COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPPINGS SATISFYING BINARY OPERATIONS IN MULTIPLICATIVE METRIC SPACES

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Abstract: In this paper, we prove some common fixed point theorems for weakly compatible mappings satisfying binary operations in multiplicative metric spaces.

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1. Introduction and Preliminaries

It is well known that the set of positive real numbers \( \mathbb{R}_+ \) is not complete according to the usual metric. To overcome this problem, in 2008, Bashirov et al. [3] introduced the concept of multiplicative metric spaces as follows:

Definition 1.1. Let \( X \) be a nonempty set. A multiplicative metric is a mapping \( d : X \times X \to \mathbb{R}_+ \) satisfying the following conditions:

(i) \( d(x, y) \geq 1 \) for all \( x, y \in X \) and \( d(x, y) = 1 \) if and only if \( x = y \);

(ii) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
(iii) \( d(x, y) \leq d(x, z) \cdot d(z, y) \) for all \( x, y, z \in X \) (multiplicative triangle inequality).

Then the mapping \( d \) together with \( X \), that is, \((X, d)\) is a multiplicative metric space.

**Example 1.2.** ([9]) Let \( \mathbb{R}_+^n \) be the collection of all \( n \)-tuples of positive real numbers. Let \( d^* : \mathbb{R}_+^n \times \mathbb{R}_+^n \rightarrow \mathbb{R} \) be defined as follows:

\[
d^*(x, y) = \frac{|x_1 y_1|^* |x_2 y_2|^* \cdots |x_n y_n|^*}{},
\]

where \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}_+^n \) and \(| \cdot |^* : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) is defined by

\[
|a|^* = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1. \end{cases}
\]

Then it is obvious that all conditions of a multiplicative metric are satisfied. Therefore \((\mathbb{R}_+^n, d)\) is a multiplicative metric space.

**Example 1.3.** ([11]) Let \( d : \mathbb{R} \times \mathbb{R} \rightarrow [1, \infty) \) be defined as \( d(x, y) = a|x-y| \), where \( x, y \in \mathbb{R} \) and \( a > 1 \). Then \( d \) is a multiplicative metric and \((X, d)\) is a multiplicative metric space. We may call it usual multiplicative metric spaces.

**Remark 1.4.** We note that the Example 1.2 is valid for positive real numbers and Example 1.3 is valid for all real numbers.

**Example 1.5.** ([11]) Let \((X, d)\) be a metric space. Define a mapping \( d_a \) on \( X \) by

\[
d_a(x, y) = a^{d(x, y)} = \begin{cases} 1 & \text{if } x = y, \\ a & \text{if } x \neq y, \end{cases}
\]

where \( x, y \in X \) and \( a > 1 \). Then \( d_a \) is a multiplicative metric and \((X, d_a)\) is known as the discrete multiplicative metric space.

**Example 1.6.** ([11]) Let \( X = C^*[a, b] \) be the collection of all real-valued multiplicative continuous functions on \([a, b] \subset \mathbb{R}_+\). Then \((X, d)\) is a multiplicative metric space with \( d \) defined by \( d(f, g) = \sup_{x \in [a, b]} \frac{|f(x)|}{g(x)} \) for arbitrary \( f, g \in X \).

**Remark 1.7.** ([11]) We note that multiplicative metrics and metric spaces are independent.

Indeed, the mapping \( d^* \) defined in Example 1.2 is multiplicative metric but
not metric as it does not satisfy triangular inequality. Consider
\[ d^* \left( \frac{1}{3}, \frac{1}{2} \right) + d^* \left( \frac{1}{2}, 3 \right) = \frac{3}{2} + 6 = 7.5 < 9 = d^* \left( \frac{1}{3}, 3 \right). \]

On the other hand the usual metric on \( \mathbb{R} \) is not multiplicative metric as it doesn't satisfy multiplicative triangular inequality, since
\[ d(2, 3) \cdot d(3, 6) = 3 < 4 = d(2, 6). \]

One can refer to [7, 9] for detailed multiplicative metric topology.

