

**KRASNER TERNARY HYPER FIELDS AND MORE  
CHARACTERIZATION OF PRIME AND MAXIMAL  
HYPER IDEALS IN KRASNER TERNARY HYPER RINGS**

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**Abstract:** In 2010, Davvaz and Mirvakili introduced a new class of hyper structure called an  $(m, n)$ -Krasner hyperring, constructed its quotient class where the hyper ideal considered in the said construction is normal, and proved the isomorphism theorems. Recently, Castillo and Vilela investigated the  $(2, 3)$ -Krasner hyperring called a Krasner ternary hyperring, and constructed its quotient class via another relation where the hyper ideal considered in the construction is not necessarily normal. In this article, a Krasner ternary hyper field is defined and characterized. Moreover, more structure results on prime and maximal hyper ideals not found in [1] are established.

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**Key Words:** Krasner ternary hyperring, Krasner  $(m, n)$ -hyperring

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## **1. Introduction**

Hyperstructure theory was introduced in 1934 by the French Mathematician F. Marty [12] at the 8th Congress of Scandinavian Mathematicians where he defined the notion of hyperoperation on groups. In a classical algebraic structure, the binary operation of two elements of a set is again an element of the same set, while in an algebraic hyperstructure, the hyperoperation of two elements, is a subset of the same set. If this hyperoperation sends two elements to a singleton then the hyperoperation coincides with the classical binary operation.

In literature, a number of different hyperstructure theories are widely studied since these represent a suitable and natural generalization of classical algebraic structures such as groups, rings and modules and for their applications to many areas of pure and applied mathematics and computer science.

In 2010, Davvaz and Mirvakili [8] defined a new class of  $n$ -ary multi-valued algebra called an  $(m, n)$ -Krasner hyperring and defined a fundamental relation on the structure from which they obtained  $(m, n)$ -rings. Recently, Anvariye [2] carried out a research on  $(m, n)$ -ary hyper modules over  $(m, n)$ -hyperring and has been able to define a strongly compatible relation on the structure and determined a sufficient condition that the said relation is transitive.

In this paper, more characterization of prime and maximal hyper ideals not explored in [1] are presented. Moreover, Krasner ternary hyper field is defined and characterized in several ways.

## 2. Preliminaries

This section presents some definitions in hyperstructure theory and the preliminary concepts of this study.

**Definition 2.1.** [3] Let  $H$  be a nonempty set and let  $\wp^*(H)$  be the set of all nonempty subsets of  $H$ . A map  $*$  :  $H \times H \rightarrow \wp^*(H)$  is called a *hyperoperation* on the set  $H$ . The couple  $(H, *)$  is called a *hypergroupoid*. If  $A$  and  $B$  are nonempty subsets of  $H$ , we define

$$\text{i. } A * B = \bigcup_{a \in A, b \in B} a * b;$$

$$\text{ii. } x * A = \bigcup_{a \in A} x * a;$$

$$\text{iii. } B * x = \bigcup_{b \in B} b * x.$$

**Example 2.2.** Let  $\mathbb{N}$  be the set of positive integers. Define a hyperoperation “ $*$ ” on  $\mathbb{N}$  by  $x * y = \{x, y\}$  for all  $x, y \in \mathbb{N}$ . Then  $(\mathbb{N}, *)$  is a hypergroupoid.

**Definition 2.3.** [3] A hypergroupoid  $(H, *)$  is *commutative* if  $a * b = b * a$  for all  $a, b \in H$ .

**Definition 2.4.** [3] A hypergroupoid  $(H, *)$  is called a *semihypergroup* if for all  $x, y, z \in H$ ,  $(x * y) * z = x * (y * z)$ .

**Definition 2.5.** [3] A nonempty subset  $B$  of a semihypergroup  $H$  is called a *subsemihypergroup* of  $H$  if  $(B, *)$  is itself a semihypergroup.

**Definition 2.6.** [3] Let  $(H, *)$  be a semihypergroup. Then  $H$  is called a *hypergroup* if it satisfies the *reproduction axiom*, that is,  $x * H = H * x = H$  for all  $x \in H$ .

