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PROPERTIES AND INVARIANTS ASSOCIATED WITH THE ACTION OF THE ALTERNATING GROUP ON UNORDERED SUBSETS

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Abstract: The transitivity, primitivity, rank and subdegrees, as well as pairing of the suborbits associated with the action of the alternating group A_n, on unordered $r-$element subsets of a set $X = \{1, 2, \cdots, n\}$ of n letters, have not received any attention. In this paper, we prove that this action is transitive. We also show that the action is imprimitive if and only if $n = 2r$. In addition, we establish that the rank associated with the action is a constant $r + 1$ if and only if $n \geq 2r$, except for $r = 2$ in which case the rank is 4 if $n = 4$, but is 3 for all $n \geq 5$. Further, we calculate the subdegrees associated with the action and arrange them according to their increasing magnitudes. Finally, we show that all the suborbits of the action, with the exception of some non-trivial suborbits corresponding to the actions of A_3 and A_4 on the set of unordered pairs, are self-paired.

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1. Introduction

The primitivity, rank and subdegrees of the action of the symmetric group S_n on

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$X^{(2)}$, that is on the set of unordered pairs from $X = \{1, 2, \cdots, n\}$, were studied by [4]. Later on, other scholars investigated the transitivity, primitivity, ranks and subdegrees, as well as pairing of the suborbits, associated with certain actions of S_n and a subgroup of S_n, namely the alternating group A_n. These are; the action of S_n on $X^{(r)}$, the unordered $r-$element subsets of the set $X = \{1, 2, \cdots, n\}$ [6], the action of S_n on $X^{[r]}$, the ordered $r-$element subsets of $X = \{1, 2, \cdots, n\}$ [7], and the action of A_n on $X^{[r]}$ [2]. However, little has been done on the action of A_n on $X^{(r)}$; a study by [5] considered the action of A_7 on $X^{(2)}$ to illustrate existence of a primitive not doubly transitive group of degree 21, which contains a non-abelian regular subgroup of order 21. The current study explores the action of A_n on $X^{(r)}$. It is presented in six sections. Section 2 of the paper gives definitions of terms, as well as some theorem, that are relevant to the study, whereas Section 3 determines transitivity and primitivity of the action. While Section 4 investigates the ranks and subdegrees of A_n on unordered pairs, triples and quadruples, Section 5 generalizes the patterns of the invariants obtained in Section 4. Finally, Section 6 examines the pairing of the suborbits corresponding to the action.

2. Notation and Preliminary Results

Definition 2.1. Let G be a group and X a non-empty set. Then G *acts on the left* of X if there exists a function $G \times X \to X$ such that $(g_1g_2)x = g_1(g_2)x$ and $ex = x$ where e is the identity in G, $x \in X$ and $g_1, g_2 \in G$. The action of G on the right of X can be defined in a similar way. In this case, X is called a $G-set$.

Definition 2.2. Suppose a group G acts on a set X. Define a relation $x \sim y$ on X if and only if there exists $g \in G$ such that $y = gx$. This defines an equivalence relation on X. The equivalence class containing x is given by $Orb_G x = \{gx | g \in G\}$, and is called the *orbit* or *transitivity class* of x. Since any set is a disjoint union of equivalence classes under an equivalence relation, it follows that if G acts on X, then X is a union of disjoint orbits.

Definition 2.3. The action of a group G on a set X is said to be *transitive* if for each $x, y \in X$, there exists $g \in G$ such that $y = gx$; in other words $Orb_G x = X$ if $x \in X$. A group which is not transitive is called *intransitive.* Now, consider an integer $k \geq 1$. Suppose that for any two ordered $k-$tuples $(x_1, \cdots, x_k)$ and $(y_1, \cdots, y_k)$ of distinct elements in X, some element of G sends x_i to y_i for all i. Then such an action is called $k-$*transitive.* An action which is $k-$transitive is $l-$transitive for $l \leq k$.

Theorem 2.4. *The alternating group A_n has a natural $(n-2)-$ transitive action on $X = \{1, 2, \cdots, n\}$ for all $n \geq 3$* [9].

