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ON MULTIPLICATIVE (GENERALIZED)-DERIVATIONS IN SEMIPRIME RINGS

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Abstract: Let R be a ring. A map $F: R \to R$ (not necessarily additive) is called a multiplicative (generalized)-derivation of R if $F(xy) = F(x)y + xf(y)$ for all $x, y \in$ R, where $f: R \to R$ is any map (not necessarily a derivation nor additive). The main purpose of this paper is to study the following situations (i)$F[x,y] \pm xy = 0$ (ii)$F[x,y] \pm yx = 0$ (iii)$F(x \circ y) \pm xy = 0$ (iv) $F(x \circ y) \pm yx = 0$ (v)$f(x)F(y) \pm xy = 0$ (vi)$f(x)F(y) \pm yx = 0$ (vii)$[F(x), y] \pm x \circ G(y) = 0$ (viii)$F(x) \circ y \pm x \circ G(y) = 0$, for all x, y in some appropriate subsets of R.

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1. Introduction

Let R denote an associative ring with center $Z(R)$. For any $x, y \in R$, we write for commutator $[x, y] = xy - yx$ and the anti-commutator $x \circ y = xy + yx$. A ring R is called prime ring if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and is called semiprime if $aRa = (0)$ implies that $a = 0$. Let S be a non-empty subset of R. A map $f: S \to R$ is called a centralizing (or commuting) map on S if $[f(x), x] \in Z(R)$ (or $[f(x), x] = 0$) for all $x \in S$. An additive map $d: R \to R$ is called a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive map $F: R \to R$ associated with a derivation $d: R \to R$

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is called a generalized derivation of R if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. If $d = 0$ then we have $F(xy) = F(x)y$, which is called a left multiplier mapping of R. In 1991, Brešar [4] introduced generalized derivation and Havala [11] gave an algebraic study of generalized derivations of prime rings. Clearly, every derivation is a generalized derivation of R. Thus generalized derivation covers both the concepts of left multiples and the concept of derivation.

The idea of multiplicative derivation was introduced in 1991 by Daif [7] as follows: A map $d : R \to R$ (not necessarily additive) is called a multiplicative derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. The complete description of these mappings was provided by Goldman and Šemrl [10]. The notion of derivation was extended to multiplicative generalized derivation by Daif and Tammam-El-Sayiad [6] as follows: A map $F : R \to R$ is said to be a multiplicative generalized derivation if there exist a derivation $d : R \to R$ such that $F(xy) = F(x)y+xd(y)$ for all $x, y \in R$. Further, Dhara and Ali[8] generalize this definition of multiplicative generalized derivation by considering d as any map from R to R. They defined in [8] that a map $F : R \to R$ (not necessarily additive) is called multiplicative (generalized)-derivation if $F(xy) = F(x)y + xf(y)$ holds for all $x, y \in R$, where $f : R \to R$ is any map (not necessarily a derivation nor additive). It is obvious that every generalized derivation is multiplicative (generalized)-derivation on R. However, the converse need not be true in general (viz. [8]).

Daif and Bell[5] proved that if a semiprime ring R admits a derivation d such that $d[x, y] \pm [x, y] = 0$ for all x, y in a non-zero ideal I of R, then R is commutative. Hongan[12] generalized these results by taking the same situations in the center of the ring R. Asma Ali et al. [2] investigated the commutativity of a prime ring admitting a generalized derivation satisfying any one of the following identities (i)$F([x, y]) \pm [x, y] \in Z(R)$ (ii)$F(x \circ y) \pm (x \circ y) \in Z(R)$ for all x, y in some appropriate subset of R. Recently, Ali et al.[3] proceed it further by taking some more identities admitting multiplicative (generalized)-derivation in prime and semiprime rings. For more references of the related work, the reader can see (Asma Ali et al.[1],Dhara and Ali[8],Khan[13], Ali et al.[3], Dhara et al.[9]). Inspired by the work of Ali et al.[3] and Khan[13], we study similar situations admitting multiplicative (generalized)-derivation on a semiprime ring.

We shall frequently use the follwing basic commutator and anti-commutator identities in the proofs of our results: $[xy, z] = x[y, z]+[x, z]y$, $[x, yz] = y[x, z]+[x, y]z$,$x \circ yz = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$, $xy \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$ for all $x, y, z \in R$.