**Definition 2.7.** [3] A nonempty set  $H$  with a hyperoperation “+” is said to be a *canonical hypergroup* if the following are satisfied:

- i. For every  $x, y \in H$ ,  $x + y = y + x$ ;
- ii. for every  $x, y, z \in H$ ,  $x + (y + z) = (x + y) + z$ ;
- iii.  $\exists 0_H \in H$  (called the *neutral element of H*) such that  $x + 0_H = \{x\}$  and  $0_H + x = \{x\}$  for all  $x \in H$ ;
- iv. for each  $x \in H$  there exists a unique element denoted by  $-x \in H$  such that  $0_H \in [x + (-x)] \cap [(-x) + x]$ ;
- v. for every  $x, y, z \in H$ ,  $z \in x + y$  implies  $y \in -x + z$  and  $x \in -y + z$ .

The pair  $(H, +)$  is called a *canonical hypergroup*.

**Definition 2.8.** [11] Let  $R$  be a nonempty set. A *ternary operation* on  $R$  is a map  $f : R \times R \times R \rightarrow R$ .

**Example 2.9.** Consider the set  $\mathbb{Z}^-$  of negative integers and “ $\cdot$ ” is the usual multiplication on  $\mathbb{Z}$ . Then “ $\cdot$ ” is a ternary operation in  $\mathbb{Z}^-$  which is not binary, since  $\mathbb{Z}^-$  is not closed under the binary product.

The remaining definitions and result in this section is taken from [5] unless otherwise specified.

**Definition 2.10.** A *Krasner ternary hyperring* is an algebraic structure  $(R, +, \cdot)$  consisting of a nonempty set  $R$ , a hyperoperation “+” and a ternary multiplication “ $\cdot$ ” satisfying the following:

- i.  $(R, +)$  is a canonical hypergroup;
- ii.  $(a \cdot b \cdot c) \cdot d \cdot e = a \cdot (b \cdot c \cdot d) \cdot e = a \cdot b \cdot (c \cdot d \cdot e)$ ;
- iii.  $(a + b) \cdot c \cdot d = a \cdot c \cdot d + b \cdot c \cdot d$ ;
- iv.  $a \cdot (b + c) \cdot d = a \cdot b \cdot d + a \cdot c \cdot d$ ;
- v.  $a \cdot b \cdot (c + d) = a \cdot b \cdot c + a \cdot b \cdot d$ ,

$$\text{vi. } a \cdot 0 \cdot 0 = 0 \cdot a \cdot 0 = 0 \cdot 0 \cdot a = 0.$$

for every  $a, b, c, d, e \in R$ . If without ambiguity,  $a \cdot b \cdot c$  can be written as  $abc$  unless otherwise specified.

**Example 2.11.** Let  $R = [0, 1]$ . We define a hyperaddition “+” on  $R$  as follows:

$$x + y = \begin{cases} [0, x], & \text{if } x = y, \\ \{\max\{x, y\}\}, & \text{if } x \neq y. \end{cases}$$

and the ternary multiplication “.” to be the ordinary multiplication. Then  $(R, +, \cdot)$  is a Krasner ternary hyperring.

**Definition 2.12.** Let  $(R, +, \cdot)$  be a Krasner ternary hyperring. A subset  $S$  of  $R$  is called a *Krasner subternary hyperring* if  $(S, +, \cdot)$  is itself a Krasner ternary hyperring. A Krasner subternary hyperring  $S$  of  $R$  such that  $S \neq R$  and  $S \neq \{0\}$  is called *proper subternary hyperring* of  $R$ .

**Definition 2.13.** A subternary hyperring  $I$  of a Krasner ternary hyperring  $R$  is called a

- i. *right hyper ideal* of  $R$  if  $abi \in I$  for all  $a, b \in R$  and  $i \in I$ ;
- ii. *left hyper ideal* of  $R$  if  $iab \in I$  for all  $a, b \in R$  and  $i \in I$ ;
- iii. *lateral hyper ideal* of  $R$  if  $aib \in I$  for all  $a, b \in R$  and  $i \in I$ ;
- iv. *two sided hyper ideal* If  $I$  is both a right and left hyper ideal;
- v. *hyper ideal* if  $I$  is left, right and lateral hyper ideal.

**Theorem 2.14. (Hyper ideal Criterion)** Let  $R$  be a Krasner ternary hyperring. A nonempty subset  $I$  of  $R$  is a right [resp. left and lateral] hyper ideal of  $R$  if and only if for all  $i, j \in I$  and  $a, b \in R$

- i.  $i + (-j) \subseteq I$ ,
- ii.  $abi \in I$  [resp.  $iab \in R$  and  $aib \in R$ ].