Definition 2.5. The subset $Stab_G x = \{g \in G | gx = x\}$ of G, also denoted by G_x, is called the *stabilizer* of x in G. It is a subgroup of G [3].

Theorem 2.6. (***Orbit-Stabilizer Theorem***) *Let X be a $G-$set and let $x \in X$. Then $|Orb_G x| = \frac{|G|}{|G_x|}$, the index of G_x in G* [3].

Definition 2.7. Let G act transitively on a finite set X. Then a subset Y of X is called a *block* or *set of imprimitivity* for the action if for each $g \in G$, either $gY = Y$ or $gY \cap Y = \phi$; i.e., if gY and Y do not overlap partially. In particular, ϕ, X and all $1-$element subsets of X are blocks, called the *trivial blocks*. The action is said to be *primitive* if the only blocks are the trivial blocks; it is *imprimitive* otherwise.

Theorem 2.8. *A $2-$transitive group is primitive* [1].

Definition 2.9. Suppose G acts transitively on X and let G_x be the stabilizer of a fixed $x \in X$. The orbits $\triangle_0 = \{x\}, \triangle_1, \triangle_2, \cdots, \triangle_{k-1}$ of G_x on X are known as *suborbits* of G. The *rank* of G is k in this case. The sizes $|\triangle_i|$, $(i = 0, 1, 2, \cdots, k-1)$ are known as the *subdegrees* of G. Both the rank and the subdegrees of G are independent of the choice of $x \in X$.

Definition 2.10. Let G be transitive on X and $\triangle$ an orbit of G_x on X. If $\triangle^* = \{gx | g \in G, x \in g\triangle\}$, then $\triangle^*$ is also an orbit of G_x called the G_x-orbit or $G-$suborbit *paired* with $\triangle$. Clearly, $\triangle^{**} = \triangle$ and $|\triangle| = |\triangle^*|$. If $\triangle = \triangle^*$, then $\triangle$ is said to be *self-paired*. The trivial suborbit of G is always self-paired, and there are other self-paired suborbits of G if and only if G has even order [8].

Notation 2.11. From this point on, G shall be reserved to denote the alternating group A_n, X the set $\{1, 2, \cdots, n\}$ and $X^{(r)}$ the set of unordered $r-$element subsets of X.

The action of G on X induces an action of G on $X^{(r)}$ that is defined by

$$g\{x_1, x_2, \cdots, x_r\} = \{g(x_1), g(x_2), \cdots, g(x_r)\} \; \forall g \in G, \; \{x_1, x_2, \cdots, x_r\} \in X^{(r)}.$$

3. Transitivity and Primitivity of G Acting on $\mathbf{X^{(r)}}$

Lemma 3.1. *The order of the stabilizer in G of an unordered $r-$element subset $\{1, 2, \cdots, r\}$ is $\frac{(n-r)!r!}{2}$ for all $n \geq r+1$.*

Proof. Clearly, the stabilizer of the subset $\{1, 2, \cdots, r\}$ is the union of the products of the even permutations of $\{1, 2, \cdots, r\}$ by the even permutations of the subset $\{r+1, r+2, \cdots, n\}$, $n \geq r+1$, and the products of the odd permutations of $\{1, 2, \cdots, r\}$ by the odd permutations of $\{r+1, r+2, \cdots, n\}$. Thus, $|G_{\{1,2,\cdots,r\}}| = 2\left\{\frac{r!}{2}\frac{(n-r)!}{2}\right\} = \frac{(n-r)!r!}{2}$. □

Theorem 3.2. *The group G acts transitively on $X^{(r)}$ for all $n \geq r+1$.*

Proof. Since $|X^{(r)}| = \binom{n}{r}$, it suffices to show that the orbit of $\{1, 2, \cdots, r\}$ has length $\binom{n}{r}$. By Theorem 2.6 and Lemma 3.1,

$$\begin{aligned} |Orb_G\{1, 2, \cdots, r\}| &= \frac{|G|}{|G_{\{1,2,\cdots,r\}}|} \\ &= \frac{\frac{n!}{2}}{\frac{(n-r)!r!}{2}} \\ &= \frac{n!}{(n-r)!r!} \\ &= \binom{n}{r}. \end{aligned}$$

