International Journal of Pure and Applied Mathematics
Volume 106 No. 1 2016, 223-235
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v106i1.17

ijpam.eu

(L, M)-DOUBLE FUZZY GRILLS

A.A. Abd El-Latif[1], Yong Chan Kim[2 §]

[1]Department of Mathematics
Faculty of Science and Arts at Belqarn
Bisha University, SAUDI ARABIA P.O. Box 60, Sabt Al-Alaya 61985

[2]Department of Mathematics
Gangneung-Wonju University
Gangneung, Gangwondo 210-702, KOREA

Abstract: The aim of this paper is to introduce the notions of (L, M)-double fuzzy grills where L and M are strictly two-sided, commutative quantale. Furthermore, we define the product of (L, M)-double fuzzy grills and investigate the image of (L, M)-double fuzzy grills.

AMS Subject Classification: 03E72, 06A15, 06F07, 54F05

Key Words: fuzzy sets, stsc-quantales, (L, M)-double fuzzy grills, product of (L, M)-double fuzzy grills

1. Introduction

In general topology the notion of grills was first proposed by Choquet [8], which was subsequently found to be an extremely useful device, like filters and nets, for the investigations of many topological notions like compactifications, proximity spaces, different types of extension problems etc. (see [7], [11], [13], [16], [17] for details). As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [2], [3]. Working under the name "intuitionistic" did not continue because doubts were thrown about the suitability of this term, especially when working in the case of complete lattice L. These doubts were quickly ended in 2005 by Garcia and Rodabaugh [9]. They proved that this

Received: September 18, 2015
Published: February 4, 2016

§Correspondence author

term is unsuitable in mathematics and applications. They concluded that they work under the name "double". Under this concept many works have been launched [5], [6].

In this paper, we introduce the notions of (L, M)-double fuzzy grills where L and M are strictly two-sided, commutative quantales. Furthermore, we define the product of (L, M)-double fuzzy grills and investigate the image of (L, M)-double fuzzy grills.

2. Preliminaries

Throughout this paper, let L, M be complete lattices with an order-reversing involution $'$ and 0_L (1_L) and 0_M (1_M) are the smallest (largest) elements of L and M respectively. Let X be a non-empty set. The family of all L-fuzzy sets on X will be denoted by L^X. The smallest element and the largest one of L^X will be denoted by 0_X and 1_X respectively. For each $\alpha \in L$, $\underline{\alpha}(x) = \alpha$, for all $x \in X$. Let $f : X \to Y$ be a crisp map. The Zadeh image and preimage operators $f^{\to} : L^X \to L^Y$ and $f^{\leftarrow} : L^Y \to L^X$ are defined by:

$$f^{\to}(\lambda)(y) = \bigvee\{\lambda(x) : x \in X, f(x) = y\},$$

$$f^{\leftarrow}(\mu) = \mu \circ f,$$

for each $\lambda \in L^X$, $\mu \in L^Y$.

Definition 2.1. [12,14,15]. A triple $(L, \leq, \odot)$ is called a strictly two-sided, commutative quantale (briefly, stsc-quantale) if it satisfies the following properties:

(L1) $L = (L, \leq, 0_L, 1_L)$ is a complete lattice,

(L2) $(L, \odot)$ is a commutative semigroup,

(L3) $a = a \odot 1_L$, for each $a \in L$,

(L4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma}(a_i \odot b).$$

A stsc-quantale $(L, \leq, \odot)$ is called a stsc-biquantale if $\odot$ is distributive over arbitrary meets, i.e.

(L) $(\bigwedge_{i\in\Gamma} a_i) \odot b = \bigwedge_{i\in\Gamma}(a_i \odot b)$.

Example 2.2. [12,14,15]. (i) Each frame is a stsc-quantatle. In particular, the unit interval $([0, 1], \leq, \vee, \wedge, 0, 1)$ is a stsc-quantale.

(ii) The unit interval with a left-continuous t-norm t, $([0,1], \leq, t)$, is a stsc-quantatle.

(iii) Every GL-monoid is a stsc-quantale .

(iv) Define a binary operation $\odot$ on $[0,1]$ by $x \odot y = max\{0, x+y-1\}$. Then $([0,1], \leq, \odot)$ is a stsc-quantale .

Definition 2.3. [12,14,15]. Let $(L, \leq, \odot)$ be a stsc-quantale. A mapping $' : L \to L$ is called an order-reversing involution, if it satisfies the following conditions:

(i) $x'' = x$, for each $x \in L$.

(ii) If $x \leq y$ then, $y' \leq x'$, for each $x, y \in L$.

In this paper, we always assume that $(L, \leq, \odot, \oplus, ')$ (respectively, $(M, \leq, \odot, \oplus, ')$) is a stsc-quantale with an order-reversing involution $'$ and the binary operation $\oplus$ is defined by:

$$x \oplus y = (x' \odot y')'$$

unless otherwise specified. For each $x, y, z \in L$ the following properties satisfies

(i) if $y \leq z$, then $(x \odot y) \leq (x \odot z)$ and $(x \oplus y) \leq (x \oplus z)$,

(ii) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$ [12].

The following equalities can be obtained easily,

(L3$'$) $a = a \oplus 0_L$, for each $a \in L$.

(L4$'$) $(\bigwedge_{i \in \Gamma} a_i) \oplus b = \bigwedge_{i \in \Gamma} (a_i \oplus b)$.

(L$'$) $(\bigvee_{i \in \Gamma} a_i) \oplus b = \bigvee_{i \in \Gamma} (a_i \oplus b)$, if $(L, \leq, \odot, \oplus, ')$ is a stsc-biquantale.

All algebraic operations on L can be extended to the set L^X and M^{L^X} as follows: for all $x \in X$, $\lambda, \mu \in L^X$, $\mathcal{U}, \mathcal{V} \in \mathcal{M}^{\mathcal{L}^{\mathcal{X}}}$;

(i) $\lambda \leq \mu$ iff $\lambda(x) \leq \mu(x)$,

(ii) For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$, for each $x \in X$,

(iii) $(\lambda \oplus \mu)(x) = \lambda(x) \oplus \mu(x)$,

(iv) $\mathcal{U} \leq \mathcal{V}$ iff $\mathcal{U}(\lambda) \leq \mathcal{V}(\lambda)$ for all $\lambda \in L^X$.

Lemma 2.4. [12]. Let $(L, \leq, \odot, \oplus, ')$ be a stsc-quantale with an order-reveresing involution $'$ and $f : X \to Y$ be a map. For each $\{\lambda_i : i \in \Gamma\} \subseteq L^X$ and $\{\mu_j : j \in J\} \subseteq L^Y$, we have the following properties:

(i) $f^{\to}(\oplus_{i \in \Gamma} \lambda_i) = \oplus_{i \in \Gamma} f^{\to}(\lambda_i)$.