**Definition 2.15.** Let  $R$  be a Krasner ternary hyperring. We say that  $R$  admits an identity provided that there exists

$$\{(e_i, f_i) \in R \times R \mid e_i, f_i \in R \text{ and } n \in \mathbb{N}\}$$

such that  $\sum_{i=1}^n e_i f_i x = \sum_{i=1}^n e_i x f_i = \sum_{i=1}^n x e_i f_i = \{x\}$  for all  $x \in R$ . In this case, the Krasner ternary hyperring is said to be a *Krasner ternary hyperring with identity*. In particular, if there exists an element  $1_R \in R$  such that  $1_R 1_R x = 1_R x 1_R = x 1_R 1_R = x$  for all  $x \in R$ , then  $1_R$  is called a *unital element* of  $R$ .

**Theorem 2.16.** *Let  $R$  be a Krasner ternary hyperring with unital element  $1_R$  and  $x \in R$ . Then the principal*

- i. *right hyper ideal generated by  $x$  is given by*

$$\langle x \rangle_r = RRx = \bigcup \left\{ \sum_{i=1}^n r_i s_i x \mid r_i, s_i \in R \text{ and } n \in \mathbb{N} \right\},$$

- ii. *left hyper ideal generated by  $x$  is given by*

$$\langle x \rangle_l = xRR = \bigcup \left\{ \sum_{i=1}^n x r_i s_i \mid r_i, s_i \in R \text{ and } n \in \mathbb{N} \right\},$$

- iii. *two-sided hyper ideal generated by  $x$  is given by*

$$\langle x \rangle_l^r = RxxR = \bigcup \left\{ \sum_{i=1}^n r_i x s_i \mid r_i, s_i \in R \text{ and } n \in \mathbb{N} \right\},$$

- iv. *lateral hyper ideal generated by  $x$  is given by*

$$\langle x \rangle_c = RxxR + RRxRR = \bigcup \left\{ \sum_{i=1}^n r_i x s_i + \sum_{j=1}^m p_j q_j x u_j v_j \right\},$$

where  $r_i, s_i, p_j, q_j, u_j, v_j \in R$  and  $m, n \in \mathbb{N}$ .

**Lemma 2.17.** *Let  $R$  be a Krasner ternary hyperring with unital element  $1_R$  and  $I$  a hyper ideal of  $R$ . If  $1_R \notin I$ , then  $I \neq R$ .*

**Remark 2.18.** *Let  $R$  be a Krasner ternary hyperring,  $I$  a hyper ideal of  $R$ , and  $x, y \in R$ . Then  $x + I = y + I$  if and only if  $x \in y + I$ .*

**Theorem 2.19.** *Let  $I$  and  $J$  be right [ resp. left and lateral ] hyper ideals of a Krasner ternary hyperring  $R$  such that  $I \subseteq J$ . Then  $J/I$  is a hyper ideal of  $R/I$ .*

**Definition 2.20.** Let  $(R_1, +, \cdot)$  and  $(R_2, \oplus, *)$  be two ternary hyperrings. A mapping  $\phi : R_1 \rightarrow R_2$  is called a *homomorphism* if the following are satisfied:

- i.  $\phi(a + b) = \phi(a) \oplus \phi(b)$  for all  $a, b \in H$ ;
- ii.  $\phi(a_1 \cdot a_2 \cdot a_3) = \phi(a_1) * \phi(a_2) * \phi(a_3)$  for all  $a_1, a_2, a_3 \in R_1$ ;
- iii.  $\phi(0_{R_1}) = 0_{R_2}$ .

**Theorem 2.21.** Let  $f : R_1 \rightarrow R_2$  be a homomorphism of Krasner ternary hyperrings. If  $B$  is a hyper ideal of  $R_2$ , then

$$f^{-1}(B) = \{a \in R_1 \mid f(a) \in B\}$$

is a hyper ideal of  $R_1$  containing  $\text{Ker } f$ .

**Definition 2.22.** Let  $(R, +, \cdot)$  be a Krasner ternary hyperring. A non-empty subset  $M$  of  $R$  is said to be *multiplicative* in  $R$  if  $abc \in M$  for all  $a, b, c \in M$ .