□

Theorem 3.3. *The group G acts imprimitively on $X^{(r)}$ if and only if $n = 2r$.*

Proof. It is adequate to prove that G acts imprimitively on $X^{(r)}$ if $n = 2r$ and primitively otherwise. Consider the case where $n = 2r$ and take a subset $Y = \{\{x_1, x_2, \cdots, x_r\}, \{y_1, y_2, \cdots, y_r\}\}$ of $X^{(r)}$ whose elements are disjoint. Suppose $g \in Stab_G\{x_1, x_2, \cdots, x_r\}$ or $g \in Stab_G\{y_1, y_2, \cdots, y_r\}$, i.e., g is a product of an even permutation of $\{x_1, x_2, \cdots, x_r\}$ by an even permutation of $\{y_1, y_2, \cdots, y_r\}$ or a product of an odd permutation of $\{x_1, x_2, \cdots, x_r\}$ by an odd permutation of $\{y_1, y_2, \cdots, y_r\}$. Then g fixes each element of Y so that $gY = Y$. Now, if r is even, then $g = (x_{\alpha_1}\ y_{\beta_1})(x_{\alpha_2}\ y_{\beta_2}) \cdots (x_{\alpha_r}\ y_{\beta_r}) \in G$ for all $\alpha_i, \beta_j \in \{1, 2, \cdots, r\}$. In this case $g\{x_1, x_2, \cdots, x_r\} = \{y_1, y_2, \cdots, y_r\}$, and vice versa, so that $gY = Y$. Any other $g \in G$ takes each element of Y to an element of $X^{(r)}$ not in Y so that $gY \cap Y = \emptyset$. Hence, Y is a non-trivial block for the action and the action is imprimitive by definition. Next, suppose $n < 2r$. If n is prime and $r = n-1$, then $|X^{(r)}| = \binom{n}{n-1} = n$ and the action will definetely have only trivial blocks. Now, consider the other cases for which $n < 2r$. Clearly, any two elements of $X^{(r)}$ are not disjoint. Hence, if Y is a proper subset of $X^{(r)}$ containing two or more elements, then there exists a permutation $g \in G$ that takes one element of Y to another and the latter to an element not in Y so that $gY \cap Y \neq \emptyset$ and $gY \neq Y$. Thus, the action lacks non-trivial blocks and is therefore primitive. On the other hand, suppose $n > 2r$. Clearly, $2 < n-2$. By Theorem 2.4, the action is $(n-2)-$transitive, and by Definition 2.3, it is $2-$transitive. Thus, by Theorem 2.8, the action is primitive.

□

4. Ranks and Subdegrees of G on $\mathbf{X^{(2)}}$, $\mathbf{X^{(3)}}$, and $\mathbf{X^{(4)}}$

Theorem 4.1. *The group G acts on $X^{(2)}$ with rank 3 and subdegrees $\binom{2}{2}\binom{n-2}{0}$, $\binom{2}{1}\binom{n-2}{1}$ and $\binom{2}{0}\binom{n-2}{2}$ for all $n \geq 5$.*

Proof. Suppose G acts on $X^{(2)}$. Then, $G_{\{1,2\}}$ has orbits each of whose every element contains exactly 2, 1, or no component from $N = \{1,2\}$ and the rest from $X - N$. These are, respectively,
$\triangle_0 = Orb_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\}$ with $|\triangle_0| = 1 = \binom{2}{2}\binom{n-2}{0}$, the number of ways of selecting 2 objects from a set of 2 distinct objects and no object from a set of $n-2$ distinct objects;
$\triangle_1 = Orb_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\},\{1,4\},\cdots,\{1,n\},\{2,3\},\{2,4\},\cdots,\{2,n\}\}$, with corresponding length $|\triangle_1| = 2(n-2) = \binom{2}{1}\binom{n-2}{1}$, the number of ways of selecting 1 object from a set of 2 distinct objects and 1 object from a set of $n-2$ distinct objects; and
$\triangle_2 = Orb_{G_{\{1,2\}}}\{3,4\} = \{\{3,4\},\cdots,\{3,n\},\{4,5\},\cdots,\{n-1,n\}\}$, in which case $|\triangle_2| = (n-3)+(n-4)+\cdots+2+1 = \frac{(n-2)(n-3)}{2!} = \binom{2}{0}\binom{n-2}{2}$, the number of ways of selecting no object from a set of 2 distinct objects and 2 objects from a set of $n-2$ distinct objects.

