International Journal of Pure and Applied Mathematics
Volume 99 No. 4 2015, 471-476
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v99i4.7

RESULTS OF GENERALIZED LOCAL COHOMOLOGY WITH RESPECT TO A PAIR OF IDEALS

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Abstract: Let (I, J) be a pair of ideals of a commutative Noetherian local ring R, and M a finitely generated module. Let t be a positive integer. We prove that (i) if $H^i_{I,J}(M)$ is minimax for all $i < t$, then $H^i_{I,J}(M)$ is (I, J)-cofinite for all $i < t$ and $\mathrm{Hom}\big(R/I, H^t_{I,J}(M)\big)$ is finitely generated; (ii) if $\mathfrak{a} \in \widetilde{W}(I, J)$ and $H^i_{I,J}(M)$ is minimax for all $i < t$, then $\mathrm{Ext}^i_R(R/\mathfrak{a}, T)$ is minimax for all $i < t$. We also prove that if $\mathrm{Supp} H^i_{I,J}(M) = \{\mathfrak{m}\}$ for all $i < t$, then $H^i_{I,J}(M)$ is Artinian and (I, J)-cofinite for all $i < t$.

AMS Subject Classification: 13D45, 13E05, 14B15
Key Words: local cohomology modules, artinian modules, cofinite modules

1. Introduction

Throughout this paper, R is denoted a commutative Noetherian ring, I, and J are denoted two ideals of R, and M is a finitely generated R-module. For notations and terminologies that is not given in this paper, the reader is referred to [1].

As a generalization of the ordinary local cohomology modules, Takahashi, Yoshino and Yoshizawa [7] introduced the local cohomology modules with respect to a pair of ideals (I, J). To be more precise, let $W(I, J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid$

Received: October 22, 2014

$I^n \subseteq \mathfrak{p} + J$ for some positive integer $n\}$. Then for an R-module M, the (I, J)-torsion submodule $\Gamma_{I,J}(M)$ of M, which consists of all elements x of M with $\mathrm{Supp}Rx \subseteq W(I, J)$, is considered. It is known, $\Gamma_{I,J}(\ )$ is a left exact additive functor from the category of all R-modules and R-homomorphism to itself. For all integer i, the i-th local cohomology functor $H^i_{I,J}$ with respect to (I, J) is defined to be the i-th right derived functor of $\Gamma_{I,J}(\ )$. The i-th local cohomology module of M with respect to (I, J) is denoted by $H^i_{I,J}(M)$. When $J = 0$, then $H^i_{I,J}$ coincides with the usual local cohomology functor H^i_I with the support in the closed subset $V(I)$.

There are many questions about ordinary local cohomology modules. In particular, Hunke [5] proposed the following question: for an integer i, when is $H^i_I(M)$ Artinian? In [3] Grothendieck conjectured that for any finite R-module M, $\mathrm{Hom}_R(R/I, H^i_I(M))$ is finite for all i. Hartshorne [4] later refined this conjecture, and proposed the following one:

Let M be a finite R-module, and let I be an ideal of R. Then

$$\mathrm{Ext}^j_R(R/I, H^i_I(M))$$

is finite, for every $i \geq 0$ and $j \geq 0$. The purpose of this paper is to investigate a similar question as above for this generalized version of local cohomology. Obviously, these results are true in the category of all graded R-modules and homogeneous R-homomorphisms.

2. The Results

Definition 1. An R-module T is called (I, J)-cofinite if $\mathrm{Supp}T \subseteq W(I, J)$ and $\mathrm{Ext}^i_R(R/I, T)$ is a finite R-modules, for every $i \geq 0$.

Lemma 2. *Let T be an arbitrary R-module. Then the following statements hold:*

- *(i) Suppose that T is a module with support in $V(I)$. Then T is (I, J)-cofinite if and only if T is I-cofinite.*
- *(ii) T is (I, J)-cofinite Artinian (minimax) if and only if T is I-cofinite Artinian (minimax).*
- *(iii) Let T be Artinian. If Hom$(R/I, T)$ is finite, then T is (I, J)-cofinite.*
- *(iv) Let Supp$T \subseteq V(I)$. If there is an element $x \in I$, such that $(0 :_T x)$ is Artinian and (I, J)-cofinite, then T is Artinian and (I, J)-cofinite.*

Proof. (i) and (ii) are clear because $V(\mathfrak{m}) \subseteq V(I) \subseteq W(I, J)$.

