \in [0,1]$ *and* $r \geq t$.

In 2003, M. Uchiyama showed the following extended form of Theorem 1.2.

Theorem 1.3. (Extended Grand Furuta Inequality, [9]) *If* $A \geq B \geq C \geq 0$ *with* $B > 0$, *then* $A^{1-t+r} \geq [A^{\frac{r}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^sA^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}}$ *holds for* p, $s \geq 1$, $t \in [0,1]$ *and* $r \geq t$.

For $S, T > 0$, operator mean of S and T is defined by F. Kubo and T. Ando in [4] as $S\sharp_\alpha T = S^{\frac{1}{2}}(S^{-\frac{1}{2}}TS^{-\frac{1}{2}})^\alpha S^{\frac{1}{2}}$, where $\alpha \in [0,1]$. Generally, if $\alpha \in \mathbb{R}$, $S^{\frac{1}{2}}(S^{-\frac{1}{2}}TS^{-\frac{1}{2}})^\alpha S^{\frac{1}{2}}$ is denoted by $S\natural_\alpha T$.

Recently, C. -S. Lin and Y. J. Cho in [5] showed characterizations of $A \geq B \geq C$ via extended grand Furuta inequality. Motived by [5], we present two characterizations of $A \geq B \geq C$ via operator mean.

2. Main Results

C.-S. Lin in 2010 showed the following results on operator mean.

Lemma 2.1. (see [6]) *For* $p, s \geq 1$, $t \in [0,1]$ *and* $r \geq 1$, *if* $A \geq B \geq C > 0$, *then*

$$C^r\sharp_{\frac{r-1}{(p-t)s+r}}(B^{\frac{t}{2}}A^{-p}B^{\frac{t}{2}})^s \leq C \leq B \leq A \leq A^r\sharp_{\frac{r-1}{(p-t)s+r}}(B^{\frac{t}{2}}C^{-p}B^{\frac{t}{2}})^s.$$

Lemma 2.2. (see [6]) *For* $p, s \geq 1$, $t \in [0,1]$ *and* $r \geq 1+t$, *if* $A \geq B \geq C > 0$, *then*

$$\begin{aligned} & C^{r-t}\sharp_{\frac{r-1-t}{(p-t)s+r}}C^{-\frac{t}{2}}B^{\frac{t}{2}}(B^{-t}\natural_sA^{-p})B^{\frac{t}{2}}C^{-\frac{t}{2}} \\ \leq\ & C \leq B \leq A \\ \leq\ & A^{r-t}\sharp_{\frac{r-1-t}{(p-t)s+r}}A^{-\frac{t}{2}}B^{\frac{t}{2}}(B^{-t}\natural_sC^{-p})B^{\frac{t}{2}}A^{-\frac{t}{2}}. \end{aligned}$$

Next we will show two characterizations of operator order for three positive definite operators via Lemma 2.1 and Lemma 2.2.

Theorem 2.1. *For $A, B, C > 0$. $A \geq B \geq C$ if and only if the following two inequalities*

$$C^r \sharp_{\frac{r-1}{(p-t)s+r}} (B^{\frac{t}{2}} A^{-p} B^{\frac{t}{2}})^s \leq C, \tag{2.1}$$

$$A \leq A^r \sharp_{\frac{r-1}{(p-t)s+r}} (B^{\frac{t}{2}} C^{-p} B^{\frac{t}{2}})^s \tag{2.2}$$

hold for $p, s \geq 1$, $t \in [0,1]$ and $r \geq 1$.