We now show that these are the only suborbits of G. Clearly, the suborbits are mutually disjoint and summing up the subdegrees, we have

$$\binom{2}{2}\binom{n-2}{0} + \binom{2}{1}\binom{n-2}{1} + \binom{2}{0}\binom{n-2}{2} = \binom{n}{2} = |X^{(2)}|.$$

Hence, $\triangle_0 \cup \triangle_1 \cup \triangle_2 = X^{(2)}$ so that $\{\triangle_0, \triangle_1, \triangle_2\}$ partitions $X^{(2)}$. Therefore, the action has exactly 3 suborbits.

Now, calculations show that the lengths of the suborbits will be arranged in increasing order of magnitude as indicated below;

$$\begin{cases} \binom{2}{2}\binom{n-2}{0} < \binom{2}{0}\binom{n-2}{2} \leq \binom{2}{1}\binom{n-2}{1} & if\ 5 \leq n \leq 7 \\ \binom{2}{2}\binom{n-2}{0} < \binom{2}{1}\binom{n-2}{1} < \binom{2}{0}\binom{n-2}{2} & if\ n \geq 8. \end{cases} \qquad \square$$

Remark 4.2. It is necessary to note that Theorem 4.1 holds only for $n \geq 5$. It, however, fails for $n = 3$ and $n = 4$ as Example 4.3 below illustrates.

Example 4.3. If $n = 3$, $X^{(2)} = \{\{1,2\},\{1,3\},\{2,3\}\}$ and $G_{\{1,2\}} = \{1\}$ partitions $X^{(2)}$ into three orbits, namely,

$$\triangle_0 = Orb_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\},$$

$$\triangle_1 = Orb_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\}\},$$

and

$$\triangle_2 = Orb_{G_{\{1,2\}}}\{2,3\} = \{\{2,3\}\}.$$

So, the rank of G on $X^{(2)}$ in this case is 3. However, the part of Theorem 4.1 regarding the subdegrees fails.

Similarly, if $n = 4$, then

$$X^{(2)} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

and $G_{\{1,2\}} = \{1,(12)(34)\}$ partitions $X^{(2)}$ into 4 orbits, namely,

$$\triangle_0 = Orb_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\},$$

$$\triangle_1 = Orb_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\},\{2,4\}\},$$

$$\triangle_2 = Orb_{G_{\{1,2\}}}\{1,4\} = \{\{1,4\},\{2,3\}\},$$

and

$$\triangle_3 = Orb_{G_{\{1,2\}}}\{3,4\} = \{\{3,4\}\}.$$

So, the rank of G on $X^{(2)}$ in this case is 4.

Theorem 4.4. *The group G acts on $X^{(3)}$ with rank 4 and subdegrees $\binom{3}{3}\binom{n-3}{0}$, $\binom{3}{2}\binom{n-3}{1}$, $\binom{3}{1}\binom{n-3}{2}$ and $\binom{3}{0}\binom{n-3}{3}$ for all $n \geq 6$.*

