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A SOLUTION OF THE THREE DIMENSIONAL NAVIER STOKES EQUATIONS

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Abstract: In this work we solved the time dependent three dimensional Navier Stokes equations using a coordinate transformation. It is found, the components of the velocity and pressure.

AMS Subject Classification: 35-XX, 34-XX, 76Dxx
Key Words: 3d Navier Stokes, coordinate transformation

1. Introduction

Navier-Stokes equations (NSEqs) are the corner stone in the physical description of the fluid dynamics phenomena. From the early works of Euler [1], Navier [2] and Stokes [3], it has been done so much research in order to develop analytical solutions, unfortunately not sufficiently general, and with not major results till today. Then, Navier-Stokes equations remain as one of the challenges of the XXI century to solve [4].

In consecuence, different computational methods have been applied to solve the time-dependent NSEqs [5]. Among them, we can find the so-called finite difference methods, which are boundary initial valued problems of NSEqs [6]-[9]. Also, intensive research in computational mathematics has been made. For instance, in reference [10] using the Picard and Newton methods in order to linearize the incompressible non-Newtonian NSEqs is created an efficient numerical solution. In the same way, in reference [11] several techniques are di-

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cussed due to the application finite element discretization to the incompressible Stokes equations. Also, an excellent review of fast solvers for incompressible NSEqs is presented in reference [12].

On the other hand, in the terrain of analytical ground, it has been found several classes of exact solutions for the three dimensional NSEqs, in some cases vanishing or disappearing the nonlinearities in order to make solvable the system. For example, in reference [13] the NSEqs are transformed into easier equations using a very interesting potential function and a transform coordinates with the purpose to find solutions. Starting from the three dimensional compressible NSEqs and a polytropic equation it is supposed and discussed a self-similar solution and its trial function [14].

This article addresses the solution of the time-dependent Navier-Stokes equations in three dimensions using a coordinate transformation. This work is organized as follows. Section (2), presents the set of equations that we solved. Also, we use the coordinate transformation and transform the of partial differential equations to a set of ordinary differential. Then, we solved the reduced system and give explicit exressions for the vector field velocity and the scalar pressure field. In section (3), we present results and concluding remarks.

2. The 3d Navier-Stokes-Equations and their solution

In Cartesian coordinates, the governing equations for incompressible three-dimensional fluid are:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+\frac{\partial p}{\partial x}=\frac{1}{R}(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}) \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial p}{\partial y}=\frac{1}{R}(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}) \tag{2}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial p}{\partial z}=\frac{1}{R}(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}) \tag{3}$$

The parameter R is called the Reynolds number and its define as $R = V_\infty L\rho_\infty/\mu_\infty$, here ρ_∞, V_∞ and μ_∞ are the density, velocity and viscosity at infinite distance from the fluid region in consideration. The incompresibility is given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

In order to search for a solution, we use the next transformation

$$\xi = x + y + z - at + \xi_0 \tag{5}$$

Where the dependence of the fields become

$$\begin{aligned} u(x,y,z,t) = u(\xi); \quad v(x,y,z,t) = v(\xi); \\ w(x,y,z,t) = w(\xi); \quad p(x,y,z,t) = p(\xi) \end{aligned} \tag{6}$$

The derivatives change like:

$$\begin{aligned} &\frac{\partial}{\partial t} = -a\frac{\partial}{\partial \xi}; \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}; \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} \\ &\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}; \quad \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \xi^2}; \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial \xi}; \quad \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \xi^2} \end{aligned} \tag{7}$$

Therefore, using eqs. (5-7) the time dependent $3d$ Navier Stokes equations eqs. (1-4) are:

$$-a\frac{du}{d\xi} + (u+v+w)\frac{du}{d\xi} + \frac{dp}{d\xi} = \frac{3}{R}\frac{d^2u}{d\xi^2} \tag{8}$$

$$-a\frac{dv}{d\xi} + (u+v+w)\frac{dv}{d\xi} + \frac{dp}{d\xi} = \frac{3}{R}\frac{d^2v}{d\xi^2} \tag{9}$$

$$-a\frac{dw}{d\xi} + (u+v+w)\frac{dw}{d\xi} + \frac{dp}{d\xi} = \frac{3}{R}\frac{d^2w}{d\xi^2} \tag{10}$$

$$\frac{du}{d\xi} + \frac{dv}{d\xi} + \frac{dw}{d\xi} = 0 \tag{11}$$

Equations (8-11) become a set of coupled ordinary differential equations. Then, integrating eqs. (8-10), and electing the integration constant as 0. And in eq. (11), choosing the integration constant as b, we obtain:

$$u + v + w = b \tag{12}$$

$$(b-a)\frac{du}{d\xi} + \frac{dp}{d\xi} = \frac{3}{R}\frac{d^2u}{d\xi^2} \tag{13}$$

$$(b-a)\frac{dv}{d\xi} + \frac{dp}{d\xi} = \frac{3}{R}\frac{d^2v}{d\xi^2} \tag{14}$$

$$(b-a)\frac{dw}{d\xi} + \frac{dp}{d\xi} = \frac{3}{R}\frac{d^2w}{d\xi^2} \tag{15}$$