**Definition 1.8.** Let \((X, d)\) be a multiplicative metric space. Then a sequence \(\{x_n\}\) in \(X\) said to be
1. a multiplicative convergent to \(x\) if for every multiplicative open ball \(B_\epsilon(x) = \{ y \mid d(x, y) < \epsilon \}, \epsilon > 1\), there exists \(N \in \mathbb{N}\) such that \(x_n \in B_\epsilon(x)\) for all \(n \geq N\), that is, \(d(x_n, x) \to 1\) as \(n \to \infty\).
2. a multiplicative Cauchy sequence if for all \(\epsilon > 1\), there exists \(N \in \mathbb{N}\) such that \(d(x_n, x_m) < \epsilon\) for all \(m, n \geq N\), that is, \(d(x_n, x_m) \to 1\) as \(n, m \to \infty\).
3. We call a multiplicative metric space complete if every multiplicative Cauchy sequence in it is multiplicative convergent to \(x \in X\).

**Remark 1.9.** The set of positive real numbers \(\mathbb{R}_+\) is not complete according to the usual metric. Let \(X = \mathbb{R}_+\) and the sequence \(\{x_n\} = \{\frac{1}{n}\}\). It is obvious \(\{x_n\}\) is a Cauchy sequence in \(X\) with respect to usual metric and \(X\) is not a complete metric space, since \(0 \notin \mathbb{R}_+\). In case of a multiplicative metric space, we take a sequence \(\{x_n\} = \{a^{\frac{1}{n}}\}\), where \(a > 1\). Then \(\{x_n\}\) is a Cauchy sequence since for \(n \geq m\),
\[
\begin{align*}
d(x_n, x_m) & = \frac{x_n}{x_m} = \frac{a^{\frac{1}{n}}}{a^{\frac{1}{m}}} = \left| a^{\frac{1}{n} - \frac{1}{m}} \right| \\
& \leq a^{\frac{1}{m} - \frac{1}{n}} < a^{\frac{1}{m}} < \epsilon \quad \text{if} \quad m > \log a, \\
\end{align*}
\]
where \(|a| = \begin{cases} a & \text{if} \ a \geq 1, \\ \frac{1}{a} & \text{if} \ a < 1. \end{cases}\)
Also, \(\{x_n\} \to 1\) as \(n \to \infty\) and \(1 \in \mathbb{R}_+\). Hence \((X, d)\) is a complete multiplicative metric space.

In 2012, Özavsar and Çevikel [9] gave the concept of multiplicative contraction mappings and proved some fixed point theorem of such mappings in a multiplicative metric space.
Definition 1.10. Let $f$ be a mapping of a multiplicative metric space $(X, d)$ into itself. Then $f$ is said to be a multiplicative contraction if there exists a real number $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq d^\lambda(x, y) \quad \text{for all } x, y \in X.$$ 

Gu et al. [6] introduced the notion of commutative and weak commutative mappings in a multiplicative metric space and proved some fixed point theorems for these mappings.

Definition 1.11. Let $f$ and $g$ be two mappings of a multiplicative metric space $(X, d)$ into itself. Then $f$ and $g$ are said to be

(1) **commutative mappings** if $fgx = gfx$ for all $x \in X$.

(2) **weak commutative mappings** if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$.

Notice that commuting mappings are obviously weakly commuting. However, the converse need not be true.

In 1996, Jungck [8] introduced the concept of weakly compatible mappings and prove fixed point theorems using these mappings in metric spaces (see [2, 4, 5, 10]).

Now, we introduce the notions in multiplicative metric spaces.

Definition 1.12. Let $f$ and $g$ be two mappings of a multiplicative metric space $(X, d)$ into itself. Then $f$ and $g$ are said to be **weakly compatible** if they commute at coincidence points, that is, $ft = gt$ for some $t \in X$ implies that $fgt = gft$.

Notice that weakly commuting mappings are obviously weakly compatible. However, the converse need not be true.

In 2007, Sedghi and Shobe [12] introduced a new binary operation $\diamond$ as follows:

Let $\diamond : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a binary operation satisfying

(i) $\diamond$ is associative and commutative,

(ii) $\diamond$ is continuous.