Proof. The group $G_{\{1,2,3\}}$ has orbits each of whose every element contains exactly 3, 2, 1, or no component from $N = \{1, 2, 3\}$. These are, respectively, $\triangle_0 = Orb_{G_{\{1,2,3\}}}\{1, 2, 3\} = \{\{1, 2, 3\}\}$, with $|\triangle_0| = 1 = \binom{3}{3}\binom{n-3}{0}$, the number of ways of selecting 3 objects from a set of 3 distinct objects and no object from a set of $n - 3$ distinct objects;

$$\begin{aligned}\triangle_1 = Orb_{G_{\{1,2,3\}}}\{1, 2, 4\} = \{&\{1, 2, 4\}, \{1, 2, 5\}, \cdots , \{1, 2, n\},\\ &\{1, 3, 4\}, \{1, 3, 5\}, \cdots , \{1, 3, n\}, \{2, 3, 4\}, \{2, 3, 5\}, \cdots , \{2, 3, n\}\},\end{aligned}$$

in which case $|\triangle_1| = 3(n - 3) = \binom{3}{2}\binom{n-3}{1}$, the number of ways of selecting 2 objects from a set of 3 distinct objects and 1 object from a set of $n-3$ distinct objects;

$$\begin{aligned}\triangle_2 = Orb_{G_{\{1,2,3\}}}\{1, 4, 5\} = \{&\{1, 4, 5\}, \cdots , \{1, 4, n\}, \{1, 5, 6\}, \cdots , \{1, 5, n\},\\ &\{1, 6, 7\}, \cdots , \{1, n-1, n\}, \{2, 4, 5\}, \cdots , \{2, n-1, n\}, \cdots , \{3, n-1, n\}\},\end{aligned}$$

with length

$$|\triangle_2| = 3[(n - 4) + (n - 5) + \cdots + 2 + 1] = \frac{3(n - 3)(n - 4)}{2} = \binom{3}{1}\binom{n-3}{2},$$

the number of ways of selecting 1 object from a set of 3 distinct objects and 2 objects from a set of $n - 3$ distinct objects; and

$$\begin{aligned}\triangle_3 = Orb_{G_{\{1,2,3\}}}\{4, 5, 6\} = \{&\{4, 5, 6\}, \cdots , \{4, 5, n\}, \{4, 6, 7\}, \cdots , \{4, 6, n\},\\ &\{4, 7, 8\}, \cdots , \{4, n-1, n\}, \{5, 6, 7\}, \cdots , \{n-2, n-1, n\}\},\end{aligned}$$

where calculations show that $|\triangle_3| = \binom{3}{0}\binom{n-3}{3}$, the number of ways of selecting no object from a set of 3 distinct objects and 3 objects from a set of $n - 3$ distinct objects.

Now, an argument similar to the one in the proof of Theorem 4.1 shows that $\{\triangle_0, \triangle_1, \triangle_2, \triangle_3\}$ is a partition of $X^{(3)}$ so that the rank is 4.

Finally, the subdegrees are arranged in increasing order of magnitude as follows:

$$\begin{cases} \binom{3}{3}\binom{n-3}{0} \leq \binom{3}{0}\binom{n-3}{3} \\ < \binom{3}{2}\binom{n-3}{1} \leq \binom{3}{1}\binom{n-3}{2} & if\ 6 \leq n \leq 8 \\ \binom{3}{3}\binom{n-3}{0} < \binom{3}{2}\binom{n-3}{1} \\ < \binom{3}{0}\binom{n-3}{3} \leq \binom{3}{1}\binom{n-3}{2} & if\ 9 \leq n \leq 14 \\ \binom{3}{3}\binom{n-3}{0} < \binom{3}{2}\binom{n-3}{1} \\ < \binom{3}{1}\binom{n-3}{2} < \binom{3}{0}\binom{n-3}{3} & if\ n \geq 15. \end{cases}$$

□

Theorem 4.5. *If $n \geq 8$, the action of G on $X^{(4)}$ has rank 5. The length of the suborbit $\triangle_i\,(i = 0, 1, 2, 3, 4)$ whose each element has exactly $4 - i$ components from the subset $\{1, 2, 3, 4\}$ is given by $|\triangle_i| = \binom{4}{4-i}\binom{n-4}{i}$ in this case. Besides, $|\triangle_i| < |\triangle_{i+1}|$ for each $i = 0, 1, 2, 3, 4$ if and only if $n \geq 24$.*

