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ON LOCAL SPECTRAL PROPERTIES OF λ-COMMUTING OPERATORS

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Abstract: Let $\mathcal{B}(X)$ be the Banach algebra of all bounded operators on a complex Banach space X, for a scalar $\lambda \in \mathbb{C}$ two operators $T, S \in \mathcal{B}(X)$ are said to λ-commute if $TS = \lambda ST$. If it holds, we show that TS and ST have many basic local spectral properties in common.

Key Words: spectrum, operator equation, λ-commutativity, local spectral properties

1. Introduction

Throughout we will denote by $\mathcal{B}(X)$ the Banach algebra of all linear operators on the complex Banach space X. For $T \in \mathcal{B}(X)$ we denote by $\sigma(T)$, $N(T)$ and $R(T)$ the spectrum, the kernel and the range of T respectively.

Recently many mathematicians have been attracted by the question: under what conditions if $T, S \in \mathcal{B}(X)$ there is $\lambda \in \mathbb{C}$ such that $TS = \lambda ST$?

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It is well known that if X is a Hilbert space and T λ-commutes with a compact operator, then T has a non-trivial hyperinvariant subspace [5].

In [2] Brooke, Busch and Pearson showed that for $T, S \in \mathcal{B}(X)$ satisfying $TS = \lambda ST$ then $\sigma(TS) = \sigma(ST) = \lambda\sigma(TS)$. If TS is not quasinilpotent then necessary $|\lambda| = 1$, and if T or S is self-adjoint then $\lambda \in \mathbb{R}$. At 2004, Yang and Du gave a simple proofs and generalizations of this results, particulary they proved that if $TS = \lambda ST$ then TS is bounded below if and only if both T and S are bounded below [9, theorem 2.5]. Schmoeger in [8] generalized this results to hermitian or normal elements of a complex Banach algebra.

Cho, Duggal, Harte and ôta generalized some Schmoeger's results and they gave the new characterization of a commutativity of Banach space operators [3, theorem 2.4 and theorem 2.2].

In [4] where X is a complex Hilbert space, Conway and Prajitura characterized the closure and the interior of the set of operators that λ-commute with a compact operator.

At 2011, Zhang, Ohwada and Cho have studied the properties of Hilbert space operators that λ-commute with a paranormal operator [10, theorem 1 and theorem 3].

In the present paper, our aim is to study some properties and concepts in local spectral theory for Banach space operators satisfying the λ-commutativity.

For $T \in \mathcal{B}(X)$, let the following notations, for detail see [1], [6], and [7]:

The spectrum of T

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\},$$

The left spectrum

$$\sigma_l(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left invertible}\},$$

The right spectrum

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not The right spectrum }\},$$

The left or right spectrum

$$\sigma_{lr}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left or right inevertible}\},$$

The ponctual spectrum

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not injective}\},$$

The surjective spectrum

$$\sigma_{su}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not surjective}\},$$

The compression spectrum

$$\sigma_{com}(T) = \{\lambda \in \mathbb{C} \ : \ R(T-\lambda) \text{ is not dense in } X\},$$

The approximate point spectrum

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C};\ \exists (x_n)_{n\in\mathbb{N}}\ of\ X\ such\ that\ \|x_n\| = 1\ and\ (T-\lambda)x_n \to 0\}.$$

Recall that T has the single-valued extension property (SVEP) at $\lambda \in \mathbb{C}$ if for any neighborhood U_λ of λ the only analytical function of $f : U_\lambda \to X$ satisfying $(T-\mu)f(\mu) = x\ \forall \mu \in U_\lambda$ is the null function $f \equiv 0$.

We set

$$\mathcal{S}(T) = \{\lambda \in \mathbb{C} \ : T \text{ does not have SVEP at } \lambda\}.$$

We say that T has SVEP if $\mathcal{S}(T) = \emptyset$.

The local resolvent $\rho_T(x)$ of T at $x \in X$ is defined as the set of all $\lambda \in \mathbb{C}$ such that there exists a neighborhood U_λ of λ and $f : U_\lambda \to X$ such that $(T-\mu)f(\mu) = x$ for all $\mu \in U_\lambda$.