Some examples of a binary operation $\diamond$ are as follows: for each $a, b \in \mathbb{R}_+$,

(1) $a \diamond b = \max\{a, b\}$,
(2) $a \diamond b = a + b$,
(3) $a \diamond b = ab$,
(4) $a \diamond b = ab + a + b$,
(5) $a \diamond b = \frac{ab}{\max\{a,b,1\}}$. 


Definition 1.13. The binary operation \( \odot \) is said to satisfy \( \alpha \)-property if there exists a positive real number \( \alpha \) such that \( a \odot b \leq \alpha \max\{a, b\} \) for all \( a, b \in \mathbb{R}_+ \).

Example 1.14. (1) If \( a \odot b = a + b \) for each \( a, b \in \mathbb{R}_+ \), then for \( \alpha \geq 2 \), we have \( a \odot b \leq \alpha \max\{a, b\} \).

(2) If \( a \odot b = ab \) for each \( a, b \in \mathbb{R}_+ \), then for \( \alpha \geq \min\{a, b\} \), we have \( a \odot b \leq \alpha \max\{a, b\} \).

(3) If \( a \odot b = ab \frac{\max\{a, b\}}{\max\{a, b, 1\}} \) for each \( a, b \in \mathbb{R}_+ \), then for \( \alpha \geq 1 \), we have \( a \odot b \leq \alpha \max\{a, b\} \).

Now we define \( \alpha \)-property in a multiplicative metric space sense as follows:

Definition 1.15. The binary operation \( \odot \) is said to satisfy a multiplicative \( \alpha \)-property if there exists a positive real number \( \alpha \) such that \( a \odot b \leq \alpha \max\{a, b\} \) for all \( a, b \in \mathbb{R}_+ \).

Example 1.16. (1) If \( a \odot b = ab \) for each \( a, b \in \mathbb{R}_+ \), then for \( \alpha \geq 2 \), we have \( a \odot b \leq (\max\{a, b\})^\alpha \).

(2) If \( a \odot b = \frac{ab}{\max\{a, b\}} \) for each \( a, b \in \mathbb{R}_+ \), then for \( \alpha \geq 1 \), we have \( a \odot b \leq (\max\{a, b\})^\alpha \).

2. Main Results

Now we prove some common fixed point theorems for weakly compatible mappings using binary operations in complete multiplicative metric spaces as follow:

Theorem 2.1. Let \((X, d)\) be a complete multiplicative metric space such that \( \odot \) satisfies a multiplicative \( \alpha \)-property with \( \alpha > 0 \). Let \( A, B, S \) and \( T \) be mappings of \((X, d)\) into itself satisfying the following conditions

\[
(C_1) \quad T(X) \subset A(X), \quad S(X) \subset B(X),
\]

\[
d(Sx, Ty) \leq (d(Ax, By) \odot d(Ax, Sx))^{k_1} \cdot (d(Ax, By) \odot d(By, Ty))^{k_2} \cdot (d(Ax, By) \odot d(Ax, Ty) \cdot d(Sx, By))^{\frac{1}{2}}^{k_3}
\]

for all \( x, y \in X \), where \( k_1, k_2, k_3 > 0 \) and \( 0 < \alpha(k_1 + k_2 + k_3) < 1 \).

Assume that

\( (C_3) \) the pairs \( A, S \) and \( B, T \) are weakly compatible,
(C₄) one of the subspace \( A(X) \) or \( B(X) \) or \( S(X) \) or \( T(X) \) is complete. Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** Let \( x₀ ∈ X \) be an arbitrary point. Since \( S(X) ⊂ B(X) \), for \( x₀ ∈ X \), there exists \( x₁ ∈ X \) such that \( Sx₀ = Bx₁ = y₀ \). Now for this point \( x₁ \) there exists \( x₂ ∈ X \) such that \( Tx₁ = Ax₂ = y₁ \). Continuing in this manner, we can inductively define a sequence \( \{yₙ\} \) such that

\[
Sx₂n = Bx₂n+1 = y₂n; \quad Tx₂n+1 = Ax₂n+2 = y₂n+1
\]

for \( n = 0, 1, 2, \ldots \).