(ii) $f^{\leftarrow}(\oplus_{j \in J} \mu_j) = \oplus_{j \in J} f^{\leftarrow}(\mu_j)$.

3. (L, M)-Double Fuzzy Grills

Definition 3.1. The pair $(\mathcal{G}, \mathcal{G}^*)$ of maps $\mathcal{G}, \mathcal{G}^* : L^X \to M$ is called an (L, M)-double fuzzy grill on X if it satisfies the following conditions:

(DFG1) $\mathcal{G}(\lambda) \le (\mathcal{G}^*(\lambda))'$, $\forall \lambda \in L^X$,

(DFG2) $\mathcal{G}(0_X) = 0_M$, $\mathcal{G}(1_X) = 1_M$ and, $\mathcal{G}^*(0_X) = 1_M$, $\mathcal{G}^*(1_X) = 0_M$,

(DFG3) $\mathcal{G}(\lambda \oplus \mu) \le \mathcal{G}(\lambda) \oplus \mathcal{G}(\mu)$,and $\mathcal{G}^*(\lambda \oplus \mu) \ge \mathcal{G}^*(\lambda) \odot \mathcal{G}^*(\mu)$ for each $\lambda, \mu \in L^X$,

(DFG4) If $\lambda \le \mu$, then $\mathcal{G}(\lambda) \le \mathcal{G}(\mu)$ and $\mathcal{G}^*(\lambda) \ge \mathcal{G}^*(\mu)$.

The triplet $(X, \mathcal{G}, \mathcal{G}^*)$ is called an (L, M)-double fuzzy grill space. If $(\mathcal{G}_1, \mathcal{G}_1^*)$ and $(\mathcal{G}_2, \mathcal{G}_2^*)$ are (L, M)-double fuzzy grills on X, we say $(\mathcal{G}_1, \mathcal{G}_1^*)$ is finer than $(\mathcal{G}_2, \mathcal{G}_2^*)$ (or $(\mathcal{G}_2, \mathcal{G}_2^*)$ is coarser than $(\mathcal{G}_1, \mathcal{G}_1^*)$) denoted by $(\mathcal{G}_2, \mathcal{G}_2^*) \le (\mathcal{G}_1, \mathcal{G}_1^*)$ iff $\mathcal{G}_2(\lambda) \le \mathcal{G}_1(\lambda)$ and $\mathcal{G}_2^*(\lambda) \ge \mathcal{G}_1^*(\lambda)$, $\forall \lambda \in L^X$.

Remark 3.2. The notion of (L, M)-double fuzzy grill can be considered as a generalization of (L, M)-grill which defined in [1].

Definition 3.3. Let $(X, \mathcal{G}_1, \mathcal{G}_1^*)$ and $(Y, \mathcal{G}_2, \mathcal{G}_2^*)$ be two (L, M)-double fuzzy grill spaces. Then a map $f : X \to Y$ is said to be:

(i) double g-map if $\mathcal{G}_1(f^{\leftarrow}(\mu)) \le \mathcal{G}_2(\mu)$ and $\mathcal{G}_1^*(f^{\leftarrow}(\mu)) \ge \mathcal{G}_2^*(\mu)$, $\forall \mu \in L^Y$.

(ii) double g-preserving map if $\mathcal{G}_1(\lambda) \ge \mathcal{G}_2(f^{\rightarrow}(\lambda))$ and $\mathcal{G}_1^*(\lambda) \le \mathcal{G}_2^*(f^{\rightarrow}(\lambda))$, $\forall \lambda \in L^X$.

Naturally, the composition of double g-maps (resp. double g-preserving maps) is a double g-map(resp. double g-preserving map).

In the following proposition we assume that $(M, \le, \odot, \oplus, ')$ is a stsc-biquantale.

Proposition 3.4. If $\{(\mathcal{G}_i, \mathcal{G}_i^*)\}_{i \in \Gamma}$ is a family of (L, M)-double fuzzy grills in a fixed set X, then $(\mathcal{G}, \mathcal{G}^*)$ is an (L, M)-double fuzzy grill in X, where $\forall \lambda \in L^X$, $\mathcal{G}(\lambda) = \bigvee_{i \in \Gamma} \mathcal{G}_i(\lambda)$ and $\mathcal{G}^*(\lambda) = \bigwedge_{i \in \Gamma} \mathcal{G}_i^*(\lambda)$

Proof. (DFG1) $\forall \lambda \in L^X$,

$$\begin{aligned} &\mathcal{G}(\lambda) = \textstyle\bigvee_{i \in \Gamma} \mathcal{G}_i(\lambda) \le \bigvee_{i \in \Gamma} (\mathcal{G}_i^*(\lambda))' \\ &= (\textstyle\bigwedge_{i \in \Gamma} \mathcal{G}_i^*(\lambda))' = (\mathcal{G}^*(\lambda))'. \end{aligned}$$

(DFG2) It is clear.

(DFG3) $\forall \lambda, \mu \in L^X$,

$$\begin{aligned} &\mathcal{G}(\lambda) \oplus \mathcal{G}(\mu) = \textstyle\bigvee_{i \in \Gamma} \mathcal{G}_i(\lambda) \oplus \bigvee_{i \in \Gamma} \mathcal{G}_i(\mu) \\ &= \textstyle\bigvee_{i \in \Gamma} (\mathcal{G}_i(\lambda) \oplus \mathcal{G}_i(\mu)) \\ &\ge \textstyle\bigvee_{i \in \Gamma} \mathcal{G}_i(\lambda \oplus \mu) = \mathcal{G}(\lambda \oplus \mu). \end{aligned}$$

$$\begin{aligned}&\mathcal{G}^*(\lambda) \odot \mathcal{G}^*(\mu) = \bigwedge_{i\in\Gamma} \mathcal{G}_i^*(\lambda) \odot \bigwedge_{i\in\Gamma} \mathcal{G}_i^*(\mu)\\ &= \bigwedge_{i\in\Gamma} (\mathcal{G}_i^*(\lambda) \odot \mathcal{G}_i^*(\mu))\\ &\le \bigwedge_{i\in\Gamma} \mathcal{G}_i^*(\lambda \oplus \mu) = \mathcal{G}^*(\lambda \oplus \mu).\end{aligned}$$

(DFG4) If $\lambda \le \mu$, then we have $\mathcal{G}_i(\lambda) \le \mathcal{G}_i(\mu)$ and $\mathcal{G}_i^*(\lambda) \ge \mathcal{G}_i^*(\mu)$, $\forall i \in \Gamma$. Then,

$$\mathcal{G}(\lambda) = \bigvee_{i\in\Gamma} \mathcal{G}_i(\lambda) \le \bigvee_{i\in\Gamma} \mathcal{G}_i(\mu) = \mathcal{G}(\mu).$$

$$\mathcal{G}^*(\lambda) = \bigwedge_{i\in\Gamma} \mathcal{G}_i^*(\lambda) \ge \bigwedge_{i\in\Gamma} \mathcal{G}_i^*(\mu) = \mathcal{G}^*(\mu).$$

Remark 3.5. If $\{(\mathcal{G}_i, \mathcal{G}_i^*)\}_{i\in\Gamma}$ is a family of (L, M)-double fuzzy grills in a fixed set X, then $(\mathcal{G}, \mathcal{G}^*)$ is not necessary an (L, M)-double fuzzy grill in X, where $\forall\lambda \in L^X$, $\mathcal{G}(\lambda) = \bigwedge_{i\in\Gamma} \mathcal{G}_i(\lambda)$ and $\mathcal{G}^*(\lambda) = \bigvee_{i\in\Gamma} \mathcal{G}_i^*(\lambda)$, as the following example shows.

