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ON ZAGREB INDICES AND ECCENTRIC CONNECTIVITY INDEX OF CERTAIN THORN GRAPHS

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Abstract: The first three Zagreb indices of a graph G denoted, $M_1(G), M_2(G)$ and $M_3(G)$, are well known. Equally well known is the eccentricity connectivity index denoted, $\xi^c(G)$. In this paper we derive closed formula for these indices for thorn cycles, thorn star graphs and thorn complete graphs, respectively. Same is repeated for the eccentricity connectivity index. The further aim of the paper is to emphasize the subtle difference between *mathematical induction* and *immediate induction* as equally valid techniques of proof. Valid immediate induction requires the rarely utilised property of many graphs namely, *well-defineness*.

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1. Introduction

For a general reference to notation and concepts of graph theory see [3]. For further reading we also refer to [4,7]. Unless mentioned otherwise all graphs will be finite, simple, connected, undirected graphs. A graph G will be of order

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(number of vertices) n and size (number of edges) q. The first three Zagreb indices of a graph G denoted, $M_1(G), M_2(G)$ and $M_3(G)$, are well known. The first two are the oldest topological irregularity measures of a graph which found application in chemical graph theory. In 1977 Alberton [2] introduced the irregularity of G as $irr(G) = \sum\limits_{e \in E(G)} imb(e)$, $imb(e) = |deg(v) - deg(u)|_{e-vu}$. To conform with the terminology of chemical graph theory Fath-Tabar [5] called Alberton's irregularity the third Zagreb index. In a more recent paper by Ado et al. [1] the notion of total irregularity was introduced. The latter can be considered the fourth Zagreb index. In more recent work Kok et al. [8] introduced four more variations by defining Fibonacci vertex labeling in terms of odd and even vertex degree. These variations are called the $\pm$-Zagreb indices. The latter were denoted, $f^{\pm}Z_i(G)$, $i = 1, 2, 3, 4$. Equally well known is the eccentricity connectivity index denoted, $\xi^c(G)$.

In this paper we derive closed formula for the first three Zagreb indices and the eccentricity connectivity index [9] for special cases of thorn cycles, thorn star graphs and thorn complete graphs which in the derivative is also called, *thorny graphs* [10]. The further aim of the paper is to emphasize the subtle difference between *mathematical induction* and *immediate induction* as equally valid techniques of proof.

2. Thorny Graphs

The first studies of a path with pendant vertices (varying in number), attached to the path vertices (later called the *spine*) were by Harary and Swenk in a series of papers. These graphs were later named *caterpillars*. Thereafter the generalisation called *thorn graphs* followed. Note that a thorn graph as defined by Gutman [6] is a graph $G^\star$ obtained from a graph G of order n by attaching $p_i \geq 0$, $i = 1, 2, \dots, n$, pendant vertices to the i^{th} vertex of G. In this paper $p_i = m$ $\forall i$, $m \in \mathbb{N}_0$. We shall utilise two methods of proof i.e. *formal mathematical induction* and *immediate induction*, which is a derivate of the former. The purpose is to demonstrate the elegancy and validity of immediate induction when applicable. Note that since $m \in \mathbb{N}_0$ we have $m \geq 0$. Generally, as much as an edgeless graph exists, a *thornless* thorn graph exists by Gutman [6]. Because $m = 0$ is defined, G itself is the *thornless* thorn graph $G^\star_{m=0}$.

2.1. Zagreb Indices for Thorny Cycles

Conventionally a cycle C_n is defined for $n \geq 3$. For the purpose of this paper we include $C_1 \simeq K_1$ called the *cell cycle* and $C_2 \simeq K_2$ called the *flat cycle*. Because a vertex v is inherently adjacent to itself we have the distance, $d(v, v) = 0$. The family of thorny cycles is defined to be $\mathbb{C} = \{C_n^\star : C_n, n \geq 1$ is a cycle with $m \geq 0$ pendant vertices attached to each vertex, $v \in V(C_n)\}$.