Now we prove \( \{yₙ\} \) is a Cauchy sequence in \( X \). Using \( (C₂) \), we have

\[
d(y₂n, y₂n+1) = d(Sx₂n, Tx₂n+1) ≤ (d(Ax₂n, Bx₂n+1) \cdot d(Ax₂n, Sx₂n))^{k₁} \cdot (d(Ax₂n, Bx₂n+1) \cdot d(Bx₂n+1, Tx₂n+1))^{k₂} \cdot (d(Ax₂n, Bx₂n+1) \cdot (d(Ax₂n, Tx₂n+1) \cdot d(Sx₂n, Bx₂n+1))^{\frac{1}{2}})^{k₃} = (d(y₂n-₁, y₂n) \cdot d(y₂n-₁, y₂n)^{k₁} \cdot (d(y₂n-₁, y₂n) \cdot d(y₂n, y₂n+1))^{k₂} \cdot (d(y₂n-₁, y₂n) \cdot (d(y₂n, y₂n+1) \cdot d(y₂n, y₂n)\right)^{\frac{1}{2}})^{k₃}.
\]

Let \( dₙ = d(yₙ, yₙ₊₁) \). Then

\[
d₂n ≤ (d₂n-₁ \cdot d₂n-₁)^{k₁} \cdot (d₂n-₁ \cdot d₂n)^{k₂} \cdot (d₂n-₁ \cdot (d₂n \cdot d₂n)^{\frac{1}{2}})^{k₃} ≤ (d₂n-₁)^{k₁ \cdot (\max\{d₂n-₁, d₂n\})^{α} \cdot ((\max\{d₂n-₁, (d₂n-₁ \cdot d₂n)^{\frac{1}{2}}\})^{α} \cdot (\max\{d₂n-₁, (d₂n-₁ \cdot d₂n)^{\frac{1}{2}}\})^{α})^{k₃}.
\]

If \( d₂n > d₂n-₁ \), then

\[
d₂n ≤ d²(αk₁+k₂+k₃),
\]

which is a contradiction. Hence \( d₂n ≤ d₂n-₁ \), so, from inequality \( (C₂) \) we have

\[
d₂n ≤ d₂n-₁,
\]

where \( k = α(k₁ + k₂ + k₃) < 1 \).

Similarly, we have

\[
d_n ≤ dⁿ_{n-1} < dⁿ_{n-2} ≤ \cdots ≤ dⁿ_{0}.
\]
Let \( m, n \in \mathbb{N} \) with \( m > n \). Then we get
\[
d(y_n, y_m) \leq d(y_n, y_{n+1}) \cdots d(y_m-1, y_m) \\
\leq d^k(y_0, y_1) \cdots d^{m-1}(y_0, y_1) \\
\leq d^{\frac{k}{m}}(y_0, y_1) \to 1
given as \( n \to \infty \). It follows that \( \{y_n\} \) is a multiplicative Cauchy sequence. Since \( X \) is complete so \( \{y_n\} \to z \in X \). Therefore, subsequences \( \{Sx_{2n}\}, \{Bx_{2n+1}\}, \{Ax_{2n}\} \) and \( \{Tx_{2n+1}\} \) also converge to \( z \in X \).

Now suppose that \( A(X) \) is complete. Then there exists \( w \in X \) such that \( Aw = z \).