Proof. It is analogous to the proofs of Theorems 4.1 and 4.4. □

5. Rank and Subdegrees of G on $\mathbf{X^{(r)}}$

Lemma 5.1. *If the action of G on $X^{(r)}$ has a suborbit whose each element contains exactly $i\,(i = 0, 1, \cdots, r)$ components from the set $\{1, 2, \cdots, r\}$, then $n \geq 2r - i$. In this case, the rank of the action is at least $r - i + 1$.*

Proof. Let $\triangle_{r-i}$ be the orbit whose each element contains exactly i components from the set $N = \{1, 2, \cdots, r\}$. Then once the first i components of an element of $\triangle_{r-i}$ have been selected from N, there remain $r - i$ components to be selected from the remaining $n - r$ elements of X. For this to happen, we must have $r - i \leq n - r$, which becomes $n \geq 2r - i$ on rewriting and it follows that $G_{\{1,2,\cdots,r\}}$ will, at least, have orbits each of whose every element contains exactly $r,\ r-1,\ r-2, \cdots,\ i+2,\ i+1$, or i elements from N. These are

$\triangle_0 = Orb_{G_{\{1,2,\cdots,r\}}}\{1,2,\cdots,r\}$, the orbit whose only element contains exactly r components from N and whose length is $|\triangle_0| = \binom{r}{r}\binom{n-r}{0}$, the number of ways of selecting r objects from r distinct objects and no object from $n-r$ distinct objects;

$\triangle_1 = Orb_{G_{\{1,2,\cdots,r\}}}\{1,2,\cdots,r-1,r+1\}$, the orbit whose each element contains exactly $r-1$ components from N, with $|\triangle_1| = \binom{r}{r-1}\binom{n-r}{1}$, the number of ways of selecting $r-1$ objects from r distinct objects and 1 object from $n-r$ distinct objects; and

$\triangle_2 = Orb_{G_{\{1,2,\cdots,r\}}}\{1,2,\cdots,r-2,r+1,r+2\}$, the orbit whose each element has exactly $r-2$ components from N, where $|\triangle_2| = \binom{r}{r-2}\binom{n-r}{2}$, the number of ways of selecting $r-2$ objects from r distinct objects and 2 objects from $n-r$ distinct objects.

The intermediate orbits $\triangle_3,\cdots,\triangle_{r-i-2}$ are described in an analogous manner. Finally, we have

$\triangle_{r-i-1} = Orb_{G_{\{1,2,\cdots,r\}}}\{1,2,\cdots,i,i+1,r+1,r+2,\cdots,2r-i-1\}$, the orbit whose each element contains exactly $i+1$ components from N and whose length is $|\triangle_{r-i-1}| = \binom{r}{i+1}\binom{n-r}{r-i-1}$, the number of ways of selecting $i+1$ objects from r distinct objects and $r-i-1$ objects from $n-r$ distinct objects; and

$\triangle_{r-i} = Orb_{G_{\{1,2,\cdots,r\}}}\{1,2,\cdots,i,r+1,r+2,\cdots,2r-i\}$, the orbit whose each element has exactly i components from N, with $|\triangle_{r-i}| = \binom{r}{i}\binom{n-r}{r-i}$, the number of ways of selecting i objects from r distinct objects and $r-i$ objects from $n-r$ distinct objects.

Clearly, the orbits do not overlap partially and they are $r-i+1$ in number. □

Theorem 5.2. *The rank of G on $X^{(r)}$ is $r+1$ if and only if $n \geq 2r$.*

Proof. Suppose $n \geq 2r$. This corresponds to $i = 0$ in Lemma 5.1 and it follows that $G_{\{1,2,\cdots,r\}}$ has orbits each of whose every element contains exactly r, $r-1$, $r-2,\cdots$, 2, 1, or no component from the set $N = \{1,2,\cdots,r\}$. The $r+1$ suborbits of G are $\triangle_0,\triangle_1,\triangle_2,\cdots,\triangle_{r-2},\triangle_{r-1}$, and $\triangle_r$ respectively. Now, to prove that G has exactly $r+1$ suborbits, it is enough to show that $\{\triangle_0,\triangle_1,\triangle_2,\cdots,\triangle_{r-1},\triangle_r\}$ is a partition of $X^{(r)}$. Clearly, $\triangle_{r-i} \neq \emptyset$ for each