(iii) Since, in view of the hypothesis, $\mathrm{Hom}(R/I, T)$ has finite length and $\mathrm{Supp}T \subseteq V(\mathfrak{m}) \subseteq V(I)$. By using [6, Proposition 4.1], we can get $\mathrm{Ext}^i_R(R/I, T)$ is finitely generated for all $i \geq 0$ thus the proof is completed.

(iv) In view of (ii), $(0 :_T x)$ is Artinian and I-cofinite. Therefore, according to [6, Proposition 4.1] T is Artinian and I-cofinite; and hence the result follows. □

Therefore, in view of lemma (2, ii) and [6, Corollary 4.4], the class of (I, J)-cofinite Artinian modules is closed under taking submodules, quotients and extensions.

Definition 3. An R-module T is said to be minimax module, if there is a finitely generated submodule T_1 of T such that the quotient module T/T_1 is Artinian. The class of minimax modules includes all finite and all Artinian modules.

Theorem 4. *Let T be an arbitrary R-module and t a positive integer and let $\mathfrak{a} \in \widetilde{W}(I, J)$, where $\widetilde{W}(I, J)$ denote the set of ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n. If $H^i_{I,J}(T)$ is Artinian (minimax) for all $0 \leq i < t$, then $\mathrm{Ext}^i_R(R/\mathfrak{a}, T)$ is Artinian (minimax)for all $0 \leq i < t$.*

Proof. We prove by induction on t. The case where $t = 1$ yields $\Gamma_{I,J}(T)$ is Artinian (minimax). In view of proof [7 ,Theorem 3.2], $\Gamma_{\mathfrak{a}}(T) \subseteq \Gamma_{I,J}(T)$. Thus $\Gamma_{\mathfrak{a}}(T)$ is Artinian (minimax). In particular, $\mathrm{Hom}(R/\mathfrak{a}, T)$ is Artinian (minimax). Assume that $t \geq 2$ and the Theorem holds true for $t - 1$. It follows from [7, Corollary 1.13] that $H^i_{I,J}(T) \cong H^i_{I,J}(T/\Gamma_{I,J}(T))$ for all $i > 0$. Also, $T/\Gamma_{I,J}(T)$ is an (I, J)-torsion-free R-module. Hence we can (and do) assume that M is an (I, J)-torsion-free R-module. Thus $\Gamma_{I,J}(T) = 0$ implies that $\Gamma_{\mathfrak{a}}(T) = 0$. Then there exists $x \in \mathfrak{a}$ such that x is an T-sequence. Now, we may consider the exact sequence $0 \longrightarrow T \xrightarrow{x} T \longrightarrow T/xT \longrightarrow 0$ to obtain the exact sequences

$$H^i_{I,J}(T) \xrightarrow{x} H^i_{I,J}(T) \longrightarrow H^i_{I,J}(T/xT) \qquad \text{and}$$

$$\mathrm{Ext}^i_R(R/\mathfrak{a}, T) \xrightarrow{x} \mathrm{Ext}^i_R(R/\mathfrak{a}, T) \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{a}, T/xT). \qquad (*)$$

Now, the above exact sequences are used in conjunction with the inductive hypothesis to see that the R- modules $H^i_{I,J}(T/xT)$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, T)$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, T/xT)$ are Artinian (minimax) for all $i < t - 1$. It is enough to show that $\operatorname{Ext}_R^{t-1}(R/\mathfrak{a}, T)$ is Artinian (minimax). Finally, note that $x \in \mathfrak{a}$, and $(*)$ exact sequence yields that the sequence

$$\operatorname{Ext}_R^{t-2}(R/\mathfrak{a}, T) \longrightarrow \operatorname{Ext}_R^{t-2}(R/\mathfrak{a}, T/xT) \longrightarrow \operatorname{Ext}_R^{t-1}(R/\mathfrak{a}, T) \longrightarrow 0,$$

is exact consequently $\operatorname{Ext}_R^{t-1}(R/\mathfrak{a}, T)$ is Artinian (minimax), as required. □

Lemma 5. *Let T is a minimax R-module with Supp in $W(I, J)$. Then the following hold:*

(i) If Hom$(R/I, T)$ is finitely generated, then T is (I, J)-cofinite.