The local spectrum $\sigma_T(x)$ of T at x is defined as $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$.

Note that the local analytical solution of the equation given in the definition of the local resolvent will be unique if T has SVEP [6].

For any subset F of $\mathbb{C}$, The local spectral space of T associated with F is defined by

$$X_T(F) = \{x \in X : \sigma_T(x) \subset F\}.$$

Obviously $X_T(F)$ is a hyper-invariant space by T, but not necessarily closed.

Recall that T has the property of Dunford (C) if $X_T(F)$ is a closed set for every closed set F of $\mathbb{C}$.

We denote by $\mathcal{O}(U,X)$ the Frchet algebra of all analytic functions from the open set U to X with the topology of uniform convergence on the compact subset in U.

We say that T satisfies the Bishop's property (β) at $\lambda \in \mathbb{C}$ if there exists $r > 0$, for every open set $U \subset D(\lambda, r)$ and for any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{O}(U,X)$ such that $\lim_{n\to\infty}(T-\mu)f_n(\mu) = 0$ in $\mathcal{O}(U,X)$, then $\lim_{n\to\infty} f_n(\mu) = 0$ in $\mathcal{O}(U,X)$.

$$\sigma_\beta(T) = \{\lambda \in \mathbb{C} \ : \ T \text{ does not satisfy the property } \ (\beta)\}.$$

T is said satisfy the property (β) if $\sigma_\beta(T) = \emptyset$

We say that T has the decomposition property (δ) if T^* satisfies property (β).

T is said decomposable on Foias's sense if and only if T satisfies (β) and (δ).

We have the following implications: Property (β) $\Rightarrow$ Dunford property (C) $\Rightarrow$ SVEP.

For every closed set F of $\mathbb{C}$, the global spectral subset $\mathcal{X}_T(F)$ is defined as the set of all point $x \in X$ such that there exists an analytic function $f : \mathbb{C} \setminus F \to X$ satisfying $(T - \lambda)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$.

Clearly $\mathcal{X}_T(F)$ is a hyper invariant subspace of T and $\mathcal{X}_T(F) \subset X_T(F)$. In addition we obtain the equality $\mathcal{X}_T(F) = X_T(F)$ for every closed set F of $\mathbb{C}$ when T has SVEP.

The algebraic core $C(T)$ of T is the largest subspace M of X satisfying $T(M) = M$. In another way,

$$C(T) = \{x \in X \ : \ \exists (x_n)_{n \geq 0} \subset X \ ; \ x_0 = x, Tx_n = x_{n-1} \ \forall n \in \mathbb{N}^*\}.$$

and the analytical core $K(T)$ of T is the set

$$K(T) = \{x \in X \ : \ \exists (x_n)_{n \geq 0} \subset X, \text{ and } \varepsilon > 0 \, ; \, x_0 = x, \\ Tx_n = x_{n-1}, \|x_n\| \leq \varepsilon^n \|x\|, \forall n \in \mathbb{N}^*\}.$$

$K(T)$ is the largest subspace of X satisfying $T(M) = M$ and it can also be shown that

$$K(T) = X_T(\mathbb{C} \setminus \{0\}) = \{x \in X \ : \ 0 \in \rho_T(x)\}.$$

Next, we need the following notations and concepts in Fredholm theory, see [1] and [7].

We denote by $N^\infty(T) = \bigcup_{n \in \mathbb{N}} N(T^n)$ the hyper-kernel of T, $R^\infty(T) = \bigcap_{n \in \mathbb{N}} R(T^n)$ the hyper-range of T and both the deficiency indices $\alpha(T) = dim N(T)$ and $\beta(T) = dim R(T)$.