Example 3.6. Let $L = M = [0, 1]$, $\oplus = \vee$ and $\odot = \wedge$. Let $X = \{a, b\}$ be a set. Define the maps $\mathcal{G}_1, \mathcal{G}_1^*, \mathcal{G}_2, \mathcal{G}_2^* : L^X \to M$ as follows:

$$\mathcal{G}_1(\lambda) = \begin{cases} 0, & \text{if} \quad \lambda \in \{\underline{0}, \chi_{\{b\}}\} \\ 1, & \text{if} \quad \lambda \in \{\underline{1}, \chi_{\{a\}}\} \end{cases} \quad \mathcal{G}_1^*(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \chi_{\{b\}}\} \\ 0, & \text{if} \quad \lambda \in \{\underline{1}, \chi_{\{a\}}\}, \end{cases}$$

$$\mathcal{G}_2(\lambda) = \begin{cases} 0, & \text{if} \quad \lambda \in \{\underline{0}, \chi_{\{a\}}\} \\ 1, & \text{if} \quad \lambda \in \{\underline{1}, \chi_{\{b\}}\} \end{cases} \quad \mathcal{G}_2^*(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \chi_{\{a\}}\} \\ 0, & \text{if} \quad \lambda \in \{\underline{1}, \chi_{\{b\}}\}. \end{cases}$$

Thus

$$(\mathcal{G}_1 \wedge \mathcal{G}_2)(\lambda) = \begin{cases} 0, & \text{if} \quad \lambda \in \{\underline{0}, \chi_{\{a\}}, \chi_{\{b\}}\} \\ 1, & \text{if} \quad \lambda = \underline{1}, \end{cases}$$

and

$$(\mathcal{G}_1^* \vee \mathcal{G}_2^*)(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda \in \{\underline{0}, \chi_{\{a\}}, \chi_{\{b\}}\} \\ 0, & \text{if} \quad \lambda = \underline{1}. \end{cases}$$

Then $(\mathcal{G}_1, \mathcal{G}_1^*)$ and $(\mathcal{G}_2, \mathcal{G}_2^*)$ are (L, M)-double fuzzy grill on X, but $(\mathcal{G}_1 \wedge \mathcal{G}_2, \mathcal{G}_1^* \vee \mathcal{G}_2^*)$ is not, since

$$\begin{aligned}&(\mathcal{G}_1 \wedge \mathcal{G}_2)(\chi_{\{a\}} \vee \chi_{\{b\}}) = 1\\ &> (\mathcal{G}_1 \wedge \mathcal{G}_2)(\chi_{\{a\}}) \vee (\mathcal{G}_1 \wedge \mathcal{G}_2)(\chi_{\{b\}}) = 0,\end{aligned}$$

$$\begin{aligned}&(\mathcal{G}_1^* \vee \mathcal{G}_2^*)(\chi_{\{a\}} \vee \chi_{\{b\}}) = 0\\ &< (\mathcal{G}_1^* \vee \mathcal{G}_2^*)(\chi_{\{a\}}) \wedge (\mathcal{G}_1^* \vee \mathcal{G}_2^*)(\chi_{\{b\}}) = 1.\end{aligned}$$

Proposition 3.7. Let $(\mathcal{G}, \mathcal{G}^*)$ be an (L, M)-double fuzzy grill on X and $f : X \to Y$ be a map. Then the pair $(f^{\Rightarrow}(\mathcal{G}), f^{\Rightarrow}(\mathcal{G}^*))$ of maps $f^{\Rightarrow}(\mathcal{G}), f^{\Rightarrow}(\mathcal{G}^*) : L^Y \to M$ defined by $f^{\Rightarrow}(\mathcal{G})(\mu) = \mathcal{G}(f^{\leftarrow}(\mu))$ and $f^{\Rightarrow}(\mathcal{G}^*)(\mu) = \mathcal{G}^*(f^{\leftarrow}(\mu))$, $\forall \mu \in L^Y$ is an (L, M)-double fuzzy grill on Y.

Proof. (DFG1) $\forall \lambda \in L^Y$,

$$f^{\Rightarrow}(\mathcal{G})(\mu) = \mathcal{G}(f^{\leftarrow}(\mu)) \le (\mathcal{G}^*(f^{\leftarrow}(\mu)))' = (f^{\Rightarrow}(\mathcal{G}^*)(\mu))'.$$

(DFG2) It is easy.
(DFG3) $\forall \lambda, \mu \in L^Y$,

$$\begin{array}{l} f^{\Rightarrow}(\mathcal{G})(\lambda \oplus \mu) = \mathcal{G}(f^{\leftarrow}(\lambda \oplus \mu)) \\ = \mathcal{G}(f^{\leftarrow}(\lambda) \oplus f^{\leftarrow}(\mu)) \qquad \text{(by Lemma 2.4(ii))} \\ \le \mathcal{G}(f^{\leftarrow}(\lambda)) \oplus \mathcal{G}(f^{\leftarrow}(\mu)) = f^{\Rightarrow}(\mathcal{G})(\lambda) \oplus f^{\Rightarrow}(\mathcal{G})(\mu). \end{array}$$

$$\begin{array}{l} f^{\Rightarrow}(\mathcal{G}^*)(\lambda \oplus \mu) = \mathcal{G}^*(f^{\leftarrow}(\lambda \oplus \mu)) \\ = \mathcal{G}^*(f^{\leftarrow}(\lambda) \oplus f^{\leftarrow}(\mu)) \qquad \text{(by Lemma 2.4(ii))} \\ \ge \mathcal{G}^*(f^{\leftarrow}(\lambda)) \odot \mathcal{G}^*(f^{\leftarrow}(\mu)) = f^{\Rightarrow}(\mathcal{G}^*)(\lambda) \odot f^{\Rightarrow}(\mathcal{G}^*)(\mu). \end{array}$$

(DFG4) If $\lambda \le \mu$, then

$$f^{\Rightarrow}(\mathcal{G})(\lambda) = \mathcal{G}(f^{\leftarrow}(\lambda)) \le \mathcal{G}(f^{\leftarrow}(\mu)) = f^{\Rightarrow}(\mathcal{G})(\mu),$$

$$f^{\Rightarrow}(\mathcal{G}^*)(\lambda) = \mathcal{G}^*(f^{\leftarrow}(\lambda)) \ge \mathcal{G}^*(f^{\leftarrow}(\mu)) = f^{\Rightarrow}(\mathcal{G}^*)(\mu).$$