2.1.1. First Zagreb Index. Recall that the first Zagreb index is defined to be

$$M_1(G) = \sum_{v \in V(G)} deg(v)^2 = \sum_{vu \in E(G)} (deg(v) + deg(u)).$$

Proposition 2.1. *For a thorny cycle $C_n^\star \in \mathbb{C}$ the First Zagreb index:*

$$M_1(C_n^\star) = \begin{cases} m(m+1), & \text{if } n = 1, \\ 2(m^2 + 3m + 1), & \text{if } n = 2, \\ n(m+4)(m+1), & \text{if } n \geq 3. \end{cases}$$

Proof. Label the vertices of C_n, v_i, $i = 1, 2, , \dots, n$ and the m pendant vertices attached to each v_i, to be $u_{i,j}$, $j = 1, 2, \dots, m$. From the definition of $M_1(G)$ it follows that:

$$M_1(C_n^\star) = \sum_{w \in V(G_n^\star)} deg(w)^2 = \sum_{i=1}^{n} deg(v_i)^2 + \sum_{i=1}^{n}\sum_{j=1}^{m} deg(u_{i,j})^2.$$

In all cases let $m \in \mathbb{N}_0$, a constant.

Case (i): If $n = 1$, $M_1(C_1^\star) = m^2 + m = m(m+1)$.

Case (ii): If $n = 2$, $M_1(C_2^\star) = 2(m+1)^2 + 2m = 2(m^2 + 3m + 1)$.

Case (iii): Let $n \geq 3$. Consider $n = 3$, therefore $M_1(C_3^\star) = 3(2+m)^2 + 3m = 3(m+4)(m+1)$. Hence the result holds for $n = 3$. Assume the result holds $3 \leq n \leq k$. So for $n = k$ we have that $M_1(C_k^\star) = k(2+m)^2 + km$.

Now consider $n = k+1$. Because the additional terms are independent from the preceding terms, the associative law for addition allows $M_1(G_{k+1}^\star) = M_1(G_k^\star) + ((2+m)^2 + m) = k(2+m)^2 + km + (2+m)^2 + m = (k+1)(2+m)^2 + (k+1)m = (k+1)(m+4)(m+1)$. Therefore the results holds for $n = k+1$ hence, through mathematical induction it holds for all $n \geq 3$, $n \in \mathbb{N}$, with m some arbitrary non-negative integer constant.

Case (iv): Because m serves as a constant in Case (i),(ii),(iii) any other constant say, $m' = m + t \geq 0$, $t \in \mathbb{Z}$ is valid. Hence, through immediate induction the result holds for $n \in \mathbb{N}$, $m \in \mathbb{N}_0$. $\square$

2.1.2. Second Zagreb Index. Recall that the Second Zagreb index is defined to be, $M_2(G) = \sum\limits_{vu \in E(G)} deg(v)deg(u)$. Because the index is defined for graphs of size at least 1, it is required that $m \geq 1$, $m \in \mathbb{N}$.

Proposition 2.2. *For a thorny cycle $C_n^\star \in \mathbb{C}$, $m \geq 1$ the Second Zagreb index:*

$$M_2(C_n^\star) = \begin{cases} m^2, & \text{if } n = 1, \\ (m+1)(3m+1), & \text{if } n = 2, \\ 2n(m+1)(m+2), & \text{if } n \geq 3. \end{cases}$$

Proof. Label the vertices of C_n, v_i, $i = 1, 2, , \dots, n$ and the m pendant vertices attached to each v_i, to be $u_{i,j}$, $j = 1, 2, \dots, m$. From the definition of $M_2(G)$ it follows that:

$$M_2(C_n^\star) = \sum_{vu \in E(G)} deg(v)deg(u).$$

In all cases let $m \in \mathbb{N}$, a constant.

Case (i): If $n = 1$, $M_2(C_1^\star) = \underbrace{m \cdot 1 + m \cdot 1 + \cdots + m \cdot 1}_{m-terms} = m(m \cdot 1) = m^2$.

Case (ii): If $n = 2$,

$$M_1(C_2^\star) = 2(\underbrace{(m+1) \cdot 1 + (m+1) \cdot 1 + \cdots + (m+1) \cdot 1}_{m-terms}) + (m+1)^2 = (m+1)(3m+1).$$

Case (iii): Let $n \geq 3$. Consider $n = 3$, therefore:

$$M_2(C_3^\star) = 3(\underbrace{(m+2) \cdot 1 + (m+2) \cdot 1 + \cdots + (m+2) \cdot 1}_{m-terms}) + 3(m+2)^2 = 3(m+2)(m + (m+2)) = 3 \cdot 2(m+1)(m+2).$$

Hence the result holds for $n = 3$. Assume the result holds $3 \leq n \leq k$. So for $n = k$ we have that $M_2(C_k^\star) = 2k(m+1)(m+2)$.

Now consider $n = k + 1$. Because the additional terms are independent from the preceding terms, the associative law for addition allows $M_2(G^\star_{k+1}) = M_2(G^\star_k) + (m+2)^2 + m(m+2) = M_2(G^\star_k) + 2(m+1)(m+2) = 2(k+1)(m+1)(m+2)$. Therefore the results holds for $n = k+1$ hence, through mathematical induction it holds for all $n \geq 3$, $n \in \mathbb{N}$, with m some arbitrary non-negative integer constant.