**Theorem 2.23.** [1] Let  $S$  be a multiplicative subset of a Krasner ternary hyperring disjoint from a hyper ideal  $I$ . Then there exist a hyper ideal  $P$  which is maximal in the set of all hyper ideals of  $R$  disjoint from  $S$  containing  $I$ . Furthermore,  $P$  is prime.

### 3. Main Results

This section presents the results generated in this study.

**Definition 3.1.** A Krasner ternary hyperring  $R$  is said to be *commutative* if for all  $a, b, c \in R$ ,  $abc = acb = bac = bca = cab = cba$ .

**Definition 3.2.** Let  $R$  be a Krasner ternary hyperring. A nonzero element  $a$  of  $R$  is said to be *right invertible* if there exist  $b \in R$  such that  $atb = t$  for all  $t \in R$ , it is called *left invertible* if for all  $t \in R$ , there exists a nonzero  $b \in R$  such that  $bat = t$  and *lateral invertible* if  $abt = t$ . If  $a$  is right, left and lateral invertible then  $a$  is called *invertible*. The element  $b$  is called the *inverse* of  $a$ .

**Definition 3.3.** A Krasner ternary hyperring  $R$  is said to be *zero divisor free* if for all  $a, b, c \in R$  with  $abc = 0$  implies  $a = 0$  or  $b = 0$  or  $c = 0$ .

**Definition 3.4.** Let  $R$  be a Krasner ternary hyperring. A nonzero element  $a$  of  $R$  is called a *unit* in  $R$  if  $a$  is invertible. A Krasner ternary hyperring in which every nonzero element is a unit is called a *zero divisor free*.

**Definition 3.5.** A commutative zero divisor free Krasner ternary hyperring is called a *Krasner ternary hyper field*.

**Definition 3.6.** A hyper ideal  $P$  of a Krasner ternary hyperring  $R$  is called a *prime hyper ideal* of  $R$ , if for every hyper ideals  $I, J$  and  $K$  of  $R$  satisfying  $IJK \subseteq P$  implies either  $I \subseteq P$  or  $J \subseteq P$  or  $K \subseteq P$ .

**Definition 3.7.** A hyper ideal  $M$  of a Krasner ternary hyperring  $R$  said to be *maximal* if the following are satisfied:

- i.  $M \subset R$ ,
- ii. if  $I$  is a hyper ideal of  $R$  such that  $M \subset I$  then,  $I = R$ .

The next result guarantees the existence of a maximal hyper ideal in every Krasner ternary hyperring.

**Theorem 3.8.** *Let  $R$  be a non trivial Krasner ternary hyperring with unital element  $1_R$ . Then a maximal right [resp. left, lateral] hyper ideal always exists and every hyper ideal in  $R$  is contained in a maximal right hyper ideal.*

*Proof.* Let  $I$  be a right hyper ideal in  $R$  and

$$\mathcal{S} = \{J_i \mid J_i \text{ is a right hyper ideal of } R, I \subseteq J \text{ and } J \neq R\}.$$

Then  $\mathcal{S} \neq \emptyset$  since  $I \in \mathcal{S}$ . Partially order  $\mathcal{S}$  by set inclusion, that is,  $J_1 \leq J_2$  if and only if  $J_1 \subseteq J_2$ . Let  $\mathcal{C} = \{C_\omega \mid \omega \in \Omega\}$  be a chain in  $\mathcal{S}$  and  $U = \bigcup_{\omega \in \Omega} C_\omega$ .

If  $a, b \in U$ , then there exists  $C_i, C_j \in \mathcal{C}$  such that  $a \in C_i$  and  $b \in C_j$ . Since  $\mathcal{C}$  is a chain, either  $C_i \subseteq C_j$  or  $C_j \subseteq C_i$ . Without loss of generality, suppose that  $C_i \subseteq C_j$ . Then  $a, b \in C_j$ . Since  $C_j$  is a right hyper ideal, by Theorem 2.14,

$a + (-b) \subseteq C_j$  and  $rs c \in C_j$  for all  $r, s \in R$  and  $c \in C_j$ . Since  $C_j \subseteq \bigcup_{\omega \in \Omega} C_\omega = U$ ,

$a + (-b) \subseteq U$  and  $rs c \in U$ . Therefore, by Theorem 2.14,  $U$  is a right hyper ideal of  $R$ . Since each  $C_\omega$  is in  $\mathcal{S}$ ,  $C_\omega \neq R$  for all  $\omega \in \Omega$ . By Remark 2.17,