Theorem 3.8. Let $\{(\mathcal{G}_i, \mathcal{G}_i^*) : i \in \Gamma\}$ be a family of (L, M)-double fuzzy grills on X satisfying the following condition:

(C) If $\mathcal{G}_i(\lambda_i) \ne 1_M$ and $\mathcal{G}_i^*(\lambda_i) \ne 0_M$ for all $i \in \Gamma$ we have $\oplus_{k \in K} \lambda_k \ne 1_X$ for every finite index subset K of Γ. We define the maps $\bigwedge_{i \in \Gamma} \mathcal{G}_i, \bigvee_{i \in \Gamma} \mathcal{G}_i^* : L^X \to M$ as follows:

$$\bigwedge_{i \in \Gamma} \mathcal{G}_i(\lambda) = \begin{cases} \bigwedge\{\oplus_{k \in K} \mathcal{G}_k(\lambda_k)\}, & \text{if } \quad \lambda = \oplus_{k \in K} \lambda_k \\ 1_M, & \text{otherwise.} \end{cases}$$

$$\bigvee_{i \in \Gamma} \mathcal{G}_i^*(\lambda) = \begin{cases} \bigvee\{\odot_{k \in K} \mathcal{G}_k^*(\lambda_k)\}, & \text{if } \quad \lambda = \oplus_{k \in K} \lambda_k \\ 0_M, & \text{otherwise.} \end{cases}$$

Where the $\bigwedge$ and $\bigvee$ are taken for every finite index subset K of Γ such that $\lambda = \oplus_{k \in K} \lambda_i$. Then $(\bigwedge_{i \in \Gamma} \mathcal{G}_i, \bigvee_{i \in \Gamma} \mathcal{G}_i^*)$ is the finest (L, M)-double fuzzy grill on X satisfies $\bigwedge_{i \in \Gamma} \mathcal{G}_i \le \mathcal{G}_i$ and $\bigvee_{i \in \Gamma} \mathcal{G}_i^* \ge \mathcal{G}_i^*$, for each $i \in \Gamma$.

Proof. Firstly, we will show that, $(\mathcal{H}, \mathcal{H}^*) = (\bigwedge_{i\in\Gamma} \mathcal{G}_i, \bigvee_{i\in\Gamma} \mathcal{G}_i^*)$ is an (L, M)-double fuzzy grill on X.

(DFG1) For every finite index subsets K of Γ such that $\lambda = \oplus_{k\in K}\lambda_K$,

$$\begin{aligned}\mathcal{H}(\lambda) &= \bigwedge\{\oplus_{k\in K}\mathcal{G}_k(\lambda_k)\} \leq \bigwedge\{\oplus_{k\in K}(\mathcal{G}_k^*(\lambda_k))'\}\\ &= (\bigvee\{\odot_{k\in K}\mathcal{G}_k^*(\lambda_k)\})' = \mathcal{H}^*(\lambda).\end{aligned}$$

(DFG2) Since $0_X = 0_X \oplus 0_X$, we have $\mathcal{H}(0_X) = 0_M$ and $\mathcal{H}^*(0_X) = 1_M$. Suppose that $\mathcal{H}(1_X) \neq 1_M$ and $\mathcal{H}^*(1_X) \neq 0_M$, then there exists a finite index subset K of Γ with $1_X = \oplus_{k\in K}\lambda_k$ such that $\mathcal{H}(1_X) \leq \oplus_{k\in K}\mathcal{G}_k(\lambda_k)$, $\oplus_{k\in K}\mathcal{G}_k(\lambda_k) \neq 1_M$ and $\mathcal{H}^*(1_X) \geq \odot_{k\in K}\mathcal{G}_k^*(\lambda_k)$, $\odot_{k\in K}\mathcal{G}_k^*(\lambda_k) \neq 0_M$. Then,

$$\bigvee_{k\in K} \mathcal{G}_k(\lambda_k) \leq \oplus_{k\in K}\mathcal{G}_k(\lambda_k) \neq 1_M,$$

$$\bigwedge_{k\in K} \mathcal{G}_k^*(\lambda_k) \geq \odot_{k\in K}\mathcal{G}_k^*(\lambda_k) \neq 0_M.$$

Thus, $G_k(\lambda_k) \neq 1_M$ and $G_k^*(\lambda_k) \neq 0_M$, $\forall k \in K$. By condition (C), $\oplus_{k\in K}\lambda_k \neq 1_X$, a contradiction. Hence $\mathcal{H}(1_X) = 1_M$ and $\mathcal{H}^*(1_X) = 0_M$.

(DFG3) For every finite index subsets K and J of Γ such that $\lambda = \oplus_{k\in K}\lambda_K$, $\mu = \oplus_{j\in J}\mu_j$ we have, $\lambda \oplus \mu = (\oplus_{k\in K}\lambda_K) \oplus (\oplus_{j\in J}\mu_j)$. Furthermore, for each $m \in K \cup J$, put $\lambda \oplus \mu = \oplus_{m\in K\cup J}\rho_m$, where

$$\rho_m = \begin{cases} \lambda_m, & \text{if } \ m \in K - (K \cap J) \\ \mu_m, & \text{if } \ m \in J - (K \cap J) \\ \lambda_m \oplus \mu_m, & \text{if } \ m \in K \cap J. \end{cases}$$

We have,

$$\begin{aligned}\mathcal{H}(\lambda \oplus \mu) &\leq \oplus_{m\in K\cup J}\mathcal{G}_m(\rho_m) \leq (\oplus_{k\in K}\mathcal{G}_k(\lambda_k)) \oplus (\oplus_{j\in J}\mathcal{G}_j(\mu_j)),\\ \mathcal{H}^*(\lambda \oplus \mu) &\geq \odot_{m\in K\cup J}\mathcal{G}_m(\rho_m) \geq (\odot_{k\in K}\mathcal{G}_k(\lambda_k)) \odot (\odot_{j\in J}\mathcal{G}_j(\mu_j)).\end{aligned}$$

By (L4′) and (L4), we have

$$\mathcal{H}(\lambda \oplus \mu) \leq \mathcal{H}(\lambda) \oplus \mathcal{H}(\mu)$$

$$\mathcal{H}^*(\lambda \oplus \mu) \geq \mathcal{H}^*(\lambda) \odot \mathcal{H}^*(\mu).$$