(ii) If there is an element $x \in I$, such that $(0 :_T x)$ is minimax and (I, J)-cofinite, then T is minimax and (I, J)-cofinite.

Proof. (i) Let T_1 be a finitely generated submodule of T, such that $T_2 = T/T_1$ is Artinian and suppose that $\operatorname{Hom}(\frac{R}{I}, T)$ is finitely generated. The exactness of

$$0 \longrightarrow \operatorname{Hom}(R/I, T_1) \longrightarrow \operatorname{Hom}(R/I, T) \longrightarrow \operatorname{Hom}(R/I, T_2) \longrightarrow \operatorname{Ext}_R^1(R/I, T_1),$$

implies that $\operatorname{Hom}(R/I, T_2)$ is finitely generated. Hence we get from Lemma (2, iii) and [2, Lemma 2.1] that T_2 is Artinian and (I, J)-cofinite, therefore T is also (I, J)-cofinite.

(ii) If $(o :_T x)$ minimax and (I, J)-cofinite, then $\operatorname{Hom}(R/I, (0 :_T x) \cong \operatorname{Hom}(R/I, T)$ is finitely generated. It is clear by (i). □

Theorem 6. *Let t be a non-negative integer, such that $H^i_{I,J}(M)$ is (I, J)-cofinite minimax for all $i < t$. Then, the R-module Hom$\big(R/I, H^i_{I,J}(M)\big)$ is finitely generated for all $i \leq t$.*

Proof. Since $H^i_{I,J}(M)$ is (I, J)-cofinite, $\operatorname{Hom}\big(R/I, H^i_{I,J}(M)\big)$ is finitely generated for all $i < t$. So it is enough to prove that $\operatorname{Hom}\big(R/I, H^t_{I,J}(M)\big)$ is finitely generated. We prove by induction on $t \geq 0$. If $t = 0$ then the result is clear. Assume that $t > 0$ and the result holds true for $t-1$. It follows from [7, Corollary 1.13] that $H^i_{I,J}(M) \cong H^i_{I,J}\big(M/\Gamma_{I,J}(M)\big)$ for all $i > 0$. Also $M/\Gamma_{I,J}(M)$ is (I, J)-torsion-free R-module. Since $\Gamma_I(M) \subseteq \Gamma_{I,J}(M)$. We can assume that M is an I-torsion-free R-module. Then there exists an element $x \in I$ which is M-regular. The exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow o$$

induces a long exact sequence

$$H^{i-1}_{I,J}(M) \longrightarrow H^{i-1}_{I,J}(M/xM) \longrightarrow H^{i}_{I,J}(M) \xrightarrow{x} H^{i}_{I,J}(M) \longrightarrow H^{i}_{I,J}(M/xM).$$

Also, by the hypothesis, $H^i_{I,J}(M/xM)$ is (I,J)-cofinite minimax for all $i < t-1$, so that, by the inductive hypothesis, $\operatorname{Hom}\big(R/I, H^{t-1}_{I,J}(M/xM)\big)$ is finitely generated. On the other hand, the exact sequence

$$0 \longrightarrow H^{t-1}_{I,J}(M)/xH^{t-1}_{I,J}(M) \longrightarrow H^{t-1}_{I,J}(M/xM) \longrightarrow (0 :_{H^t_{I,J}(M)} x) \longrightarrow 0,$$

induces the exact sequence

$$0 \longrightarrow \operatorname{Hom}\Big(R/I, H^{t-1}_{I,J}(M)/xH^{t-1}_{I,J}(M)\Big) \longrightarrow \operatorname{Hom}\Big(R/I, H^{t-1}_{I,J}(M/xM)\Big)$$
$$\longrightarrow \operatorname{Hom}\Big(R/I, (0 :_{H^t_{I,J}(M)} x)\Big) \longrightarrow \operatorname{Ext}^1\Big(R/I, H^{t-1}_{I,J}(M)/xH^{t-1}_{I,J}(M)\Big).$$