the ascent, the descent and the index of T are respectively

$$asc(T) = inf\{n \in \mathbb{N} \ : \ N(T^n) = N(T^{n+1})\},$$

$$des(T) = inf\{n \in \mathbb{N} \ : \ R(T^n) = R(T^{n+1})\},$$

$$ind(T) = \alpha(T) - \beta(T).$$

The ascent spectrum and the descent spectrum are the sets

$$\sigma_{asc}(T) = \{\lambda \in \mathbb{C} : \ asc(T - \lambda) = \infty\},$$

$$\sigma_{des}(T) = \{\lambda \in \mathbb{C} : \ des(T - \lambda) = \infty\}.$$

Let the sets of Fredholm operators, upper semi-Fredholm, lower semi-Fredholm, left semi-Fredholm , right semi-Fredholm, Weyl, upper semi-Weyl, lower semi-Weyl, left semi Weyl, right semi-Weyl, Browder, upper semi-Browder, lower semi-Browder, left semi-Browder and right semi-Browder respectively with their associated spectrums:

$$\Phi(X) := \{T \in \mathcal{B}(X) \,:\, \alpha(T) < \infty \, and \, \beta(T) < \infty\},\ \ \sigma_e(T),$$

$$\Phi_+(X) := \{T \in \mathcal{B}(X) \,:\, \alpha(T) < \infty \text{ and } R(T) \text{ is closed},\ \ \sigma_{SF^+}(T),$$

$$\Phi_-(X) := \{T \in \mathcal{B}(X) \,:\, \beta(T) < \infty\},\ \ \sigma_{SF_-}(T),$$

$$\Phi_l(X) := \{T \in \mathcal{B}(X) \,:\, \alpha(T) < \infty \text{ and } R(T) \text{ is closed and admits an complemented in } X\},\ \ \sigma_{le}(T),$$

$$\Phi_r(X) := \{T \in \mathcal{B}(X) \,:\, N(T) \text{ admits an complemented in X and } \beta(T) < \infty\},\ \ \sigma_{re}(T),$$

$$\Phi_0(X) := \{T \in \Phi(X) \,:\, ind(T) = 0\},\ \ \sigma_w(T),$$

$$\Phi_+^-(X) := \{T \in \Phi_+(X) \,:\, ind(T) \leq 0\},\ \ \sigma_{aw}(T),$$

$$\Phi_-^+(X) := \{T \in \Phi_-(X) \,:\, ind(T) \geq 0\},\ \ \sigma_{sw}(T),$$

$$\Phi_{lw}(X) := \{T \in \Phi_l(X) \,:\, ind(T)0\},\ \ \sigma_{lw}(T),$$

$$\Phi_{rw}(X) := \{T \in \Phi_r(X) \,:\, ind(T) \geq 0\},\ \ \sigma_{rw}(T),$$

$$\Phi_b(X) := \{T \in \Phi(X) \,:\, asc(T) = des(T) < \infty\},\ \ \sigma_b(T),$$

$$\Phi_{ab}(X) := \{T \in \Phi_+(X) \,:\, asc(T) < \infty\},\ \ \sigma_{ab}(T),$$

$$\Phi_{sb}(X) := \{T \in \Phi_-(X) \,:\, des(T) < \infty\},\ \ \sigma_{sb}(T),$$

$$\Phi_{lb}(X) := \{T \in \Phi_l(X) \,:\, asc(T) < \infty\},\ \ \sigma_{lb}(T),$$

$$\Phi_{rb}(X) := \{T \in \Phi_r(X) \,:\, des(T) < \infty\},\ \ \sigma_{rb}(T),$$

Also we consider the following operators with their associated spectrum:

$$R_1(X) = \{T \in \mathcal{B}(X) \,:\, des(T) < \infty, R(T^{des(T)}) \text{ is closed}\},\ \sigma_{rD}(T),$$

$$R_2(X) = \{T \in \mathcal{B}(X) \,:\, asc(T) < \infty, R(T^{des(T)+1}) \text{ is closed}\},\ \sigma_{lD}(T),$$

$$SF_0(T) = \{T \in \Phi_+(X) \cup \Phi_-(X) \,:\, \alpha(T) = 0 \text{ or } \beta(T) = 0\},\ \sigma_{SF_0}(T),$$

$$D(X) = \{T \in \mathcal{B}(X) \ : \ R(T) \text{ is closed and } \ N(T) \subset R^{\infty}(T)\}, \ \sigma_{se}(T),$$