$i = 0, 1, 2, \cdots, r$ and $\triangle_i \cap \triangle_j = \emptyset$ unless $i = j$, $(i, j = 0, 1, 2, \cdots, r)$. Also,

$$[i = 0]r\sum|\triangle_i| = [i = 0]r\sum\binom{r}{r-i}\binom{n-r}{i} = \binom{n}{r} = |X^{(r)}|$$

so that $\triangle_0 \cup \triangle_1 \cup \cdots \cup \triangle_r = X^{(r)}$. Thus, $\{\triangle_0, \triangle_1, \triangle_2, \cdots, \triangle_{r-1}, \triangle_r\}$ partitions $X^{(r)}$.

Conversely, suppose the rank is $r + 1$. Then there exists a suborbit $\triangle_r$ whose each element contains no component from N. This suborbit corresponds to $i = 0$ in Lemma 5.1. Its length is $\binom{r}{0}\binom{n-r}{r}$ wherein the factor $\binom{n-r}{r}$ is defined only if $n - r \geq r$, which becomes $n \geq 2r$ on rewriting. □

Theorem 5.3. *If $n \geq 2r$ and $\triangle_i\,(i = 0, 1, 2, \cdots, r)$ is the suborbit of G whose each element contains exactly $r - i$ components from $\{1, 2, \cdots, r\}$, then $|\triangle_i| = \binom{r}{r-i}\binom{n-r}{i}$. Furthermore, $|\triangle_i| < |\triangle_{i+1}|$ for all $n \geq r(r + 2)$.*

Proof. Let $\triangle_i$ be the suborbit of G whose each element has $r - i$ elements from the subset $\{1, 2, \cdots, r\}$. From Lemma 5.1, $|\triangle_i| = \binom{r}{r-i}\binom{n-r}{i}$, the number of ways of selecting $r - i$ objects from r distinct objects and i objects from $n - r$ distinct objects.

The proof of the other part of the theorem is by mathematical induction. If $r = 2$, then from Theorem 4.1, $\binom{2}{2-i}\binom{n-2}{i} < \binom{2}{2-i-1}\binom{n-2}{i+1}$, $(i = 0, 1, 2)$ for all $n \geq 8$. So, the statement is true for $r = 2$. Now, suppose it is true for $r = k$ for an integer $k \geq 3$. That is, if $n \geq k(k + 2)$,

$$\binom{k}{k-i}\binom{n-k}{i} < \binom{k}{k-i-1}\binom{n-k}{i+1}$$

$$\Rightarrow \frac{k!}{i!(k-i)!}\frac{(n-k)!}{(n-k-i)!i!} < \frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k)!}{(n-k-i-1)!(i+1)!}$$

$$\Rightarrow \quad \frac{k!}{i!(k-i)!}\frac{(n-k)(n-k-1)!}{(n-k-i)(n-k-i-1)!i!} < \frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k)(n-k-1)!}{(n-k-i-1)(n-k-i-2)!(i+1)!}$$

$$\Rightarrow \quad \frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!}\left(\frac{n-k}{n-k-i}\right) <$$

$$\frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{n-k}{n-k-i-1}\right)$$

$$\Rightarrow \quad \frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} <$$
$$\frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{n-k-i}{n-k-i-1}\right).$$

For $r = k+1$, we need to show that

$$\binom{k+1}{k-i+1}\binom{n-k-1}{i} < \binom{k+1}{k-i}\binom{n-k-1}{i+1}$$

$$\Rightarrow \quad \frac{(k+1)!}{i!(k-i+1)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} <$$
$$\frac{(k+1)!}{(i+1)!(k-i)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}$$

$$\Rightarrow \quad \frac{(k+1)k!}{i!(k-i+1)(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} <$$
$$\frac{(k+1)k!}{(i+1)!(k-i)(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}$$

$$\Rightarrow \quad \frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!}\left(\frac{k+1}{k-i+1}\right) <$$
$$\frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{k+1}{k-i}\right)$$

$$\Rightarrow \quad \frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} <$$
$$\frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{k-i+1}{k-i}\right).$$