$1_R \notin C_\omega$  for all  $\omega \in \Omega$ . Hence,  $1_R \notin \bigcup_{\omega \in \Omega} C_\omega = U$ . Hence, by Remark 2.17,

$U \neq R$ . Thus,  $U \in \mathcal{S}$ . Since  $C_\omega \subseteq U$  for all  $\omega \in \Omega$ ,  $U$  is an upperbound of the chain  $\mathcal{C}$ . Thus, every chain in  $\mathcal{S}$  has an upperbound in  $\mathcal{S}$ . So, by Zorn's Lemma,  $\mathcal{S}$  has a maximal element. But a Maximal element of  $\mathcal{S}$  is a maximal hyper ideal of  $R$  that contains  $I$ . □

**Theorem 3.9.** *Let  $M$  be a hyper ideal in a Krasner ternary hyperring  $R$  with unital element  $1_R \neq 0$ .*

- i. If  $M$  is maximal and  $R$  is commutative, then  $R/M$  is a Krasner ternary hyper field.
- ii. If  $R/M$  is zero divisor free, then  $M$  is maximal.

*Proof.*

- i. Let  $a+M \in R/M$  such that  $a+M \neq M$ . Then  $M$  is properly contained in  $\langle a \rangle_l + M$ . By maximality of  $M$ ,  $\langle a \rangle_l + M = R$ . Since  $1_R \in R$ ,  $1_R \in \langle a \rangle_l + M$ . Theorem 2.16,  $1_R \in \sum_{i=1}^n ar_i s_i + m$  for some  $m \in M$ ,  $n \in \mathbb{N}$ , and  $r_i, s_i \in R$ . Let  $t + M \in R/M$ . Then  $t \in R$  and for some  $m \in M$ ,

$$\begin{aligned}
 t &= 1_R 1_R t \in \left[ \left( \sum_{i=1}^n ar_i s_i \right) + m \right] \left[ \left( \sum_{i=1}^n ar_i s_i \right) + m \right] (t) \\
 &= (ar_1 s_1 + \dots + ar_n s_n + m) (ar_1 s_1 + \dots + ar_n s_n + m) (t) \\
 &= (ar_1 s_1 ar_1 s_1 t) + \dots + (ar_n s_n ar_1 s_1 t) + (mar_1 s_1 t) \\
 &\quad + (ar_1 s_1 ar_2 s_2 t) + \dots + (ar_n s_n ar_2 s_2 t) + (mar_2 s_2 t) + \dots \\
 &\quad + (ar_1 s_1 ar_n s_n t) + \dots + (ar_n s_n ar_n s_n t) + (mar_n s_n t) + mmt \\
 &= \sum_{i,j} a (r_i s_i ar_j s_j) t + \sum_{i=1}^n mar_i s_i t \\
 &\subseteq a \left( \sum_{i,j} (r_i s_i ar_j s_j) \right) t + M + M.
 \end{aligned}$$

Therefore,  $t \in abt + M$  for some  $b \in \sum_{i,j} r_i s_i ar_j s_j$ . By Remark 2.18,  $abt \in t + M$  for all  $t + M \in R/M$ . Hence,

$$(a + M)(b + M)(t + M) = t + M.$$

Therefore,  $a + M$  is invertible. Since  $R$  is commutative, it follows that  $R/M$  is a Krasner ternary hyper field.

- ii. Let  $N$  be a hyper ideal of  $R$  such that  $M \subset N \subseteq R$ . Then there exists an  $n \in N \setminus M$ . Since  $R/M$  is zero divisor free, for all  $M \neq a + M \in R/M$ , there exists  $b + M \in R/M$  such that  $(a + M)(b + M)(t + M) = t + M$ ,



for all  $t + M \in R/M$ . Thus,  $abt + M = t + M$ , for all  $t \in R$ . Hence, by Remark 2.18,  $abt \in t + M$ . In particular, since  $1_R \in R$ , there exists  $b \in R$  such that  $nb1_R \in 1_R + M$ . Hence,  $nb1_R \in 1_R + m$  for some  $m \in N$ . By Definition 2.7(v),  $1_R \in nb1_R + (-m)$ . Since  $n \in N$ ,  $nb1_R \in N$ . Also,  $m \in N$  since  $M \subseteq N$ . Consequently,  $1_R \in nb1_R + (-m) \subseteq N$ . By Theorem 2.17,  $N = R$ . Therefore,  $M$  is maximal.  $\square$

**Theorem 3.10.** *The following conditions in a commutative Krasner ternary hyperring  $R$  with unital  $1_R \neq 0$  are equivalent.*

- i.  $R$  is a Krasner ternary hyper field,
- ii.  $R$  has no nontrivial hyper ideals,
- iii.  $0$  is a maximal hyper ideal in  $R$ .