Recall that $T \in \mathcal{B}(X)$) is Drazin reversible if there is $T^D \in \mathcal{B}(X)$ and some $k \in \mathbb{N}$

$$TT^D = T^D T, \ \ T^D T T^D = T^D, \ \ T^{k+1} T^D = T^k,$$

The Drazin spectrum of T is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \ \text{ is not Drazin reversible }\}.$$

2. Main Results

We begin by the following theorem

Theorem 2.1. *Let $T, S \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}^*$ such that*

$$TS = \lambda ST.$$

Then we have :

1. $\sigma_p(TS) = \lambda \sigma_p(ST)$
2. $\sigma_{su}(TS) = \lambda \sigma_{su}(ST)$
3. $\sigma_{com}(TS) = \lambda \sigma_{com}(ST)$
4. $\sigma_{ap}(TS) = \lambda \sigma_{ap}(ST)$
5. $\sigma_l(TS) = \lambda \sigma_l(ST)$
6. $\sigma_r(TS) = \lambda \sigma_r(ST)$

Proof. 1. Let $\lambda \in \mathbb{C}^*$, it is clear that

$$\begin{aligned} \mu \in \sigma_p(TS) &\Leftrightarrow TS - \mu \ \text{ is not injective} \\ &\Leftrightarrow \lambda ST - \mu = \lambda(ST - \frac{\mu}{\lambda}) \ \text{ is not injective} \\ &\Leftrightarrow ST - \frac{\mu}{\lambda} \ \text{ is not injective} \\ &\Leftrightarrow \frac{\mu}{\lambda} \in \sigma_p(ST) \end{aligned}$$

Hence $\sigma_p(TS) = \lambda \sigma_p(ST)$

2. Again if $\lambda \in \mathbb{C}^*$ then

$$\begin{aligned}
\mu \notin \sigma_{su}(TS) &\Leftrightarrow TS - \mu \text{ is surjective} \\
&\Leftrightarrow R(TS-\mu) = X \\
&\Leftrightarrow R(\lambda ST - \mu) = R(\lambda(ST - \frac{\mu}{\lambda})) = X \\
&\Leftrightarrow R(ST - \frac{\mu}{\lambda}) = X \\
&\Leftrightarrow \frac{\mu}{\lambda} \notin \sigma_{su}(ST)
\end{aligned}$$

Therefore $\sigma_{su}(TS) = \lambda\sigma_{su}(ST)$

3. Let $\lambda \in \mathbb{C}^*$ then

$$\begin{aligned}
\mu \notin \sigma_{com}(TS) &\Leftrightarrow R(TS-\mu) \text{ is dense in } X \\
&\Leftrightarrow \overline{R(TS-\mu)} = X \\
&\Leftrightarrow \overline{R(\lambda ST - \mu)} = \overline{R(\lambda(ST - \frac{\mu}{\lambda}))} = X \\
&\Leftrightarrow \overline{R(ST - \frac{\mu}{\lambda})} = X \\
&\Leftrightarrow \frac{\mu}{\lambda} \notin \sigma_{com}(ST)
\end{aligned}$$

This shows that $\sigma_{com}(TS) = \lambda\sigma_{com}(ST)$

4. Let $\mu \in \sigma_{ap}(TS)$ then

$$\begin{aligned}
\mu \in \sigma_{ap}(TS) &\Leftrightarrow \exists (x_n)_{n\in\mathbb{N}} \subset X \textit{ such that } \|x_n\| = 1 \textit{ and} \\
&\quad \lim_{n\longrightarrow+\infty} (TS-\mu)x_n = 0 \\
&\Leftrightarrow \exists (x_n)_{n\in\mathbb{N}} \subset X \textit{ such that } \|x_n\| = 1 \textit{ and} \\
&\quad \lim_{n\longrightarrow+\infty} \lambda(ST - \frac{\mu}{\lambda})x_n = 0 \\
&\Leftrightarrow \exists (x_n)_{n\in\mathbb{N}} \subset X \textit{ such that } \|x_n\| = 1 \textit{ and} \\
&\quad \lim_{n\longrightarrow+\infty} (ST - \frac{\mu}{\lambda})x_n = 0 \\
&\Leftrightarrow \frac{\mu}{\lambda} \in \sigma_{ap}(ST).
\end{aligned}$$