Now, we claim that \( Sw = z \). Let \( Sw \neq z \). On putting \( x = w \) and \( y = x_{2n+1} \) in inequality (C2), we get
\[
d(Sw, y_{2n+1}) \\
= d(Sw, Tx_{2n+1}) \\
\leq (d(Aw, Bx_{2n+1}) \circ d(Aw, Sw))^{k_1} \\
\cdot (d(Aw, Bx_{2n+1}) \circ d(Bx_{2n+1}, Tx_{2n+1}))^{k_2} \\
\cdot (d(Aw, Bx_{2n+1}) \circ (d(Aw, Tx_{2n+1}) \cdot d(Sw, Bx_{2n+1})))^{k_3}.
\]
As \( n \to \infty \), we have
\[
d(Sw, z) \leq (d(z, z) \circ d(z, Sw))^{k_1} \cdot (d(z, z) \circ d(z, z))^{k_2} \\
\cdot (d(z, z) \circ (d(z, z) \cdot d(Sw, z))^{\frac{1}{2}})^{k_3} \\
\leq ((\max\{1, d(z, Sw)\})^{\alpha})^{k_1} \cdot ((\max\{1, 1\})^{\alpha})^{k_2} \\
\cdot ((\max\{1, d^{\frac{1}{2}}(Sw, z)\})^{\alpha})^{k_3} \\
= d^{\alpha(k_1 + \frac{k_3}{2})}(Sw, z),
\]
which is a contradiction. Hence \( Sw = z \). This implies \( z = Sw = Aw \). Therefore, \( w \) is a coincidence point of \( A \) and \( S \). Since \( z = Sw \in S(X) \subset B(X) \), there exists \( v \in X \) such that \( z = Bv \).

Next, we claim that \( Tv = z \). Let \( Tv \neq z \). On putting \( x = x_{2n} \) and \( y = v \) in inequality (C2), we have
\[
d(Sx_{2n}, Tv) \\
\leq (d(Ax_{2n}, Bv) \circ d(Ax_{2n}, Sx_{2n}))^{k_1} \cdot (d(Ax_{2n}, Bv) \circ d(Bv, Tv))^{k_2} \\
\cdot (d(Ax_{2n}, Bv) \circ (d(Ax_{2n}, Tv) \cdot d(Sx_{2n}, Bv)))^{\frac{1}{2}})^{k_3}.
\]
As \( n \to \infty \), we have
\[
d(z, T v) \leq (d(z, z) \odot d(z, z))^{k_1} \cdot (d(z, z) \odot d(z, T v))^{k_2} \\
\quad \cdot (d(z, z) \odot (d(z, T v) \cdot d(z, z))^\frac{1}{2})^{k_3} \\
\leq ((\max\{1, 1\})^{\alpha})^{k_1} \cdot ((\max\{1, d(T v, z)\})^{\alpha})^{k_2} \\
\quad \cdot ((\max\{1, d^\frac{1}{2}(z, T v)\})^{\alpha})^{k_3} \\
= d^{\alpha(k_2 + k_3)}(z, T v),
\]
which is a contradiction, therefore, \( z = T v = B v \). Hence \( v \) is a coincidence point of \( B \) and \( T \). Since the pairs \( A, S \) and \( B, T \) are weakly compatible, we have
\[
S z = S(Aw) = A(Sw) = Az
\]
and
\[
T z = T(Bv) = B(T v) = Bz.
\]
Next, we claim that \( S z = z \). Let \( S z \neq z \). Then using inequality (C_2) and on putting \( x = z \) and \( y = x_{2n+1} \), we have
\[
d(Sz, T x_{2n+1}) \\
\leq (d(Az, Bx_{2n+1}) \odot d(Az, S z))^{k_1} \\
\quad \cdot (d(Az, Bx_{2n+1}) \odot d(Bx_{2n+1}, T x_{2n+1}))^{k_2} \\
\quad \cdot (d(Az, Bx_{2n+1}) \odot (d(Az, T x_{2n+1}) \cdot d(Sz, Bx_{2n+1}))^\frac{1}{2})^{k_3}.
\]
Letting \( n \to \infty \), we have
\[
d(Sz, z) \leq (d(Sz, z) \odot 1)^{k_1} \cdot (d(Sz, z) \odot d(z, z))^{k_2} \\
\quad \cdot (d(Sz, z) \odot (d(Sz, z) \cdot d(Sz, z))^\frac{1}{2})^{k_3} \\
\leq ((\max\{d(Sz, z), 1\})^{\alpha})^{k_1} \cdot ((\max\{d(Sz, z), 1\})^{\alpha})^{k_2} \\
\quad \cdot ((\max\{d(Sz, z), d(Sz, z)\})^{\alpha})^{k_3} \\
= d^{\alpha(k_1 + k_2 + k_3)}(Sz, z),
\]
which is a contradiction. So, we have \( Sz = z \) and hence we have \( Sz = Az = z \).