2. Main Results

Theorem 1. *Let R be a semiprime ring and I a non zero left ideal of R. If $F : R \to R$ is a multiplicative (generalized)-derivation associated with a map $f : R \to R$ such that $F[x, y] \pm xy = 0$ for all $x, y \in I$, then $I[x, f(x)] = (0)$ for all $x \in I$.*

Proof. Assume that

$$F[x, y] \pm xy = 0 \tag{1}$$

for all $x, y \in I$. Replace y by yx and using (1), we obtain

$$[x, y]f(x) = 0 \tag{2}$$

for all $x, y \in I$. Substituting $f(x)y$ for y and using (2), we get

$$[x, f(x)]yf(x) = 0 \tag{3}$$

for all $x, y \in I$. Replace y by yx in (3), we obtain

$$[x, f(x)]yxf(x) = 0 \tag{4}$$

for all $x, y \in I$. Right multiply (3) by x and subtract it from (4), we get $[x, f(x)]y[x, f(x)] = 0$ for all $x, y \in I$. On replacing y by ry, we obtain $[x, f(x)]ry[x, f(x)] = 0$ for all $x, y \in I, r \in R$ and hence, $y[x, f(x)]Ry[x, f(x)] = (0)$ for all $x, y \in I$. The semiprimeness of R yields that, $y[x, f(x)] = 0$ for all $x, y \in I$. Therefore,$I[x, f(x)] = (0)$ for all $x \in I$. □

Theorem 2. *Let R be a semiprime ring and I a non zero left ideal of R. If $F : R \to R$ is a multiplicative (generalized)-derivation associated with a map $f : R \to R$ such that $F[x, y] \pm yx = 0$ for all $x, y \in I$, then $I[x, f(x)] = (0)$ for all $x \in I$.*

Proof. Assume that

$$F[x, y] \pm yx = 0 \tag{5}$$

for all $x, y \in I$. Replace y by yx and using (5), we obtain $[x, y]f(x) = 0$ for all $x, y \in I$. Further, proceed as Theorem 1, we obtain $I[x, f(x)] = (0)$ for all $x \in I$. □

Theorem 3. *Let R be a semiprime ring and I a non zero left ideal of R. If $F : R \to R$ is a multiplicative (generalized)-derivation associated with a map $f : R \to R$ such that $F(x \circ y) \pm xy = 0$ for all $x, y \in I$, then $I[x, f(x)] = (0)$ for all $x \in I$.*

Proof. We assume that,

$$F(x \circ y) \pm xy = 0 \tag{6}$$

for all $x, y \in I$. Replace y by yx and using (6), we get

$$(x \circ y)f(x) = 0 \tag{7}$$

for all $x, y \in I$. Substituting $f(x)y$ for y and using (7), we obtain

$$[x, f(x)]yf(x) = 0 \tag{8}$$

for all $x, y \in I$. Right multiply (8) by x, we get

$$[x, f(x)]yf(x)x = 0 \tag{9}$$

for all $x, y \in I$. Replace y by yx in (8), we obtain

$$[x, f(x)]yxf(x) = 0 \tag{10}$$

for all $x, y \in I$. Subtract (9) from (10), we get $[x, f(x)]y[x, f(x)] = 0$ for all $x, y \in I$. On replacing y by ry, we obtain $[x, f(x)]ry[x, f(x)]$ $= 0$ for all $x, y \in I, r \in R$ and hence, $y[x, f(x)]Ry[x, f(x)] = (0)$ for all $x, y \in I$. Now, Semiprimeness of R forces to $y[x, f(x)] = 0$ for all $x, y \in I$. $I[x, f(x)] =$ (0) for all $x \in I$. □

Theorem 4. *Let R be a semiprime ring and I a non zero left ideal of R. If $F : R \to R$ is a multiplicative (generalized)-derivation associated with a map $f : R \to R$ such that $F(x \circ y) \pm yx = 0$ for all $x, y \in I$, then $I[x, f(x)] = (0)$ for all $x \in I$.*

Proof. We assume that,

$$F(x \circ y) \pm yx = 0 \tag{11}$$

for all $x, y \in I$. Replace y by yx and using (11), we get

$$(x \circ y)f(x) = 0 \tag{12}$$

for all $x, y \in I$. Further, proceed as Theorem 3, we obtain $I[x, f(x)]$ $= (0)$, for all $x \in I$. □

Corollary 5. *Let R be a semiprime ring admitting a multiplicative (generalized)-derivation $F : R \to R$ associated with a map $f : R \to R$. If R satisfies any one of the following identities*

1. $F[x,y] \pm xy = 0$
2. $F[x,y] \pm yx = 0$
3. $F(x \circ y) \pm xy = 0$
4. $F(x \circ y) \pm yx = 0$

for all $x, y \in R$,then the map f is a commuting map on R.