*Proof.* (i  $\Rightarrow$  ii). Suppose to the contrary that  $R$  has a nontrivial hyper ideal  $I$ . Then there exists  $0 \neq t \in R \setminus I$ . Let  $i \in I$ . Since  $R$  is a Krasner ternary hyper field, there exists  $b \in R$  such that  $ibr = r$  for all  $r \in R$ . In particular,  $ibt = t$ . Since  $I$  is a hyper ideal,  $t = ibt \in I$ , a contradiction. Therefore,  $R$  has no proper hyper ideal.

(ii  $\Rightarrow$  iii). Let  $I$  be a hyper ideal of  $R$  such that  $0 \subset I \subseteq R$ . Since  $R$  has no proper hyper ideal,  $0 = I$  or  $I = R$ . Thus,  $\{0\}$  is maximal.

(iii  $\Rightarrow$  i). Suppose that  $\{0\}$  is a maximal hyper ideal of  $R$ . Therefore, by Theorem 3.9(i),  $R = R/\{0\}$  is a Krasner ternary hyper field.  $\square$

**Theorem 3.11.** *Let  $R$  be a Krasner ternary hyperring with unital element  $1_R$ ,  $a, b, c$  are arbitrary elements of  $R$  but fixed, and  $P$  a hyper ideal of  $R$ . The following are equivalent:*

- i.  $P$  is a prime hyper ideal of  $R$ ,
- ii.  $\{ar_i br_j c \mid r_i, r_j \in R\} \subseteq P$  if and only if  $a \in P$  or  $b \in P$  or  $c \in P$ ,
- iii. If  $\langle a \rangle \langle b \rangle \langle c \rangle \subseteq P$ , then either  $a \in P$  or  $b \in P$  or  $c \in P$ .

*Proof.* (i  $\Rightarrow$  ii). Let  $P$  be a prime hyper ideal of  $R$ . Suppose that either  $a \in P$  or  $b \in P$  or  $c \in P$ . Without loss of generality, suppose that  $a \in P$ . Since  $P$  is a hyper ideal,  $ar_i br_j c \in P$  for all  $r_i, r_j \in R$ . So,

$$\{ar_i br_j c \mid r_i, r_j \in R\} \subseteq P.$$

Conversely, assume that  $\{ar_i br_j c \mid r_i, r_j \in R\} \subseteq P$  and suppose that  $a \notin P$  and  $b \notin P$ . Let  $I = \langle a \rangle_l$ ,  $J = \langle b \rangle_l^r$  and  $K = \langle c \rangle_r$ . Then  $a \in I$  and  $b \in J$ . Hence,  $I \not\subseteq P$  and  $J \not\subseteq P$ . Thus,

$$\begin{aligned} IJK &= (aRR)(RbR)(RRc) \\ &= a(RRR)b(RRR)c \\ &\subseteq aRbRc \\ &\subseteq P. \end{aligned}$$

Since  $P$  is prime, then either  $I = \langle a \rangle_l \subseteq P$  or  $J = \langle b \rangle_l^r \subseteq P$  or  $K = \langle c \rangle_r \subseteq P$ . But  $I \not\subseteq P$  and  $J \not\subseteq P$ . Therefore,  $K \subseteq P$  and  $c \in P$ . (ii  $\Rightarrow$  iii). Suppose that  $\langle a \rangle \langle b \rangle \langle c \rangle \subseteq P$ . Since  $a \in \langle a \rangle$ ,  $c \in \langle c \rangle$ , and for all  $r_i, r_j \in R$ ,  $r_i br_j \in \langle b \rangle$ , it follows that  $a(r_i br_j)c \in \langle a \rangle \langle b \rangle \langle c \rangle \subseteq P$ . Hence,  $\{ar_i br_j c \mid r_i, r_j \in R\} \subseteq P$ . By (ii), either  $a \in P$  or  $b \in P$  or  $c \in P$ . (iii  $\Rightarrow$  i). Let  $I, J$  and  $K$  be hyper ideals of  $R$  such that  $IJK \subseteq P$ . Suppose that  $I \not\subseteq P$  and  $J \not\subseteq P$ . Then there exists  $i \in I$  and  $j \in J$  such that  $i, j \notin P$ . Since  $\langle i \rangle \subseteq I$  and  $\langle j \rangle \subseteq J$  and for all  $k \in K$ ,  $\langle k \rangle \subseteq K$ . Thus,  $\langle i \rangle \langle j \rangle \langle k \rangle \subseteq IJK \subseteq P$ . By condition (iii), either  $i \in I$ ,  $j \in J$  or  $k \in K$ . Since  $i \notin I$  and  $j \notin J$ , it follows that  $k \in K$  and  $K \subseteq P$ . Consequently,  $P$  is prime.  $\square$