Hence $\sigma_{ap}(TS) = \lambda\sigma_{ap}(ST)$

5. Let $\lambda \in \mathbb{C}^*$, then

$$\mu \notin \sigma_l(TS) \Leftrightarrow TS - \mu \text{ is left inversible}$$

$$\begin{aligned}
&\Leftrightarrow \quad \exists\, T_1 \in \mathcal{B}(X) \text{ such that} \\
&\qquad T_1(TS-\mu) = I \\
&\Leftrightarrow \quad \exists\, T_1 \in \mathcal{B}(X) \text{ such that} \\
&\qquad T_1(\lambda ST-\mu) = (\lambda T_1)(ST-\frac{\mu}{\lambda}) = I \\
&\Leftrightarrow \quad \exists\, T_1' \in \mathcal{B}(X) \text{ such that} \\
&\qquad T_1'(ST-\frac{\mu}{\lambda}) = I \text{ with } T_1' = \lambda T_1 \\
&\Leftrightarrow \quad ST-\frac{\mu}{\lambda} \text{ is also left inversible} \\
&\Leftrightarrow \quad \frac{\mu}{\lambda} \notin \sigma_l(ST)
\end{aligned}$$

Hence $\sigma_l(TS) = \lambda\sigma_l(ST)$

6. Similarly.

□

We obtain the following corollary, see [2, lemma 2.1].

Corollary 2.1. *Let $T, S \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}^*$ such that*

$$TS = \lambda ST.$$

Then :

$$\sigma(TS) = \lambda\sigma(ST).$$

Proof. Using theorem 2.1 we have

$$\begin{aligned}
\sigma(TS) &= \sigma_l(TS) \cap \sigma_r(TS) \\
&= [\lambda\sigma_l(ST)] \cap [\lambda\sigma_r(ST)] \\
&= \lambda[\sigma_l(ST) \cap \sigma_r(ST)] \\
&= \lambda\sigma(ST).
\end{aligned}$$

□

We now establish the relationship between the local spectrum and β-spectrum for operators that λ-commute.

Theorem 2.2. *Let $T, S \in \mathcal{B}(X)$, $\mu \in \mathbb{C}$ and $\lambda \in \mathbb{C}^*$ such that*

$$TS = \lambda ST.$$

Then we have:

1. $\sigma_{TS}(x) = \lambda\sigma_{ST}(x)$
2. $\sigma_\beta(TS) = \lambda\sigma_\beta(ST)$
3. TS *has SVEP at* μ *if and only if* ST *has it at* $\frac{\mu}{\lambda}$. *Otherwise we have* $\mathcal{S}(TS) = \lambda\mathcal{S}(ST)$
4. TS *has SVEP if and only if* ST *has it*

Proof. 1. Suppose that $TS = \lambda ST$ and $\mu_0 \notin \sigma_{TS}(x)$, then there exists a neighborhood U of μ_0 and $f \in \mathcal{O}(U, X)$ such that

$$(TS - \mu)f(\mu) = x \text{ for all } \mu \in U$$

Since $TS = \lambda ST$, then

$$\begin{aligned}(TS - \mu)f(\mu) = x \text{ for all } x \in U \quad &\Leftrightarrow \quad (\lambda ST - \mu)f(\mu) = x \text{ for all } \mu \in U\\ &\Leftrightarrow \quad \lambda(ST - \frac{\mu}{\lambda})f(\mu) = x \text{ for all } \mu \in U\\ &\Leftrightarrow \quad (ST - \frac{\mu}{\lambda})[\lambda f(\mu)] = x \text{ for all } \mu \in U\end{aligned}$$

We define the following two bijections:
$S : X \to X$ with $S(x) = \lambda x$ and $s : \mathbb{C} \to \mathbb{C}$ with $s(z) = \lambda z$, then

$$\lambda f(\mu) = (S \circ f)(\mu) = (S \circ f)(s(\frac{\mu}{\lambda})) \text{ for all } \mu \in U.$$