Then, substracting eq (14) from (13):

$$(b-a)\frac{d(u-v)}{d\xi} = \frac{3}{R}\frac{d^2(u-v)}{d\xi^2} \tag{16}$$

Substracting eq (15) from (14):

$$(b-a)\frac{d(v-w)}{d\xi} = \frac{3}{R}\frac{d^2(v-w)}{d\xi^2} \tag{17}$$

Substracting eq (13) from (15):

$$(b-a)\frac{d(w-u)}{d\xi} = \frac{3}{R}\frac{d^2(w-u)}{d\xi^2} \tag{18}$$

Now, relabelling variables

$$f = u - v \tag{19}$$

$$g = v - w \tag{20}$$

$$h = w - u \tag{21}$$

Taking into account that

$$f + g + h = 0 \tag{22}$$

Then, eqs. (16-18) using eqs. (19-21), become:

$$(b-a)\frac{df}{d\xi} = \frac{3}{R}\frac{d^2 f}{d\xi^2} \tag{23}$$

$$(b-a)\frac{dg}{d\xi} = \frac{3}{R}\frac{d^2 g}{d\xi^2} \tag{24}$$

$$(b-a)\frac{dh}{d\xi} = \frac{3}{R}\frac{d^2 h}{d\xi^2} \tag{25}$$

Or

$$\frac{3}{R}\frac{d^2 f}{d\xi^2} + (a-b)\frac{df}{d\xi} + (0)f = 0 \tag{26}$$

$$\frac{3}{R}\frac{d^2 g}{d\xi^2} + (a-b)\frac{dg}{d\xi} + (0)g = 0 \tag{27}$$

$$\frac{3}{R}\frac{d^2 h}{d\xi^2} + (a-b)\frac{dh}{d\xi} + (0)h = 0 \tag{28}$$

Then, solving eqs. (26-28), using a standard method in ordinary differential equations, we obtain:

$$f = c_1 \exp(r_1\xi) + c_2 \exp(r_2\xi) \tag{29}$$

$$g = e_1 \exp(r_1\xi) + e_2 \exp(r_2\xi) \tag{30}$$

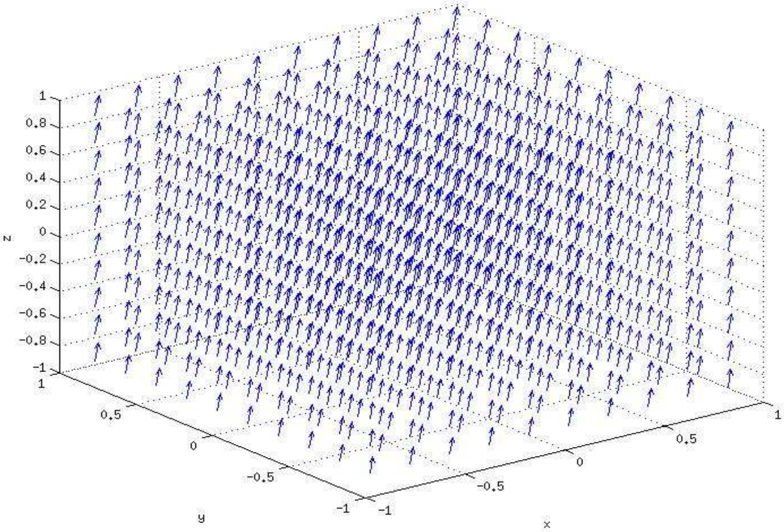

Figure 1: Vectorial velocity field, eqs. (37-39).

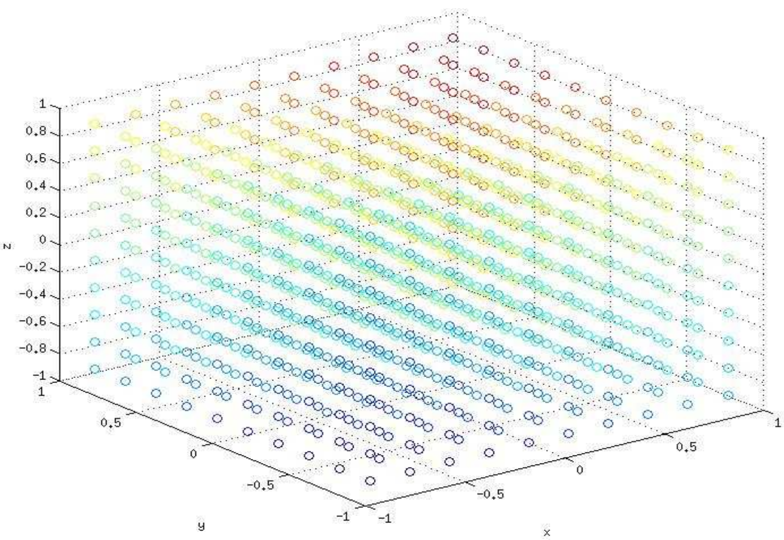

Figure 2: Scalar speed field, $U^2 = u^2 + v^2 + w^2$.