Theorem 6. *Let R be a semiprime ring and I a non zero left ideal of R. If $F : R \to R$ is a multiplicative (generalized)-derivation associated with a map $f : R \to R$ such that $f(x)F(y) \pm xy = 0$ for all $x, y \in I$,then $I[x, f(x)]_2 = (0)$ for all $x \in I$.*

Proof. We assume that

$$f(x)F(y) \pm xy = 0 \tag{13}$$

for all $x, y \in I$. Replace y by yx and using (13) we get

$$f(x)yf(x) = 0 \tag{14}$$

for all $x, y \in I$. Substituting $[x, f(x)]y$ for y in (14), we get

$$f(x)[x, f(x)]yf(x) = 0 \tag{15}$$

for all $x, y \in I$. Left multiply (10) by $[x, f(x)]$ and subtract (15) from it,we obtain

$$[[x, f(x)], f(x)]yf(x) = 0 \tag{16}$$

for all $x, y \in I$. Again replacing y by yx in (16), we get

$$[[x, f(x)], f(x)]yxf(x) = 0 \tag{17}$$

for all $x, y \in I$. Right multiply (16) by x and subtract it from (17), we get

$$[[x, f(x)], f(x)]y[x, f(x)] = 0 \tag{18}$$

for all $x, y \in I$. Right multiply (16) by $[x, f(x)]$ we get

$$[[x, f(x)], f(x)]yf(x)[x, f(x)] = 0 \tag{19}$$

for all $x, y \in I$. Right multiply (18) by $f(x)$ we get

$$[[f(x), x], f(x)]y[x, f(x)]f(x) = 0 \tag{20}$$

for all $x, y \in I$. Subtract (19) from (20) we get $[[x, f(x)], f(x)]y[[x, f(x)], f(x)] = 0$ for all $x, y \in I$. On replacing y by ry, we obtain $[[x, f(x), f(x)]]ry[[x, f(x)], f(x)] = 0$ for all $x, y \in I, r \in R$ and hence, $y[[x, f(x)], f(x)]Ry[[x, f(x)], f(x)] = (0)$ for all $x, y \in I$. The semiprimeness of R implies $y[[x, f(x)], f(x)] = 0$ for all $x, y \in I$. That is, $I[x, f(x)]_2 = (0)$ for all $x \in I$. □

Theorem 7. *Let R be a semiprime ring and I a non zero left ideal of R. If $F : R \to R$ is a multiplicative (generalized)-derivation associated with a map $f : R \to R$ such that $f(x)F(y) \pm yx = 0$ for all $x, y \in I$,then $I[x, f(x)]_2 = 0$ for all $x \in I$.*

Proof. Assume that,

$$f(x)F(y) \pm yx = 0 \tag{21}$$

for all $x, y \in I$. Replace y by yx and using (21),we get $f(x)yf(x) = 0$ for all $x, y \in I$. Further, the proof follows from Theorem 6, after equation (14). □

Corollary 8. *Let R be semiprime ring and I a nonzero ideal in R.If $F : R \to R$ is a multiplicative (generalized)-derivation associated with a map $f : R \to R$. If I satisfies any one of the identities $f(x)F(y) \pm xy = 0$ and $f(x)F(y) \pm yx = 0$ for all $x, y \in I$, then f is commuting on I.*

Proof. By equation (14) in theorem 7, we have $f(x)yf(x) = 0$ for all $x, y \in I$. Therefore, $[f(x), x]y[f(x), x] = 0$ for all $x, y \in I$. That is , $[f(x), x]I[f(x), x] = (0)$ for all $x, y \in I$. Since I is an ideal of a semiprime ring therefore $[f(x), x] = 0$ for all $x \in I$, thus f is commuting on I. □

Theorem 9. *Let R be a semiprime ring and I a non zero left ideal of R. If $F, G : R \to R$ are multiplicative (generalized)-derivations associated with maps $f, g : R \to R$ respectively. If $[F(x), y] \pm x \circ G(y) = 0$ for all $x, y \in I$, then $I[x, g(x)] = (0)$ and $I[x, f(x)] = (0)$ for all $x \in I$.*