(DFG4) Let $\lambda, \mu \in L^X$ with $\lambda \leq \mu$. By the definition of $\mathcal{H}(\mu)$ and $\mathcal{H}^*(\mu)$, there exists a finite index subset J of Γ with $\mu = \oplus_{j\in J}\mu_j$ such that $\mathcal{H}(\mu) \leq \oplus_{j\in J}\mathcal{G}_j(\mu_j)$ and $\mathcal{H}^*(\mu) \geq \odot_{j\in J}\mathcal{G}_j(\mu_j)$. On the other hand since $\lambda = \lambda \wedge \mu = \oplus_{j\in J}(\lambda \wedge \mu_j)$, we have

$$\mathcal{H}(\lambda) \leq \oplus_{j\in J}\mathcal{G}_j(\lambda \wedge \mu_j) \leq \oplus_{j\in J}\mathcal{G}_j(\mu_j),$$

$$\mathcal{H}^*(\lambda) \geq \odot_{j\in J}\mathcal{G}_j^*(\lambda \wedge \mu_j) \geq \odot_{j\in J}\mathcal{G}_j^*(\mu_j).$$

By (L4′) and (L4), we have $\mathcal{H}(\lambda) \leq \mathcal{H}(\mu)$ and $\mathcal{H}^*(\lambda) \geq \mathcal{H}^*(\mu)$.

We will show that $\mathcal{H}(\lambda) \leq \mathcal{G}_i(\lambda)$ and $\mathcal{H}^*(\lambda) \geq \mathcal{G}_i^*(\lambda)$, $\forall i \in \Gamma$, from the following:

If $\mathcal{G}_i(\lambda) = 1_M$ and $\mathcal{G}_i^*(\lambda) = 0_M$, it is trivial.

If $\mathcal{G}_i(\lambda) \neq 1_M$ and $\mathcal{G}_i^*(\lambda) \neq 0_M$, for $\lambda = \lambda \oplus 0_X$, we have

$$\mathcal{H}(\lambda) \leq \mathcal{G}_i(\lambda \oplus 0_X) \leq \mathcal{G}_i(\lambda) \oplus \mathcal{G}_i(0_X) = \mathcal{G}_i(\lambda),$$

$$\mathcal{H}^*(\lambda) \geq \mathcal{G}_i^*(\lambda \oplus 0_X) \geq \mathcal{G}_i^*(\lambda) \odot \mathcal{G}_i^*(0_X) = \mathcal{G}_i^*(\lambda).$$

Now, let $(\mathcal{K}, \mathcal{K}^*)$ be another (L, M)-double fuzzy grill satisfies $\mathcal{K} \leq \mathcal{G}_i$ and $\mathcal{K}^* \geq \mathcal{G}_i^*$, $\forall i \in \Gamma$. We will show that $\mathcal{K}(\lambda) \leq \mathcal{H}(\lambda)$ and $\mathcal{K}^*(\lambda) \geq \mathcal{H}^*(\lambda)$, $\forall \lambda \in L^X$. By the definition of $\mathcal{H}(\lambda)$ and $\mathcal{H}^*(\lambda)$, there exists a finite index subset K of Γ with $\lambda = \oplus_{k\in K}\lambda_k$ such that $\mathcal{H}(\lambda) \leq \oplus_{k\in K}\mathcal{G}_k(\lambda_k)$ and $\mathcal{H}^*(\lambda) \geq \odot_{k\in K}\mathcal{G}_k^*(\lambda_k)$. On the other hand, since $\mathcal{K} \leq \mathcal{G}_i$ and $\mathcal{K}^* \geq \mathcal{G}_i^*$, $\forall i \in \Gamma$, we have:

$$\mathcal{K}(\lambda) = \mathcal{K}(\oplus_{k\in K}\lambda_k) \leq \oplus_{k\in K}\mathcal{K}(\lambda_k) \leq \oplus_{k\in K}\mathcal{G}_k(\lambda_k),$$

$$\mathcal{K}^*(\lambda) = \mathcal{K}^*(\oplus_{k\in K}\lambda_k) \geq \odot_{k\in K}\mathcal{K}^*(\lambda_k) \geq \odot_{k\in K}\mathcal{G}_k^*(\lambda_k).$$

By (L4′) and (L4), we have $\mathcal{K}(\lambda) \leq \mathcal{H}(\lambda)$ and $\mathcal{K}^*(\lambda) \geq \mathcal{H}^*(\lambda)$, $\forall \lambda \in L^X$.

Proposition 3.9. Let $f : X \to Y$ be a map and $(\mathcal{G}, \mathcal{G}^*)$ be an (L, M)-double fuzzy grill on Y. Define the maps $f^{\Leftarrow}(\mathcal{G}), f^{\Leftarrow}(\mathcal{G}^*) : L^X \to M$ as follows:

$$f^{\Leftarrow}(\mathcal{G})(\mu) = \begin{cases} \bigwedge\{\mathcal{G}(\nu) : \mu = f^{\leftarrow}(\nu)\}, & \text{if} \quad \mu \neq 1_X \\ 1_M, & \text{if} \quad \mu = 1_X, \end{cases}$$

$$f^{\Leftarrow}(\mathcal{G}^*)(\mu) = \begin{cases} \bigvee\{\mathcal{G}^*(\nu) : \mu = f^{\leftarrow}(\nu)\}, & \text{if} \quad \mu \neq 1_X \\ 0_M, & \text{if} \quad \mu = 1_X. \end{cases}$$

Then $(\phi^{\Leftarrow}(\mathcal{G}), \phi^{\Leftarrow}(\mathcal{G}^*))$ is an (L, M)-double fuzzy grill on X.

Proof. (DFG1) $\forall \mu \in L^X$

$$\begin{aligned} f^{\Leftarrow}(\mathcal{G})(\mu) &= \bigwedge\{\mathcal{G}(\nu) : \mu = f^{\leftarrow}(\nu)\} \geq \bigwedge\{(\mathcal{G}^*(\nu))' : \mu = f^{\leftarrow}(\nu)\} \\ &= (\bigvee\{\mathcal{G}^*(\nu) : \mu = f^{\leftarrow}(\nu)\})' = (f^{\Leftarrow}(\mathcal{G}^*)(\mu))'. \end{aligned}$$

(DFG2) It is clear.