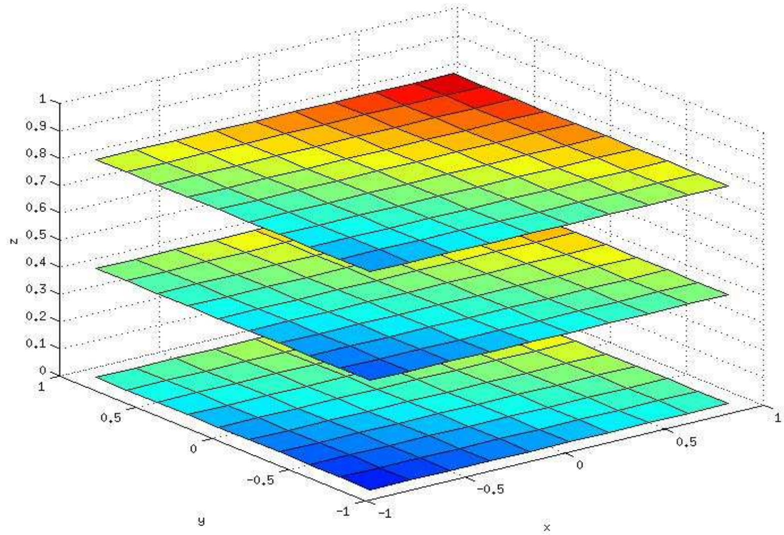

Figure 3: Equal parallel slices of the scalar speed field.

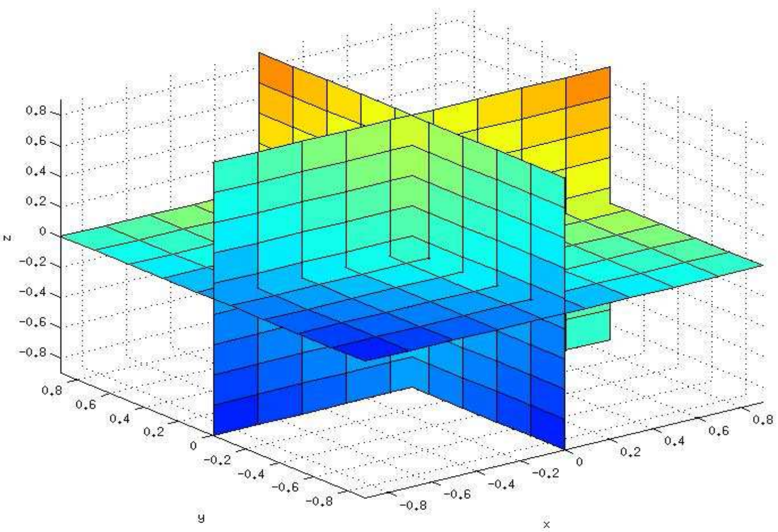

Figure 4: Perpendicular slices of the scalar field speed.

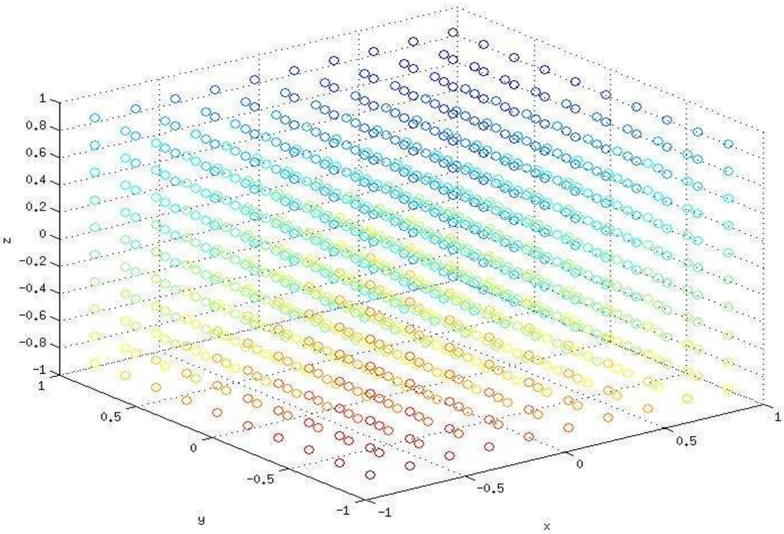

Figure 5: The scalar field pressure eq. (49).