Hence $\lambda f(\mu) = (S \circ f \circ s)(\frac{\mu}{\lambda})$
And Since μ course the neighborhood U of μ_0 then $\frac{\mu}{\lambda}$ also course the neighborhood V of $\frac{\mu_0}{\lambda}$, hence by replacing $\lambda f(\mu)$ by $(S \circ f \circ s)(\frac{\mu}{\lambda})$ and by noting $g = S \circ f \circ s$ wich is analytic on V, we obtain:

$$\begin{aligned}(TS - \mu)f(\mu) = x \text{ for all } x \in U \quad &\Leftrightarrow \quad (ST - \frac{\mu}{\lambda})[\lambda f(\mu)] = x \quad \text{for all } \mu \in U\\ &\Leftrightarrow \quad (ST - \frac{\mu}{\lambda})[S \circ f \circ s)(\frac{\mu}{\lambda})] = x\\ &\qquad \text{for all } \mu \in U\\ &\Leftrightarrow \quad (ST - \mu')g(\mu') = x \text{ for all } \mu' \in V\end{aligned}$$

Finally $\frac{\mu_0}{\lambda} \notin \sigma_{ST}(x)$ and therefore $\sigma_{TS}(x) = \lambda\sigma_{ST}(x)$

2. Let $\mu_0 \in \sigma_\beta(TS)$, then there exists $r > 0$ for every open set $U \subset D(\mu_0, r)$ and for all sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{O}(U, X)$ such that

$$\lim_{n\to\infty}(TS - \mu)f_n(\mu) = 0 \Rightarrow \lim_{n\to\infty} f_n(\mu) = 0 \text{ in } \mathcal{O}(U, X)$$

To show that $\frac{\mu_0}{\lambda} \in \sigma_\beta(ST)$, let $r' > 0$, $V \subset D(\frac{\mu_0}{\lambda}, r')$ and $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{O}(U, X)$ such that $\lim_{n\to\infty}(ST - \mu')g_n(\mu') = 0$. We have

$$\begin{aligned}
& \lim_{n\to\infty}(ST - \mu')g_n(\mu') = 0 \text{ in } \mathcal{O}(V, X)\\
\Leftrightarrow\ & \lim_{n\to\infty}\lambda(ST - \mu')g_n(\mu') = 0 \text{ in } \mathcal{O}(V, X)\\
\Leftrightarrow\ & \lim_{n\to\infty}\lambda(ST - \mu')g_n(\mu') = 0 \ \text{ in } \mathcal{O}(V, X)\\
\Leftrightarrow\ & \lim_{n\to\infty}(\lambda ST - \lambda\mu')g_n(\mu') = 0 \text{ in } \mathcal{O}(V, X)\\
\Leftrightarrow\ & \lim_{n\to\infty}(TS - \lambda\mu')g_n(\frac{1}{\lambda}\lambda\mu') = 0 \text{ in } \mathcal{O}(V, X)\\
\Leftrightarrow\ & \lim_{n\to\infty}(TS - \lambda\mu')[g_n \circ s^{-1}](\lambda\mu') = 0 \text{ in} \mathcal{O}(V, X)\\
\Leftrightarrow\ & \mu = \lambda\mu' \quad \lim_{n\to\infty}(TS - \mu)[g_n \circ s^{-1}](\mu) = 0\\
\Rightarrow\ & \lim_{n\to\infty} g_n \circ s^{-1}(\mu) = 0 \text{ in } \ \mathcal{O}(U, X)\\
\Rightarrow\ & \lim_{n\to\infty} g_n(\mu') = 0 \text{ in } \ \mathcal{O}(V, X)
\end{aligned}$$

Hence $\frac{\mu_0}{\lambda} \in \sigma_\beta(ST)$. Similarly we can show the other inclusion. Finally $\sigma_\beta(TS) = \lambda\sigma_\beta(ST)$

3. By the same argument as 2.

4. As $\mathcal{S}(TS) = \lambda\mathcal{S}(ST)$ Then:

$$TS \text{ has } SVEP \Leftrightarrow \mathcal{S}(TS) = \varnothing \Leftrightarrow \mathcal{S}(ST) = \varnothing.$$

□

Lemma 2.1. *Let $T, S \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}^*$ such that*

$$TS = \lambda ST.$$

We have :