Finally, we claim that \( T z = z \). Let \( T z \neq z \). Then on putting \( x = x_{2n} \) and \( y = z \) in inequality (C_2), we have
\[
d(Sx_{2n}, T z) \leq (d(Ax_{2n}, B z) \odot d(Ax_{2n}, S x_{2n}))^{k_1} \\
\quad \cdot (d(Ax_{2n}, B z) \odot d(B z, T z))^{k_2} \\
\quad \cdot (d(Ax_{2n}, B z) \odot (d(Ax_{2n}, T z) \cdot d(Sx_{2n}, B z))^\frac{1}{2})^{k_3}.
\]
Letting $n \to \infty$, we have
\[
d(z, Tz) \leq (d(z, Tz) \diamond 1)^{k_1} \cdot (d(z, Tz) \diamond 1)^{k_2} \\
\cdot (d(z, Tz) \diamond (d(z, Tz) \cdot d(z, Tz))^{\frac{1}{2}})^{k_3} \\
\leq ((\max\{d(z, Tz), 1\})^{\alpha})^{k_1} \cdot ((\max\{d(Tz, z), 1\})^{\alpha})^{k_2} \\
\cdot ((\max\{d(Tz, z), d(Tz, z)\})^{\alpha})^{k_3} \\
= d^{\alpha(k_1+k_2+k_3)}(Tz, z),
\]
which is a contradiction, therefore, we have $Tz = z$ and hence we conclude that $Tz = Bz = z$. Therefore, $z$ is a common fixed point of $A, B, S$ and $T$.

Similarly we can complete the proof for cases in which $B(X)$ or $S(X)$ or $T(X)$ is complete. Uniqueness can easily follows from inequality $(C_2)$. This completes the proof.

In Theorem 2.1, if we put $S = T$, then we obtain the following corollary.

**Corollary 2.2.** Let $(X, d)$ be a complete multiplicative metric space such that $\diamond$ satisfies a multiplicative $\alpha$-property with $\alpha > 0$. Let $A, B$ and $S$ be mappings of $(X, d)$ into itself satisfying the following conditions

\[(C_5) \quad S(X) \subset A(X), \quad S(X) \subset B(X),\]

\[(C_6) \quad d(Sx, Sy) \leq (d(Ax, By) \diamond d(Ax, Sx))^{k_1} \cdot (d(Ax, By) \diamond d(By, Sy))^{k_2} \cdot (d(Ax, By) \diamond (d(Ax, Sy) \cdot d(Sx, By))^{\frac{1}{2}})^{k_3}\]

for all $x, y \in X$, where $k_1, k_2, k_3 > 0$ and $0 < \alpha(k_1 + k_2 + k_3) < 1$.

Assume that

\[(C_7) \quad \text{the pairs } A, S \text{ and } B, S \text{ are weakly compatible},\]

\[(C_8) \quad \text{one of the subspaces } A(X) \text{ or } B(X) \text{ or } S(X) \text{ is complete}.\]

Then $A, B$ and $S$ have a unique common fixed point in $X$.

In Theorem 2.1, if we put $A = B = I$, then we obtain the following corollary.

**Corollary 2.3.** Let $(X, d)$ be a complete multiplicative metric space such that $\diamond$ satisfies a multiplicative $\alpha$-property with $\alpha > 0$. Let $S$ and $T$ be mappings of $(X, d)$ into itself satisfying the following conditions

\[(C_9) \quad d(Sx, Ty) \leq (d(x, y) \diamond d(x, Sx))^{k_1} \cdot (d(x, y) \diamond d(y, Ty))^{k_2} \cdot (d(x, y) \diamond (d(x, Ty) \cdot d(Sx, y))^{\frac{1}{2}})^{k_3}\]
for all $x, y \in X$, where $k_1, k_2, k_3 > 0$ and $0 < \alpha(k_1 + k_2 + k_3) < 1$.

Assume that

$(C_{10})$ one of the subspaces $S(X)$ or $T(X)$ is complete.

Then $S$ and $T$ have a unique common fixed point in $X$.