$$h = l_1 \exp(r_1 \xi) + l_2 \exp(r_2 \xi) \tag{31}$$

With r_1 and r_2 given by:

$$r_1 = \frac{-1 + \frac{\mathbf{Re}(\mathbf{a}-\mathbf{b})}{3}}{2} \tag{32}$$

$$r_2 = \frac{-1 - \frac{\mathbf{Re}(\mathbf{a}-\mathbf{b})}{3}}{2} \tag{33}$$

Now using a bit of algebra, and using eq. (12), we obtain:

$$\frac{g + 2f + b}{3} = u \tag{34}$$

$$\frac{h + 2g + b}{3} = v \tag{35}$$

$$\frac{f+2h+b}{3}=w \tag{36}$$

Then, the components of the velocity are:

$$u=\frac{(e_1+2c_1)\exp(r_1\xi)+(e_2+2c_2)\exp(r_2\xi)+b}{3} \tag{37}$$

$$v=\frac{(l_1+2e_1)\exp(r_1\xi)+(l_2+2e_2)\exp(r_2\xi)+b}{3} \tag{38}$$

$$w=\frac{(c_1+2l_1)\exp(r_1\xi)+(c_2+2l_2)\exp(r_2\xi)+b}{3} \tag{39}$$

Deriving $v(\xi)$, with respect to ξ, we obtain:

$$\frac{3}{R}\frac{dv}{d\xi}=\frac{3}{R}\frac{d}{d\xi}(\frac{(h_0+2g_0)e^{(-\frac{Ra\xi}{3})}}{3})=-a\frac{(h_0+2g_0)e^{(-\frac{Ra\xi}{3})}}{3} \tag{40}$$

$$av=a\frac{(h_0+2g_0)e^{(-\frac{Ra\xi}{3})}}{3} \tag{41}$$

$$uv=\frac{(g_0+2f_0)e^{(-\frac{Ra\xi}{3})}}{3}\frac{(h_0+2g_0)e^{(-\frac{Ra\xi}{3})}}{3} \tag{42}$$

$$vw=\frac{(h_0+2g_0)e^{(-\frac{Ra\xi}{3})}}{3}\frac{(f_0+2h_0)e^{(-\frac{Ra\xi}{3})}}{3} \tag{43}$$

$$v^2=\frac{(h_0+2g_0)e^{(-\frac{Ra\xi}{3})}}{3}\frac{(h_0+2g_0)e^{(-\frac{Ra\xi}{3})}}{3} \tag{44}$$

Asking for p in eq. (14), we obtain the pressure:

$$p = av - uv - v^2 - vw - \frac{3}{R}\frac{dv}{d\xi} \tag{45}$$

$$p = 2a\frac{(h_0 + 2g_0)e^{(-\frac{Ra\xi}{3})}}{3} - uv - v^2 - vw \tag{46}$$

$$\begin{aligned} p = \quad & 2a\frac{(h_0 + 2g_0)e^{(-\frac{Ra\xi}{3})}}{3} - ((g_0 + 2f_0)(h_0 + 2g_0) \\ & +(h_0 + 2g_0)(f_0 + 2h_0) + (h_0 + 2g_0)(h_0 + 2g_0))\frac{e^{(-\frac{2Ra\xi}{3})}}{9} \end{aligned} \tag{47}$$

$$\begin{aligned} p = \quad & 2a\frac{(h_0 + 2g_0)e^{(-\frac{Ra\xi}{3})}}{3} \\ & -((h_0 + 2g_0)(f_0 + g_0 + h_0 + 2(f_0 + h_0 + g_0)))\frac{e^{(-\frac{2Ra\xi}{3})}}{9} \end{aligned} \tag{48}$$

And using eq. (25), finally the pressure is:

$$p = a\frac{(h_0 + 2g_0)e^{(-\frac{Ra\xi}{3})}}{3} \tag{49}$$

3. Conclusions

In this work we found one analytical solution for the three dimensional Navier-Stokes. Here we note that, both the equation (4) the fluid incompressibility and the coordinates transformation, eq. (5), are the key arguments to reduce the equations (1-3) to a system of solvable ordinary differential equations. Consequently, we are able to build new exact solutions to the incompressible three dimensional Navier-Stokes equations. Figures (1-5) show the structure of the vector velocity field, scalar speed field and scalar pressure field at different slices. As far as we know, these solutions are not knwon in current literature. Also, the whole method is easily understood by science and ingeniering students, who are begining and introductory course in differential equations or fluid mechanics.

At last, the method can be easily extended to higher dimensions in order to search for solutions of Navier-Stokes formulations in n-dimensions.

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