Since $\operatorname{Ext}^1_R\Big(R/I, H^{t-1}_{I,J}(M)/xH^{t-1}_{I,J}(M)\Big)$ is finitely generated, it follows from the above exact sequence that $\operatorname{Hom}\Big(R/I, (0 :_{H^t_{I,J}(M)} x)\Big)$ is finitely generated R-module. Now, as $x \in I$ we have $\operatorname{Hom}\big(R/I, H^t_{I,J}(M)\big)$ is finitely generated, as required. □

Lemma 7. *Let t be a non-negative integer, such that $H^i_{I,J}(M)$ is minimax for all $i < t$. Then $H^i_{I,J}(M)$ is (I,J)-cofinite for all $i < t$ and $\operatorname{Hom}\big(R/I, H^t_{I,J}(M)\big)$ is finitely generated.*

Proof. By Theorem (6), we need only to prove that $H^i_{I,J}(M)$ is (I,J)-cofinite for all $i < t$. We proceed by induction on i. It is clear that $H^0_{I,J}(M)$ is (I,J)-cofinite. Assume that $i > 0$ and the result holds true for smaller values than i. Thus we obtain that $H^i_{I,J}(M)$ is (I,J)-cofinite minimax for all $j \leq i-1$ by the inductive hypothesis. It follows by Theorem (6) that $\operatorname{Hom}\big(R/I, H^i_{I,J}(M)\big)$ is finitely generated. Therefore by Lemma (5), $H^i_{I,J}(M)$ is (I,J)-cofinite minimax. Now the assertion follows from Theorem (6). □

Corollary 8. *Let t be a non-negative integer such that $\operatorname{Supp} H^i_{I,J}(M) = \{\mathfrak{m}\}$ for $i < t$. Then $H^i_{I,J}(M)$ is Artinian and (I,J)-cofinite for $i < t$. Moreover $\operatorname{Hom}\big(R/I, H^t_{I,J}(M)\big)$ is finitely generated.*

Proof. We now prove the Lemma by induction on t. It $t = 1$ then it is clear that $\Gamma_{I,J}(M)$ is Artinian and (I,J)-cofinite. Assume that $t > 1$ and the result holds true for $t-1$. By the inductive hypothesis, the R-module $H^i_{I,J}(M)$

is Artinian and (I, J)-cofinite for all $i < t-1$. Therefore, by Theorem (6) $\mathrm{Hom}\big(R/I, H^{t-1}_{I,J}(M)\big)$ is finitely generated R-module. Since $\mathrm{Supp}H^{t-1}_{I,J}(M) = \{\mathfrak{m}\}$, it follows that it yields from Lemma (2, iv) that, $H^{t-1}_{I,J}(M)$ is Artinian and (I, J)-cofinite. Now assertion follows from Theorem (6). □

Acknowledgments

The author would like to thank the referees for their careful reading.

References

[1] M. P. Brodmann and R. Y Sharp, *Local cohomology: An Algebraic introduction with geometric applications*, Cambridge Univ. Press 1998.

[2] F. Dehghani-Zadeh , *On the finiteness properties of generalized local cohomology modules*, International Electronic Journal of Algebra, **10** (2011), 113-122.

[3] A. Grothendieck, *Cohomological local des faisceaur coherents et théoremes de Lefschetz locaux et globaus (SGA2), North-Holland*, Amsterdam, 1968.

[4] R. Hartshorne, ' *Affine duality and cofiniteness* ', Invent. Math, (1970), 145-164.

[5] C. Hunkek, *Problems on local cohomology, in: Free resolutions in commutative algebra and algebraic geometry*, Research Notes In Math, ed, by Eisenbud. and Huneke C., Jones and Bartlett, Boston, 1992, 93-108.

[6] L. Melkersson, *Modules cofinite with respect to an ideal*, J. Algebra, **285** (2005), 649-668.

[7] R. Takahashi and Y. Yoshino and T. Yoshinawa, *Local cohomology based on a nonclosed support defined by a pair of ideals*, J. Pure Apple. Algebra, **213** (2009), 582-600.