Proof. Firstly, assume that

$$[F(x), y] - x \circ G(y) = 0 \tag{22}$$

for all $x, y \in I$. Replace y by yx in (22), we get

$$y[F(x), x] + ([F(x), y] - x \circ G(y))x - x \circ yg(x) = 0 \tag{23}$$

for all $x, y \in I$. Using (22) in (23), we obtain

$$y[F(x),x] - x \circ yg(x) = 0$$

for all $x, y \in I$.

$$y[F(x),x] - (x \circ y)g(x) + y[x,g(x)] = 0 \tag{24}$$

for all $x, y \in I$. Replace y by $g(x)y$ in (24), we get

$$g(x)y[F(x),x] - (x \circ g(x)y)g(x) + g(x)y[x,g(x)] = 0$$

for all $x, y \in I$.

$$g(x)y[F(x),x] - g(x)(x \circ y)g(x) - [x,g(x)]yg(x) + g(x)y[x,g(x)] = 0 \tag{25}$$

for all $x, y \in I$. Left multiply (24) by $g(x)$ and subtract from (25), we get

$$[x,g(x)]yg(x) = 0 \tag{26}$$

for all $x, y \in I$. Replace y by yx in (26), we get

$$[x,g(x)]yxg(x) = 0 \tag{27}$$

for all $x, y \in I$. Right multiply (26) by x and subtract it from (27), we get $[x,g(x)]y[x,g(x)] = 0$ for all $x, y \in I$ implies $y[x,g(x)]Ry[x, g(x)] = (0)$ for all $x, y \in I$. Semiprimeness of R forces that $y[x,g(x)] = 0$ for all $x, y \in I$. Therefore, $I[x,g(x)] = (0)$ for all $x \in I$.

Now replacing x by xy in (22) and using the same argument as above, we conclude that $I[x,f(x)] = (0)$ for all $x \in I$. The same technique can be followed in case $[F(x),y] + x \circ G(y) = 0$ for all $x, y \in I$. This completes the proof. □

Theorem 10. *Let R be a semiprime ring and I a non zero left ideal of R. If $F, G : R \to R$ are multiplicative (generalized)- derivations associated with maps $f, g : R \to R$ respectively. If $F(x) \circ y \pm x \circ G(y) = 0$ for all $x, y \in I$, then $I[x,g(x)] = (0)$ and $I[x,f(x)] = (0)$ for all $x \in I$.*

Proof. Firstly we assume,

$$F(x) \circ y - x \circ G(y) = 0 \tag{28}$$

for all $x, y \in I$. Replace y by yx and using (28), we obtain

$$y[x,g(x)] - y[F(x),x] - (x \circ y)g(x) = 0 \tag{29}$$

for all $x, y \in I$. Replace y by $g(x)y$ in (29), we get

$$g(x)y[x,g(x)] - g(x)y[F(x),x] - g(x)(x \circ y)g(x) - [x,g(x)]yg(x) = 0 \tag{30}$$

for all $x, y \in I$. Left multiply (29) by $g(x)$ and subtract it from (30), we obtain $[x, g(x)]yg(x) = 0$ for all $x, y \in I$. Further, the proof follows from Theorem 9, after equation (26). The same technique can be followed in case $F(x) \circ y + x \circ G(y) = 0$ for all $x, y \in I$. □

Corollary 11. *Let R be a semiprime ring admitting a multiplicative (generalized)-derivation $F, G : R \to R$ associated with the maps $f, g : R \to R$ respectively. If R satisfies any one of the identities $[F(x), y] \pm x \circ G(y) = 0$ and $F(x) \circ G(y) \pm x \circ G(y) = 0$ for all $x, y \in R$,then f and g both are commuting maps on R.*

3. Eamples

In this section we develop an example to show that the condition of semiprimeness of the ring in our results is crucial.

Example 12. *Consider* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in S \right\}$, *where S is any ring. We define maps $F, f : R \to R$ by* $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $f\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ac & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $G, g : R \to R$ *by* $G\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$g\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^2 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then it is verified that F and G are multiplicative (generalized)-derivations associated with the maps f and g respectively. It is easy to see that the identities $[F(x), y] \pm x \circ G(y) = 0$ and $F(x) \circ G(y) \pm x \circ G(y) = 0$ are satisfied for all $x, y \in R$. Here R is not semiprime ring because* $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$. *Note that f and g are not commuting maps on R. Hence, the semiprimeness in Cor.11 is crucial.*

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