Now we prove common fixed point theorems for weakly compatible mappings in multiplicative metric spaces without completeness of $X$ as follow:

**Theorem 2.4.** Let $(X, d)$ be a multiplicative metric space such that $\diamond$ satisfies a multiplicative $\alpha$-property with $\alpha > 0$. Let $A, B, S$ and $T$ be mappings of $(X, d)$ into itself satisfying the conditions $(C_1)$-$(C_4)$. Then $A, B, S$ and $T$ have a unique common fixed point.

**Proof.** From the proofs of Theorem 2.1, $\{y_n\}$ is a multiplicative Cauchy sequence. Suppose that $A(X)$ is complete. Then there exists $u \in A(X)$ such that

$$y_{2n+1} = Ax_{2n+2} = Tx_{2n+1} \to u \text{ as } n \to \infty.$$ 

Consequently, we can find $v \in X$ such that $Av = u$. Further a multiplicative Cauchy sequence $\{y_n\}$ has a convergent subsequence $\{y_{2n+1}\}$, therefore, the sequence $\{y_n\}$ also converges and hence a subsequence $\{y_{2n}\}$ also converges. Thus we have

$$y_{2n} = Bx_{2n+1} = Sx_{2n} \to u \text{ as } n \to \infty.$$ 

Now, we claim $Sv = u$. Let $Sv \neq u$. Then putting $x = v$ and $y = x_{2n+1}$ in inequality $(C_2)$, we get

$$d(Sv, y_{2n+1}) = d(Sv, Tx_{2n+1}) \leq (d(Av, Bx_{2n+1}) \diamond d(Av, Sv))^{k_1} \cdot (d(Av, Bx_{2n+1}) \diamond d(Bx_{2n+1}, Tx_{2n+1}))^{k_2} \cdot (d(Av, Bx_{2n+1}) \diamond (d(Av, Tx_{2n+1}) \cdot d(Sv, Bx_{2n+1}))^{\frac{1}{2}})^{k_3}.$$ 

Letting $n \to \infty$, we have

$$d(Sv, u) \leq (d(u, u) \diamond d(u, Sv))^{k_1} \cdot (d(u, u) \diamond d(u, Sv))^{k_2} \cdot (d(u, u) \diamond (d(Sv, u))^{\frac{1}{2}})^{k_3} \leq ((\max\{1, d(u, Sv)\})^{\alpha})^{k_1} \cdot ((\max\{1, 1\})^{\alpha})^{k_2} \cdot ((\max\{1, d(u, Sv)\})^{\alpha})^{k_3} = d^{\alpha(k_1 + \frac{k_3}{2})}(u, Sv).$$
which is a contradiction. Hence $Sv = u$ and hence $u = Av = Sv$, that is $v$ is a coincidence point of $A$ and $S$.

Since $u = Sv \in S(X) \subset B(X)$, there exists $w \in X$ such that $u = Bw$.

Next, we claim $Tw = u$. Let $Tw \neq u$. Then on putting $x = v$ and $y = w$ in inequality $(C_2)$, we have

\[
d(u, Tw) = d(Sv, Tw) \leq (d(Av, Bw) \diamond d(Av, Sv))^{k_1} \cdot (d(Av, Bw) \diamond d(Bw, Tw))^{k_2} \\
\quad \cdot (d(Av, Bw) \diamond (d(Av, Tw) \cdot d(Sv, Bw))^{\frac{1}{2}})^{k_3} \\
= d(u, u) \diamond d(u, u))^{k_1} \cdot (d(u, u) \diamond d(u, Tw))^{k_2} \\
\quad \cdot (d(u, u) \diamond d(u, Tw) \cdot d(u, u))^{\frac{1}{2}})^{k_3} \leq ((\max\{1, 1\})^{\alpha})^{k_1} \cdot ((\max\{1, d(u, Tw)\})^{\alpha})^{k_2} \\
\quad \cdot ((\max\{1, d^{\frac{1}{2}}(u, Tw)\})^{\alpha})^{k_3} \\
= d^{\alpha(k_1 + k_2 + k_3)}(u, Tw),
\]

which is a contradiction, which implies that $u = Tw$ and hence $u = Bw = Tw$, that is, $w$ is a coincidence point of $B$ and $T$. Since the pairs $A, S$ and $B, T$ are weakly compatible, we have

\[Su = S(Av) = A(Sv) = Au = w_1 \quad \text{(say)}\]

and

\[Tu = T(Bw) = B(Tw) = Bu = w_2 \quad \text{(say)}.
\]