(DFG3) $\forall \lambda, \mu \in L^X$,

$$\begin{aligned}
&f^{\Leftarrow}(\mathcal{G})(\lambda) \oplus f^{\Leftarrow}(\mathcal{G})(\mu) \\
&= (\bigwedge\{\mathcal{G}(\lambda_1) : \lambda = f^{\leftarrow}(\lambda_1)\}) \oplus (\bigwedge\{\mathcal{G}(\mu_1) : \mu = f^{\leftarrow}(\mu_1)\}) \\
&= \bigwedge\{\mathcal{G}(\lambda_1) \oplus \mathcal{G}(\mu_1) : \lambda = f^{\leftarrow}(\lambda_1), \mu = f^{\leftarrow}(\mu_1)\} \\
&\geq \bigwedge\{\mathcal{G}(\lambda_1 \oplus \mu_1) : \lambda \oplus \mu = f^{\leftarrow}(\lambda_1) \oplus f^{\leftarrow}(\mu_1)\} \\
&= \bigwedge\{\mathcal{G}(\lambda_1 \oplus \mu_1) : \lambda \oplus \mu = f^{\leftarrow}(\lambda_1 \oplus \mu_1)\} \\
&= f^{\Leftarrow}(\mathcal{G})(\lambda \oplus \mu).
\end{aligned}$$

$$\begin{aligned}
&f^{\Leftarrow}(\mathcal{G}^*)(\lambda) \odot f^{\Leftarrow}(\mathcal{G}^*)(\mu) \\
&= (\bigvee\{\mathcal{G}^*(\lambda_1) : \lambda = f^{\leftarrow}(\lambda_1)\}) \odot (\bigvee\{\mathcal{G}^*(\mu_1) : \mu = f^{\leftarrow}(\mu_1)\}) \\
&= \bigvee\{\mathcal{G}^*(\lambda_1) \odot \mathcal{G}^*(\mu_1) : \lambda = f^{\leftarrow}(\lambda_1), \mu = f^{\leftarrow}(\mu_1)\} \\
&\leq \bigvee\{\mathcal{G}^*(\lambda_1 \oplus \mu_1) : \lambda \oplus \mu = f^{\leftarrow}(\lambda_1) \oplus f^{\leftarrow}(\mu_1)\} \\
&= \bigvee\{\mathcal{G}^*(\lambda_1 \oplus \mu_1) : \lambda \oplus \mu = f^{\leftarrow}(\lambda_1 \oplus \mu_1)\} \\
&= f^{\Leftarrow}(\mathcal{G}^*)(\lambda \oplus \mu).
\end{aligned}$$

(DFG4) Let $\lambda, \mu \in L^X$ with $\lambda \leq \mu$. Then

$$\begin{aligned}
f^{\Leftarrow}(\mathcal{G})(\lambda) &= \bigwedge\{\mathcal{G}(\nu) : \nu = f^{\leftarrow}(\lambda)\} \\
&\leq \bigwedge\{\mathcal{G}(\nu) : \nu = f^{\leftarrow}(\mu)\} = f^{\Leftarrow}(\mathcal{G})(\mu),
\end{aligned}$$

$$\begin{aligned}
f^{\Leftarrow}(\mathcal{G}^*)(\lambda) &= \bigvee\{\mathcal{G}^*(\nu) : \nu = f^{\leftarrow}(\lambda)\} \\
&\geq \bigvee\{\mathcal{G}^*(\nu) : \nu = f^{\leftarrow}(\mu)\} = f^{\Leftarrow}(\mathcal{G}^*)(\mu).
\end{aligned}$$

Theorem 3.10. Let $\{(\mathcal{G}_i, \mathcal{G}_i^*) : i \in \Gamma\}$ be a family of (L, M)-double fuzzy grills on X_i. Let $X = \Pi_{i\in\Gamma} X_i$ be a product set and $\pi_i : X \to X_i$ a projection map, for each $i \in \Gamma$. We define the maps $\mathcal{G}, \mathcal{G}^* : L^X \to M$ as follows:

$$\mathcal{G}(\mu) = \begin{cases} \bigwedge\{\oplus_{i\in K}\mathcal{G}_i(\mu_i)\}, & \text{if } \quad \mu = \oplus_{i\in K}\pi_i^{\leftarrow}(\mu_i) \\ 1_M, & \text{otherwise,} \end{cases}$$

$$\mathcal{G}^*(\mu) = \begin{cases} \bigvee\{\odot_{i\in K}\mathcal{G}_i^*(\mu_i)\}, & \text{if } \quad \mu = \oplus_{i\in K}\pi_i^{\leftarrow}(\mu_i) \\ 0_M, & \text{otherwise,} \end{cases}$$

where the $\bigwedge$ and $\bigvee$ are taken for every finite index subset K of Γ such that $\mu = \oplus_{i\in K}\pi_i^{\leftarrow}(\mu_i)$. Then:

(i) For $\mu \in L^X$, we have $\mathcal{G}(\mu) = \bigwedge_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i)(\mu)$ and $\mathcal{G}^*(\mu) = \bigvee_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i^*)(\mu)$.

(ii) $(\mathcal{G}, \mathcal{G}^*)$ is the finest (L, M)-double fuzzy grill on X for which each projection map $\pi_i : X \to X_i$ is a double g-map.

(iii) A map $f : (Y, \mathcal{H}, \mathcal{H}^*) \to (X, \mathcal{G}, \mathcal{G}^*)$ is a double g-map if and only if $\pi_i \circ f : (Y, \mathcal{H}, \mathcal{H}^*) \to (X_i, \mathcal{G}_i, \mathcal{G}^*)$ is a double g-map, for each $i \in \Gamma$.

Proof. (i) From Proposition 3.9, each $(\pi_i^{\Leftarrow}(\mathcal{G}_i), \pi_i^{\Leftarrow}(\mathcal{G}_i^*))$ is an (L, M)-double fuzzy grill on X. Firstly, we show that $\bigwedge_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i)$ and $\bigvee_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i^*)$ are exist, that is, they satisfy the condition (C) of Theorem 3.8.

(C) If $\pi_i^{\Leftarrow}(\mathcal{G}_i)(\mu_i) \neq 1_M$ and $\pi_i^{\Leftarrow}(\mathcal{G}_i^*)(\mu_i) \neq 0_M$ for all $i \in \Gamma$, then there exists ν_i with $\mu_i = \pi_i^{\leftarrow}(\nu_i)$ such that $\mathcal{G}_i(\nu_i) \neq 1_M$ and $\mathcal{G}_i^*(\nu_i) \neq 0_M$. It implies that $\nu_i \neq 1_X$, that is, there exists $x_i \in X_i$ with $\nu_i(x_i) \neq 1_L$. For every finite index subset K of Γ, put

$$x = \begin{cases} \pi_k(x) = x_k, & \text{if} \quad x_k \in X_k \quad \text{for each } k \in K, \\ \pi_j(x) = x_j, & \text{if} \quad x_j \in X_j \quad \text{for each } j \in \Gamma - K. \end{cases}$$

Then we have

$$\oplus_{i\in K}\mu_i(x) = \oplus_{i\in K}\pi_i^{\leftarrow}(\nu_i)(x) = \oplus_{i\in K}\nu_i(x_i) \neq 1_L.$$