Now, since $n-2k \geq k^2$, then $(n-k-i)-(k-i+1) \geq k^2-1 > 0$ so that $n-k-i > k-i+1$. From the inductive hypothesis, that is,

$$\frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} <$$

$$\frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{n-k-i}{n-k-i-1}\right),$$

and the fact that $\frac{k-i+1}{k-i} > \frac{n-k-i}{n-k-i-1}$, then

$$\frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} <$$
$$\frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{k-i+1}{k-i}\right),$$

proving that $\binom{k+1}{k-i+1}\binom{n-k-1}{i} < \binom{k+1}{k-i}\binom{n-k-1}{i+1}$. So, the statement is true for $r = k+1$ whenever true for $r = k$. Therefore, by the principle of mathematical induction, it is true for all $r \geq 2$. □

Remark 5.4. Theorems 5.2 and 5.3 hold only for $n \geq 5$ if $r = 2$. This is clear from Theorem 4.1

6. Pairing of the of Suborbits of G on $\mathbf{X^{(r)}}$

Theorem 6.1. *All the suborbits of the action of G on $X^{(r)}$, except for some non-trivial suborbits corresponding to the actions of A_3 and A_4 on $X^{(2)}$, are self-paired.*

Proof. Consider the general suborbit $\triangle_{r-i}$ $(i = 0, 1, \cdots, r)$ whose each element contains exactly i components from the subset $\{1, 2, \cdots, r\}$. Then

$$\{1, 2, \cdots, i, r+1, r+2, \cdots, 2r-i-1, 2r-i\} \in \triangle_{r-i}.$$

If r and i are both even or both odd, then

$$g = (i+1\ \ r+1)(i+2\ \ r+2)\cdots(r-1\ \ 2r-i-1)(r\ \ 2r-i) \in G$$

and if one of r and i is even, and the other odd, then

$$g = (1\ \ 2)(i+1\ \ r+1)(i+2\ \ r+2)\cdots(r-1\ \ 2r-i-1)(r\ \ 2r-i) \in G.$$

In any of these cases

$$g\{1, 2, \cdots, i, r+1, r+2, \cdots, 2r-i-1, 2r-i\}$$
$$= \{1, 2, \cdots, i, i+1, i+2, \cdots, r-1, r\} \in g\triangle_{r-i}$$

and

$$g\{1, 2, \cdots, i, i+1, i+2, \cdots, r-1, r\}$$
$$= \{1, 2, \cdots, i, r+1, r+2, \cdots, 2r-i-1, 2r-i\} \in \triangle_{r-i},$$

so that $\triangle^*_{r-i} = \triangle_{r-i}$ (it is easy to verify that no such g exists for cases where $n = 3$ and $n = 4$, while $r = 2$ and $i = 1$; see Remark 6.2 and Example 6.3 below for more details).

□

Remark 6.2. Theorem 6.1 partially fails for the action of A_3 on $X^{(2)}$. The two non-trivial suborbits of the action (revisit Example 4.3) are paired (see Example 6.3 below for the proof of this claim). The theorem also partially fails for the action of A_4 on $X^{(2)}$ where the two suborbits each of whose every element contains exactly one of 1 and 2 (revisit Example 4.3) are paired (see Example 6.3 for further details).

Example 6.3. Let $G = A_3$ act on $X^{(2)}$. The non-trivial suborbits of G are $\triangle_1 = \{\{1, 3\}\}$ and $\triangle_2 = \{\{2, 3\}\}$. Now, for the permutation $g = (1\ 2\ 3)$, $g\{1, 3\} = \{1, 2\} \in g\triangle_1$ but $g\{1, 2\} = \{2, 3\} \in \triangle_2$. So, $\triangle^*_1 = \triangle_2$.

Similarly, if A_4 acts on $X^{(2)}$, the non-trivial suborbits $\triangle_1$, $\triangle_2$ and $\triangle_3$ seen in Example 4.3 have the property that $\triangle^*_1 = \triangle_2$ while $\triangle^*_3 = \triangle_3$.

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