**Theorem 3.12.** *A hyper ideal  $P$  of a Krasner ternary hyperring  $R$  is prime if and only if for every  $a, b, c \in P^c$ , there exist  $r_i, r_j \in R$  such that  $ar_i br_j c \in P^c$ .*

*Proof.* Suppose that  $P$  is a prime hyper ideal of  $R$  and  $a, b, c \in P^c$ . Then  $a, b, c \notin P$ . By Theorem 3.11,  $\{ar_i br_j c \mid r_i, r_j \in R\} \not\subseteq P$ . Thus, there exist  $r_i, r_j \in R$  such that  $ar_i br_j c \notin P$ . Therefore,  $ar_i br_j c \in P^c$ . Conversely, let  $a, b, c \notin P$ . By assumption, there exist  $r_i, r_j \in R$  such that  $ar_i br_j c \in P^c$ , that is,  $ar_i br_j c \notin P$ . Thus,  $\{ar_i br_j c \mid r_i, r_j \in R\} \not\subseteq P$ . Therefore, by Theorem 3.11,  $P$  is a prime hyper ideal.  $\square$

**Theorem 3.13.** *A maximal hyper ideal of a Krasner ternary hyperring  $R$  with unital element  $1_R$  is prime.*

*Proof.* Let  $R$  be a Krasner ternary hyperring,  $M$  a maximal hyper ideal of  $R$ , and  $a, b$  and  $c$  are arbitrary elements of  $R$  but fixed. Let  $\{ar_i bs_j c \mid r_i, s_i \in R\} \subseteq M$ . Suppose that  $a \notin M$  and  $b \notin M$ . Then  $M \neq R$  and  $M$  is properly contained in  $\langle a \rangle_l + M$  and  $\langle b \rangle_l^r + M$ . Hence,  $\langle a \rangle_l + M = R$  and  $\langle b \rangle_l^r + M = R$ , by maximality of  $M$ . By Theorem 2.16,  $\langle a \rangle_l = iRR$  and  $\langle b \rangle_l^r = RjR$ . Thus, there exists  $r_i, q_i, u_j, v_j \in R$ ,  $n_1, n_2 \in \mathbb{N}$  and  $m_1, m_2 \in M$  such that

$$1_R \in \left( \sum_{i=1}^{n_1} ar_iq_i \right) + m_1 \text{ and } 1_R \in \left( \sum_{j=1}^{n_2} u_jbv_j \right) + m_2. \text{ Hence,}$$

$$\begin{aligned} k = 1_R 1_R k &\in \left[ \left( \sum_{i=1}^{n_1} ar_iq_i \right) + m_1 \right] \left[ \left( \sum_{j=1}^{n_2} u_jbv_j \right) + m_2 \right] (c) \\ &= (ar_1q_1 + \dots + ar_{n_1}q_{n_1} + m_1)(u_1bv_1 + \dots + u_{n_2}bv_{n_2} + m_2)(c) \\ &= \sum_{i,j} a(r_iq_iu_j)bv_jc + (m_1m_2c) \\ &\subseteq M. \end{aligned}$$

Therefore, by Theorem 3.11,  $M$  is prime. □

**Lemma 3.14.** *Let  $R$  be a Krasner ternary hyperring. If  $P$  is a prime hyper ideal of  $R$ , then  $P^c$  is a multiplicative subset of  $R$ .*