Now, we will show that $\mathcal{G} = \bigwedge_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i)$ and $\mathcal{G}^* = \bigvee_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i^*)$. For every finite index subset K of Γ with $\mu = \oplus_{k\in K}\pi_k^{\leftarrow}(\mu_k)$ we have:

$$\begin{aligned} &\bigwedge_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i)(\mu) = \bigwedge_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i)(\oplus_{k\in K}\pi_k^{\leftarrow}(\mu_k)). \\ &\leq \pi_k^{\Leftarrow}(\mathcal{G}_k)(\pi_k^{\leftarrow}(\oplus_{k\in K}\mu_k)) \\ &\leq \mathcal{G}_k(\oplus_{k\in K}\mu_k) \qquad \text{(by Proposition 3.9)} \\ &\leq \oplus_{k\in K}\mathcal{G}_k(\mu_k), \end{aligned}$$

$$\begin{aligned} &\bigvee_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i^*)(\mu) = \bigvee_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i^*)(\oplus_{k\in K}\pi_k^{\leftarrow}(\mu_k)). \\ &\geq \pi_k^{\Leftarrow}(\mathcal{G}_k^*)(\pi_k^{\leftarrow}(\oplus_{k\in K}\mu_k)) \\ &\geq \mathcal{G}_k^*(\oplus_{k\in K}\mu_k) \qquad \text{(by Proposition 3.9)} \\ &\geq \odot_{k\in K}\mathcal{G}_k^*(\mu_k) \end{aligned}$$

By (L4′) and (L4), we have

$$\bigwedge_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i)(\mu) \leq \mathcal{G}(\mu)$$

$$\bigvee_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i^*)(\mu) \geq \mathcal{G}^*(\mu). \tag{3.1}$$

By Theorem 3.8, for every finite index subset J of Γ with $\mu = \oplus_{j\in J}\mu_j$, we have:

$$\bigwedge_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i)(\mu) \leq \oplus_{j\in J}\pi_j^{\Leftarrow}(\mathcal{G}_j)(\mu_j)$$

$$\bigvee_{i\in\Gamma} \pi_i^{\Leftarrow}(\mathcal{G}_i^*)(\mu) \geq \odot_{j\in J}\pi_j^{\Leftarrow}(\mathcal{G}_j^*)(\mu_j).$$

There exists $\nu_j \in L^{X_j}$ with $\mu_j = \pi_j^{\leftarrow}(\nu_j)$ such that

$$\oplus_{j\in J}\pi_j^{\Leftarrow}(\mathcal{G}_j)(\mu_j) \leq \oplus_{j\in J}\mathcal{G}_j(\nu_j)$$

$$\odot_{j\in J}\pi_j^{\Leftarrow}(\mathcal{G}_j^*)(\mu_j) \geq \odot_{j\in J}\mathcal{G}_j^*(\nu_j).$$

Then,

$$\bigwedge_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{G}_i)(\mu) \leq \oplus_{j\in J}\mathcal{G}_j(\nu_j)$$

$$\bigvee_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{G}_i^*)(\mu) \geq \odot_{j\in J}\mathcal{G}_j^*(\nu_j).$$

On the other hand for $\mu = \oplus_{j\in J}\mu_j = \oplus_{j\in J}\pi_j^{\leftarrow}(\nu_j)$, we have:

$$\mathcal{G}(\mu) \leq \oplus_{j\in J}\mathcal{G}_j(\nu_j) \text{ and } \mathcal{G}^*(\mu) \geq \odot_{j\in J}\mathcal{G}_j^*(\nu_j).$$

By (L4$'$) and (L4), we have

$$\mathcal{G}(\mu) \leq \bigwedge_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{G}_i)(\mu) \text{ and } \mathcal{G}^*(\mu) \geq \bigvee_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{G}_i^*)(\mu). \tag{3.2}$$

From (3.1) and (3.2), we have

$$\mathcal{G}(\mu) = \bigwedge_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{G}_i)(\mu), \quad \mathcal{G}^*(\mu) = \bigvee_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{G}_i^*)(\mu).$$

(ii) From (i), Theorems 3.8, and Proposition 3.9, $(\mathcal{G}, \mathcal{G}^*)$ is an (L, M)-double fuzzy grill on X. For each $i \in \Gamma$ and $\nu_i \in L^{X_i}$, by the definition of $\mathcal{G}$ and $\mathcal{G}^*$, we have

$$\mathcal{G}(\pi_i^{\leftarrow}(\nu_i)) \leq \mathcal{G}_i(\nu_i),\ \mathcal{G}^*(\pi_i^{\leftarrow}(\nu_i)) \geq \mathcal{G}_i^*(\nu_i).$$

Hence, $\pi_i : (X, \mathcal{G}, \mathcal{G}^*) \to (X_i, \mathcal{G}_i, \mathcal{G}_i^*)$ is a double g-map.

Let $\pi_i : (X, \mathcal{K}, \mathcal{K}^*) \to (X_i, \mathcal{G}_i, \mathcal{G}_i^*)$ be a double g-map for each $i \in \Gamma$, that is

$$\mathcal{K}(\pi_i^{\leftarrow}(\nu_i)) \leq \mathcal{G}_i(\nu_i) \text{ and } \mathcal{K}^*(\pi_i^{\leftarrow}(\nu_i)) \geq \mathcal{G}_i^*(\nu_i).$$

For all finite index subset K of Γ with $\mu = \oplus_{k\in K}\pi_k^{\leftarrow}(\nu_k)$, we have

$$\begin{aligned} \mathcal{K}(\mu) &= \mathcal{K}(\oplus_{k\in K}\pi_k^{\leftarrow}(\nu_k)) \leq \oplus_{k\in K}\mathcal{K}(\pi_k^{\leftarrow}(\nu_k)) \leq \oplus_{k\in K}\mathcal{G}_k(\nu_k), \\ \mathcal{K}^*(\mu) &= \mathcal{K}^*(\oplus_{k\in K}\pi_k^{\leftarrow}(\nu_k)) \geq \odot_{k\in K}\mathcal{K}^*(\pi_k^{\leftarrow}(\nu_k)) \geq \odot_{k\in K}\mathcal{G}_k^*(\nu_k). \end{aligned}$$

By (L4$'$) and (L4), we have $\mathcal{K}(\mu) \leq \mathcal{G}(\mu)$ and $\mathcal{K}^*(\mu) \geq \mathcal{G}^*(\mu)$, for all $\mu \in L^X$.

(iii) For each $\mu \in L^{X_i}$, we have

$$\begin{aligned}&\mathcal{H}(\pi_i \circ f)^{\leftarrow}(\mu) = \mathcal{H}(f^{\leftarrow}(\pi_i^{\leftarrow}(\mu)))\\&\leq \mathcal{G}(\pi_i^{\leftarrow}(\mu)) \qquad \text{(since } f \text{ is a double } g\text{-map)}\\&\leq \mathcal{G}_i(\mu), \qquad \text{(since } \pi_i \text{ is a double } g\text{-map)}\end{aligned}$$

$$\begin{aligned}&\mathcal{H}^*(\pi_i \circ f)^{\leftarrow}(\mu) = \mathcal{H}^*(f^{\leftarrow}(\pi_i^{\leftarrow}(\mu)))\\&\geq \mathcal{G}^*(\pi_i^{\leftarrow}(\mu)) \quad \text{(since } f \text{ is a double } g\text{-map)}\\&\geq \mathcal{G}_i^*(\mu). \qquad \text{(since } \pi_i \text{ is a double } g\text{-map)}\end{aligned}$$

Then, $\pi_i \circ f$ is a double g-map.