From inequality $(C_2)$, we have

\[
d(w_1, w_2) = d(Su, Tu) \leq (d(Au, Bu) \diamond d(Au, Su))^{k_1} \cdot (d(Au, Bu) \diamond d(Bu, Tu))^{k_2} \\
\quad \cdot (d(Au, Bu) \diamond (d(Au, Tu) \cdot d(Su, Bu))^{\frac{1}{2}})^{k_3} \\
= (d(w_1, w_2) \diamond d(w_1, w_1))^{k_1} \cdot (d(w_1, w_2) \diamond d(w_2, w_2))^{k_2} \\
\quad \cdot (d(w_1, w_2) \diamond (d(w_1, w_2) \cdot d(w_1, w_2))^{\frac{1}{2}})^{k_3} \leq ((\max\{d(w_1, w_2), 1\})^{\alpha})^{k_1} \cdot ((\max\{d(w_1, w_2), 1\})^{\alpha})^{k_2} \\
\quad \cdot ((\max\{d(w_1, w_2), d(w_1, w_2)\})^{\alpha})^{k_3} \\
= d^{\alpha(k_1 + k_2 + k_3)}(w_1, w_2),
\]
which is a contradiction, that is, $w_1 = w_2$. Therefore, we have $Su = Au = Tu = Bu$

Again using inequality $(C_2)$, we have

$$d(Sv, Tu) \leq (d(Av, Bu) \diamond d(Av, Sv))^{k_1} \cdot (d(Av, Bu) \diamond d(Bu, Tu))^{k_2} \cdot (d(Av, Bu) \diamond (d(Av, Tu) \cdot d(Sv, Bu))^\frac{1}{2})^{k_3}$$

$$= (d(Sv, Tu) \diamond d(Av, Sv))^{k_1} \cdot (d(Sv, Tu) \diamond d(Tu, Tu))^{k_2} \cdot (d(Sv, Tu) \diamond (d(Sv, Tu) \cdot d(Sv, Tu))^\frac{1}{2})^{k_3}$$

$$\leq ((\max\{d(Sv, Tu), 1\})^\alpha)^{k_1} \cdot ((\max\{d(Sv, Tu), 1\})^\alpha)^{k_2} \cdot ((\max\{d(Sv, Tu), d(Sv, Tu)\})^\alpha)^{k_3}$$

$$= d^{\alpha(k_1 + k_2 + k_3)}(Sv, Tu),$$

which is a contradiction, this implies that $Sv = Tu$, that is, $u = Tu$. Therefore $u$ is a common fixed point of $A, B, S$ and $T$.

Similarly we can complete the proof for cases in which $B(X)$ or $S(X)$ or $T(X)$ is complete.

Uniqueness can easily follows from inequality $(C_2)$. This completes the proof. \qed

In Theorem 2.4, if we put $S = T$, then we obtain the following corollary.

**Corollary 2.5.** Let $(X, d)$ be a multiplicative metric space such that $\diamond$ satisfies a multiplicative $\alpha$-property with $\alpha > 0$. Let $A, B$ and $S$ be mappings of $(X, d)$ into itself satisfying the conditions $(C_5)-(C_8)$. Then $A, B$ and $S$ have a unique common fixed point in $X$.

In Theorem 2.4, if we put $A = B = I$, then we obtain the following corollary.

**Corollary 2.6.** Let $(X, d)$ be a multiplicative metric space such that $\diamond$ satisfies a multiplicative $\alpha$-property with $\alpha > 0$. Let $S$ and $T$ be mappings of $(X, d)$ into itself satisfying the conditions $(C_9)$ and $(C_{10})$. Then $S$ and $T$ have a unique common fixed point in $X$.

References