*Proof.* Let  $i, j, k \in P^c$ . Then  $i, j, k \notin P$ . Since  $P$  is prime,  $ijk \notin P$ . Thus,  $ijk \in P^c$ . Therefore,  $P^c$  is multiplicative. □

*Proof.* Let  $I, J$  and  $K$  be hyper ideals of  $R$  such that  $P \subset I, P \subset J$  and  $P \subset K$ . Since  $P \cap M = \emptyset$  and  $I \cap M \neq \emptyset$  implies  $I \cap M \neq \emptyset, J \cap M \neq \emptyset$  and  $K \cap M \neq \emptyset$ . Thus, there exists  $i \in I \cap M, j \in J \cap M$  and  $k \in K \cap M$ . Since  $M$  is multiplicative,  $ijk \in M$  therefore,  $ijk \in IJK \cap M$ . Since  $M \cap P = \emptyset, ijk \notin P$  consequently,  $IJK \not\subseteq P$ . Therefore,  $P$  is prime. □

**Theorem 3.15.** *Let  $R$  be a Krasner ternary hyperring. Then the complement of every proper hyper ideal of  $R$  contains a nonempty multiplicative subset of  $R$  if and only if every proper hyper ideal of  $R$  is properly contained in a prime hyper ideal.*

*Proof.* Let  $I$  be a proper hyper ideal in  $R$  and  $M$  a nonempty multiplicative subset of  $R$  such that  $M \subseteq I^c$ . Then  $I \cap M = \emptyset$ . By Theorem 2.23, there exists a hyper ideal  $P$  in  $R$  such that  $I \subset P, P \cap M = \emptyset$  and  $J \cap M \neq \emptyset$  for every hyper ideal  $J$  properly containing  $P$ . Therefore, by Theorem 2.23,  $P$  is prime. Conversely, suppose that  $I$  is a proper hyper ideal of  $R$  and  $P$ , a prime hyper ideal such that  $I \subset P$ . Thus,  $P^c \subset I^c$ . Since  $P$  is prime, by Lemma 3.14,  $P^c$  is multiplicative and  $P^c \neq \emptyset$  since  $P \neq R$ . Therefore,  $I^c$  contains a nonempty multiplicative set. □

**Theorem 3.16.** *Let  $R$  and  $S$  be Krasner ternary hyperrings,  $Q$  a prime hyper ideal of  $S$ , and  $f : R \rightarrow S$  be a homomorphism. Then  $f^{-1}(Q)$  is a prime hyper ideal of  $R$  which contains  $\text{Ker}f$ .*

*Proof.* By Theorem 2.21,  $f^{-1}(Q)$  is a hyper ideal of  $R$  which contains  $\text{Ker}\phi$ . Suppose  $x_1, x_2, x_3 \in R$  such that  $x_1x_2x_3 \in f^{-1}(Q)$ . Then  $f(x_1x_2x_3) \in Q$ . Since  $f$  is a homomorphism and  $Q$  is prime, either  $f(x_1) \in Q$ ,  $f(x_2) \in Q$ , or  $f(x_3) \in Q$ . Hence, either  $x_1 \in f^{-1}(Q)$ ,  $x_2 \in f^{-1}(Q)$ , or  $x_3 \in f^{-1}(Q)$ . Therefore,  $f^{-1}(Q)$  is prime.  $\square$

**Theorem 3.17.** *Let  $R$  be a Krasner ternary hyperring,  $I$  a hyper ideal of  $R$  and  $P$  a prime hyper ideal of  $R$  containing  $I$ . Then  $P/I$  is a prime hyper ideal of  $R/I$ .*

*Proof.* Since  $P$  is prime,  $P$  is properly contained in  $R$ . Thus, there exists  $x \in R$  such that  $x \notin P$ . Since  $I \subseteq P$ ,  $x \notin I$ . Thus  $x + I \neq I$  and  $x + I \notin P/I$ . Hence  $P/I \neq R/I$ . On the other hand, by Theorem 2.19,  $P/I$  is a hyper ideal of  $R/I$ . Let  $x + I, y + I, z + I \in R/I$  such that  $(x + I)(y + I)(z + I) \in P/I$ . Then  $xyz + I \in P/I$  and  $xyz \in P$ . Since  $P$  is prime, either  $x \in P$ ,  $y \in P$ , or  $z \in P$ . Thus, either  $x + I \in P/I$ ,  $y + I \in P/I$ , or  $z + I \in P/I$ .  $\square$

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