1. *$R(TS) = R(ST)$, for all $\mu \in \mathbb{C}$ $\quad R(TS - \mu) = R(ST - \frac{\mu}{\lambda})$*
2. *$N(TS) = N(ST)$, for all $\mu \in \mathbb{C}$ $\quad N(TS - \mu) = N(ST - \frac{\mu}{\lambda})$*

Proof. 1. ⇒) Let $y \in R(TS)$ then there exists $x \in X$ such that $TS(x) = y$

$$TS(x) = y \quad \Leftrightarrow \quad \lambda ST(x) = y$$

$$\begin{aligned} &\Leftrightarrow \quad ST(\lambda x) = y \\ &\Leftrightarrow \quad ST(x') = y \quad \textit{with} \quad x' = \lambda x \end{aligned}$$

Where $y \in R(ST)$.

$\Leftarrow$) Similarly we show the reverse inclusion. Finally $R(TS) = R(ST)$

2. Let $x \in N(TS)$, then

$$\begin{aligned} x \in N(TS) \quad &\Leftrightarrow \quad TS(x) = 0 \\ &\Leftrightarrow \quad \lambda ST(x) = 0 \\ &\Leftrightarrow \quad ST(x) = 0 \quad \forall \lambda \in \mathbb{C}^* \\ &\Leftrightarrow \quad x \in N(ST) \end{aligned}$$

we then conclude that $N(TS) = N(ST)$.

□

As a straightforward consequence of Lemma 2.1 we easily obtain the following corollary

Corollary 2.2. *Let $T, S \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}^*$ such that*

$$TS = \lambda ST.$$

Then we have for all $n \in \mathbb{N}^*$:

1. $R((TS)^n) = R((ST)^n)$ *and for all* $\mu \in \mathbb{C}$ $R[(TS - \mu)^n] = R[(ST - \frac{\mu}{\lambda})^n]$
2. $N((TS)^n) = N((ST)^n)$ *and for all* $\mu \in \mathbb{C}$ $N[(TS - \mu)^n] = N[(ST - \frac{\mu}{\lambda})^n]$
3. $\alpha(TS) = \alpha(ST)$, $\beta(TS) = \beta(ST)$ *and* $ind(TS) = ind(ST)$
4. *For all* $\mu \in \mathbb{C}$ $\alpha(TS - \mu) = \alpha(ST - \frac{\mu}{\lambda})$, $\beta(TS - \mu) = \beta(ST - \frac{\mu}{\lambda})$ *and* $ind(TS - \mu) = ind(ST - \frac{\mu}{\lambda})$
5. $asc(TS) = asc(ST)$ *and* $des(TS) = des(ST)$
6. *For all* $\mu \in \mathbb{C}$ $asc(TS - \mu) = asc(ST - \frac{\mu}{\lambda})$ *and* $des(TS - \mu) = des(ST - \frac{\mu}{\lambda})$
7. $R^\infty(TS) = R^\infty(ST)$
8. *For all* $\mu \in \mathbb{C}$ $R^\infty(TS - \mu) = R^\infty(ST - \frac{\mu}{\lambda})$
9. $N^\infty(TS) = N^\infty(ST)$

10. *For all* $\mu \in \mathbb{C}$ $N^{\infty}(TS-\mu) = N^{\infty}(ST-\frac{\mu}{\lambda})$

We establish the following theorem, the proof is easily by using corollary 2.2

Theorem 2.3. *Let* $T, S \in \mathcal{B}(X)$ *and* $\lambda \in \mathbb{C}^*$ *such that* $TS = \lambda ST$. *Then we have the following equalities :*