Conversely, for every finite index subset K of Γ with $\mu = \oplus_{k\in K}\pi_k^{\leftarrow}(\nu_k)$, since for each $i \in \Gamma$, $\pi_i \circ f : (Y, \mathcal{H}, \mathcal{H}^*) \to (X_i, \mathcal{G}_i, \mathcal{G}_i^*)$ is a double g-map, for all $\nu_i \in L^{X_i}$,

$$\mathcal{H}(f^{\leftarrow}(\pi_i^{\leftarrow}(\nu_i))) \leq \mathcal{G}_i(\nu_i),\ \mathcal{H}^*(f^{\leftarrow}(\pi_i^{\leftarrow}(\nu_i))) \geq \mathcal{G}_i^*(\nu_i).$$

It follows, for all $k \in K$,

$$\mathcal{H}(f^{\leftarrow}(\pi_k^{\leftarrow}(\nu_k))) \leq \mathcal{G}_k(\nu_k),\ \mathcal{H}^*(f^{\leftarrow}(\pi_k^{\leftarrow}(\nu_k))) \geq \mathcal{G}_k^*(\nu_k).$$

This implies that:

$$\begin{aligned}&\mathcal{H}(f^{\leftarrow}(\mu)) = \mathcal{H}(f^{\leftarrow}(\oplus_{k\in K}\pi_k^{\leftarrow}(\nu_k)))\\&= \mathcal{H}(\oplus_{k\in K}f^{\leftarrow}(\pi_k^{\leftarrow}(\nu_k))) \qquad \text{(by Lemma 2.4(ii))}\\&\leq \oplus_{k\in K}\mathcal{H}(f^{\leftarrow}(\pi_k^{\leftarrow}(\nu_k)))\\&\leq \oplus_{k\in K}\mathcal{G}_k(\nu_k),\end{aligned}$$

$$\begin{aligned}&\mathcal{H}^*(f^{\leftarrow}(\mu)) = \mathcal{H}^*(f^{\leftarrow}(\oplus_{k\in K}\pi_k^{\leftarrow}(\nu_k)))\\&= \mathcal{H}^*(\oplus_{k\in K}f^{\leftarrow}(\pi_k^{\leftarrow}(\nu_k))) \qquad \text{(by Lemma 2.4(ii))}\\&\geq \odot_{k\in K}\mathcal{H}^*(f^{\leftarrow}(\pi_k^{\leftarrow}(\nu_k)))\\&\geq \odot_{k\in K}\mathcal{G}_k^*(\nu_k).\end{aligned}$$

By (L4$'$) and (L4), for all $\mu \in L^X$,

$$\mathcal{H}(f^{\leftarrow}(\mu)) \leq \mathcal{G}(\mu),\ \ \mathcal{H}^*(f^{\leftarrow}(\mu)) \geq \mathcal{G}^*(\mu)$$

Hence $f : (Y, \mathcal{H}, \mathcal{H}^*) \to (X, \mathcal{G}, \mathcal{G}^*)$ is a double g-map.

From Theorem 3.10, we can define a product of (L, M)-double fuzzy grills.

Definition 3.11. Let $\{(\mathcal{G}_i, \mathcal{G}_i^*)\}_{i\in\Gamma}$ be a family of (L, M)-double fuzzy grills on X_i, $X = \Pi_{i\in\Gamma}X_i$ a product set and for each $i \in \Gamma$, $\pi_i : X \to X_i$ a projection map. The product of (L, M)-double fuzzy grills is the finest (L, M)-double fuzzy grill on X for which all $\pi_i, i \in \Gamma$ are double g-maps.

References

[1] S.E. Abbas, Images and preimages of (L, M)-grillbases, *Jour Hacet. J. Math. Stat.* **40** (5) (2011), 737-747.

[2] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, **20** (1) (1986), 87-96.

[3] K. Atanassov, New operator defined over the intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, **61** (1993), 131-142.

[4] R. Bělohlávek,*Fuzzy Relational Systems*, Kluwer Academic Publishers, New York (2002), **doi:** 10.1007/978-1-4615-0633-1.

[5] V. Çetkin and H. Aygün, On (L, M)-double fuzzy ideals, *International Journal of Fuzzy Systems,* **14** (1), (2012), 166-174.

[6] V. Çetkin and H. Aygün, On double fuzzy preuniformity, *J. Nonlinear Sci. Appl.,* **6** (2013), 263-278.

[7] K.C. Chattopadhyay, U.K. Mukherjee, S.K. Samanta, Fuzzy proximities structures and fuzzy grills, *Fuzzy Sets and Systems,* **79** (1996), 383-393 .

[8] G. Choquet, Sur les notions de Filtre et grille, *Comptes Rendus Acad. Sci.,* **224** (1974), 171-173.

[9] J.G. Garcia, S.E. Rodabaugh, Order-theoretic, topological, categorical redundancides of interval-valued sets, grey sets, vague sets, interval-valued "intuitionistic" sets, "intuitionistic" fuzzy sets and topologies, *Fuzzy Sets and Systems,* **156** (2005), 445-484, **doi:** 10.1016/j.fss.2005.05.023.

[10] U. Höhle, S.E. Rodabaugh, *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory*, The Handbooks of Fuzzy Sets Series 3, Kluwer Academic Publishers, Boston, 1999, **doi:** 10.1007/978-1-4615-5079-2.

[11] M. Khare, R. Singh, Complete ξ-grills and (L, n)-merotopies, *Fuzzy Sets and Systems,* **159** (2008), 620-628, **doi:** 10.1016/j.fss.2007.10.007.

[12] Y.C. Kim, J.M. Ko, Images and preimages of L-filterbases, *Fuzzy Sets and Systems,* **157** (2006), 1913-1927, **doi:** 10.1016/j.fss.2005.12.012.

[13] M.N. Mukherjee, S. Das, Fuzzy grills and induced fuzzy topology, *Mat. Vesn.,* **62(4)** (2010), 285-297.

[14] C.J. Mulvey, Quantales, *Suppl. Rend. Cric. Mat. Palermo Ser. II,* **12** (1986), 99-104.

[15] S.E. Rodabaugh, E.P. Klement, *Topological and Algebraic Structures In Fuzzy Sets*, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003, **doi:** 10.1007/978-94-017-0231-7.

[16] B. Roy, M.N. Mukherjee, On a typical topology induced by a grill, *Soochow Journal of Athematics,* **33** (4) (2007), 771-786.

[17] B. Roy, M.N. Mukherjee, Concerning topologies induced by principal grills, *An. Stiint. Univ. Al. I. Cuza Iasi, Ser. Noua, Mat.,* **55** (2) (2009), 285-294.