Theorem 2.2. *For $A, B, C > 0$. $A \geq B \geq C$ if and only if the following two inequalities*

$$C^{r-t} \sharp_{\frac{r-1-t}{(p-t)s+r}} C^{-\frac{t}{2}} B^{\frac{t}{2}} (B^{-t} \natural_s A^{-p}) B^{\frac{t}{2}} C^{-\frac{t}{2}} \leq C, \tag{2.3}$$

$$A \leq A^{r-t} \sharp_{\frac{r-1-t}{(p-t)s+r}} A^{-\frac{t}{2}} B^{\frac{t}{2}} (B^{-t} \natural_s C^{-p}) B^{\frac{t}{2}} A^{-\frac{t}{2}} \tag{2.4}$$

hold for $p, s \geq 1$, $t \in [0,1]$ and $r \geq 1+t$.

Proof of Theorem 2.1. The necessity is obviously by Lemma 2.1. We only need to prove the sufficiency. We adopt the same method as in [5].

Putting $p = t = 1$, $r = 2$ in (2.1), we have $C^2 \sharp_{\frac{1}{2}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^s \leq C$. By the definition of $\sharp$, the following inequality holds.

$$(C^{-1}(B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^s C^{-1})^{\frac{1}{2}} \leq C^{-1}. \tag{2.5}$$

Because $C > 0$ and C is bounded, there exist two positive numbers m_C and n_C such that $m_C I \geq C \geq n_C I > 0$. According to Theorem 6 in [1] ($X \geq Y \geq 0$ with $mI \geq X \geq nI > 0 \Rightarrow \frac{(m+n)^2}{4mn} X^2 \geq Y^2$), we have

$$C^{-1}(B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^s C^{-1} \leq \frac{(m_c^{-1} + n_c^{-1})^2}{4 m_c^{-1} n_c^{-1}} C^{-2}. \tag{2.6}$$

Deleting C^{-1} in the both side of the inequality above, and applying Löwner-Heinz inequality, the following inequality holds.

$$B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} \leq \Big(\frac{(m_c^{-1} + n_c^{-1})^2}{4 m_c^{-1} n_c^{-1}} \Big)^{\frac{1}{s}} I. \tag{2.7}$$

Letting $s \to +\infty$ above, then $A^{-1} \leq B^{-1}$, which ensures $A \geq B$.

By the same way, we can obtain $B \geq C$ from (2.2). □

Proof of Theorem 2.2. The necessity is obviously by Lemma 2.2. We only need to prove the sufficiency.

Putting $p = t = 1$, $r = 4$ in (2.3), we have $C^3 \sharp_{\frac{1}{2}} C^{-\frac{1}{2}} B^{\frac{1}{2}} (B^{-1} \natural_s A^{-1}) B^{\frac{1}{2}} C^{-\frac{1}{2}} \leq C$. By the definitions of $\sharp$ and $\natural$, the following inequality holds.

$$(C^{-2}(B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^s C^{-2})^{\frac{1}{2}} \leq C^{-2}. \tag{2.8}$$

According to Theorem 6 in [1], we have

$$C^{-2}(B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^s C^{-2} \leq \Big(\frac{(m_c^{-2} + n_c^{-2})^2}{4m_c^{-2} n_c^{-2}}\Big)^{\frac{1}{s}} C^{-4}. \tag{2.9}$$

Deleting C^{-2} in the both side of the inequality above, and applying Löwner-Heinz inequality, the following inequality holds.

$$B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} \leq \Big(\frac{(m_c^{-2} + n_c^{-2})^2}{4m_c^{-2} n_c^{-2}}\Big)^{\frac{1}{s}} I. \tag{2.10}$$

Letting $s \to +\infty$ above, then $A^{-1} \leq B^{-1}$, which ensures $A \geq B$.

By the same way, we can obtain $B \geq C$ from (2.4). □

Remark 2.1. We can also obtain $A \geq B$ from (2.5) and (2.8) by Theorem 3.1 in [10](For $C, D > 0$, $r > 0$, $\delta > -r$ and $0 < w \leq 1$, if $C^{\delta+r} \geq (C^{\frac{r}{2}} D^s C^{\frac{r}{2}})^w$ holds for any $s > 1$, then $D \leq I$). We leave the details to readers.

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