1. $\sigma_{asc}(TS) = \lambda\sigma_{asc}(ST), \quad \sigma_{des}(TS) = \lambda\sigma_{des}(ST)$
2. $\sigma_{e}(TS) = \lambda\sigma_{e}(ST)$
3. $\sigma_{SF^+}(TS) = \lambda\sigma_{SF^+}(ST), \quad \sigma_{SF^-}(TS) = \lambda\sigma_{SF^-}(ST)$
4. $\sigma_{le}(TS) = \lambda\sigma_{le}(ST), \quad \sigma_{re}(TS) = \lambda\sigma_{re}(ST)$
5. $\sigma_{w}(TS) = \lambda\sigma_{w}(ST)$
6. $\sigma_{aw}(TS) = \lambda\sigma_{aw}(ST), \quad \sigma_{sw}(TS) = \lambda\sigma_{sw}(ST)$
7. $\sigma_{lw}(TS) = \lambda\sigma_{lw}(ST), \quad \sigma_{rw}(TS) = \lambda\sigma_{rw}(ST)$
8. $\sigma_{b}(TS) = \lambda\sigma_{b}(ST)$
9. $\sigma_{ab}(TS) = \lambda\sigma_{ab}(ST), \quad \sigma_{sb}(TS) = \lambda\sigma_{sb}(ST)$
10. $\sigma_{lb}(TS) = \lambda\sigma_{lb}(ST), \quad \sigma_{rb}(TS) = \lambda\sigma_{rb}(ST)$
11. $\sigma_{se}(TS) = \lambda\sigma_{se}(ST)$
12. $\sigma_{BF}(TS) = \lambda\sigma_{BF}(ST), \quad \sigma_{BW}(TS) = \lambda\sigma_{BW}(ST)$
13. $\sigma_{rD}(TS) = \lambda\sigma_{rD}(ST), \quad \sigma_{lD}(TS) = \lambda\sigma_{lD}(ST)$
14. $\sigma_{SF_0}(TS) = \lambda\sigma_{SF_0}(ST)$

The connection between the Drazin spectrum for operators satisfying the λ-commutativity is the following

Theorem 2.4. *Let* $T, S \in \mathcal{B}(X)$ *and* $\lambda \in \mathbb{C}^*$ *such that* $TS = \lambda ST$. *Then*

$$\sigma_D(TS) = \lambda\sigma_D(ST)$$

Proof. Let $\mu \notin \sigma_D(TS)$ (*), as $TS = \lambda ST$ then we have the following equivalents :

$$
\begin{aligned}
(*) \quad &\Leftrightarrow \quad TS - \mu \text{ is Drazin invertible} \\
&\Leftrightarrow \quad \exists R \in \mathcal{B}(X), (TS-\mu)R = R(TS-\mu), R(TS-\mu)R = R \\
&\qquad \text{ and } (TS-\mu)^{n+1}R = (TS-\mu)^n \\
&\Leftrightarrow \quad (\lambda ST-\mu)R = R(\lambda ST-\mu), R(\lambda ST-\mu)R = R \\
&\qquad \text{ and } (\lambda ST-\mu)^{n+1}R = (\lambda ST-\mu)^n \\
&\Leftrightarrow \quad (ST-\frac{\mu}{\lambda})[\lambda R] = [\lambda R](ST-\frac{\mu}{\lambda}), [\lambda R](ST-\frac{\mu}{\lambda})R = R, \\
&\qquad \lambda^{n+1}(ST-\frac{\mu}{\lambda})^{n+1}R = \lambda^n(ST-\frac{\mu}{\lambda})^n \\
&\Leftrightarrow \quad (ST-\frac{\mu}{\lambda})[\lambda R] = [\lambda R](ST-\frac{\mu}{\lambda}), [\lambda R](ST-\frac{\mu}{\lambda})[\lambda R] = [\lambda R], \\
&\qquad (ST-\frac{\mu}{\lambda})^{n+1}[\lambda R] = (ST-\frac{\mu}{\lambda})^n \\
&\Leftrightarrow \quad ST-\frac{\mu}{\lambda} \text{ is Drazin invertible} \\
&\Leftrightarrow \quad \frac{\mu}{\lambda} \notin \sigma_D(ST).
\end{aligned}
$$

□

Using the previous results we obtain the following properties on local spectral space, global spectral, analytical core and the property (C) for operators satisfying the λ-commutativity.

Theorem 2.5. *Let $T, S \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}^*$ such that $TS = \lambda ST$. Then*

1. $X_{TS}(F) = X_{ST}(\frac{F}{\lambda})$ *and* $\mathcal{X}_T(F) = \mathcal{X}_T(\frac{F}{\lambda})$
2. *TS has the property (C) if and only if ST also has the property (C)*
3. $K(TS) = K(ST)$

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