Case (iv): Because m serves as a constant in Case (i),(ii),(iii) any other constant say, $m' = m + t \geq 0$, $t \in \mathbb{Z}$ is valid. Hence, through immediate induction the result holds for $n \in \mathbb{N}$, $m \in \mathbb{N}_0$. $\square$

2.1.3. Third Zagreb Index. Recall that the Third Zagreb index is defined to be, $M_3(G) = \sum\limits_{vu \in E(G)} |deg(v) - deg(u)|$. Because the index is defined for graphs of size at least 1, it is required that $m \geq 1$, $m \in \mathbb{N}_0$.

Proposition 2.3. *For a thorny cycle $C^\star_n \in \mathbb{C}$, $m \geq 1$ the Third Zagreb index:*

$$M_3(C^\star_n) = \begin{cases} m(m-1), & \text{if } n = 1, \\ 2m^2, & \text{if } n = 2, \\ nm(m+1), & \text{if } n \geq 3. \end{cases}$$

Proof. Label the vertices of C_n, v_i, $i = 1, 2, , \dots, n$ and the m pendant vertices attached to each v_i, to be $u_{i,j}$, $j = 1, 2, \dots, m$. From the definition of $M_3(G)$ it follows that:

$$M_3(C^\star_n) = \sum_{vu \in E(G)} |deg(v) - deg(u)|.$$

In all cases let $m \in \mathbb{N}_0$, a constant.

Case (i): If $n = 1$, $M_3(C^\star_1) = \underbrace{(m-1) + (m-1) + \cdots + (m-1}_{m-terms} = m(m-1)$.

Case (ii): If $n = 2$,

$$M_3(C^\star_2) = 2\underbrace{(((m+1)-1) + ((m+1)-1) + \cdots + ((m+1)-1))}_{m-terms} + ((m+1) - (m+1)) = 2m^2.$$

Case (iii): Let $n \geq 3$. Consider $n = 3$, therefore:

$$M_3(C_3^\star) = 3(\underbrace{((m+2)-1)+((m+2)-1+\cdots+((m+2)-1}_{m-terms}) + 3((m+2)-(m+2)) = 3m(m+1).$$

Hence the result holds for $n=3$. Assume the result holds $3 \leq n \leq k$. So for $n=k$ we have that $M_3(C_k^\star) = km(m+1)$.

Now consider $n = k+1$. Because the additional terms are independent from the preceding terms, the associative law for addition allows $M_3(G_{k+1}^\star) = M_3(G_k^\star)+m(m+1)+((m+2)-(m+2)) = M_3(G_k^\star)+m(m+1) = (k+1)m(m+1)$. Therefore the results holds for $n = k+1$ hence, through mathematical induction it holds for all $n \geq 3$, $n \in \mathbb{N}$, with m some arbitrary non-negative integer constant.

Case (iv): Because m serves as a constant in Case (i),(ii),(iii) any other constant say, $m' = m+t \geq 0$, $t \in \mathbb{Z}$ is valid. Hence, through immediate induction the result holds for $n \in \mathbb{N}$, $m \in \mathbb{N}_0$. □

3. Well-Defineness in Graphs

Consider the sets $\mathbb{N}_0$ and $\mathbb{N}$. We have the axioms that both $\mathbb{N}_0, \mathbb{N}$ are well-defined. We also have the axioms that the "+" and "-" operations are well-defined. We now define two identical sets $\mathbb{N}_0 = \mathfrak{N}_0$ and $\mathbb{N} = \mathfrak{N}$ and act blind to the fact of identical sets. Both the sets $\mathfrak{N}_0 = \{n_i : n_0 = 0, n_1 = 1, n_i = n_{i-1} + n_1, i \in \mathbb{N}_0, i \geq 2\}$ and $\mathfrak{N} = \{n_i : n_1 = 1, n_2 = 2, n_i = n_{i-1} + n_1, i \in \mathbb{N}, i \geq 3\}$ are well-defined since all consecutive numbers are unique (unambigiously distinct). The latter is true because, if for $\mathfrak{N}_0$ we assume to the contrary that for some $i \in \mathbb{N}_0$ both $n_i = n_{i-1} + n_1$ and $n_i' = n_{i-1} + n_1$ but $n_i \neq n_i'$, it means that n_{i-1} and n_{i-1}', $n_{i-1} \neq n_{i-1}'$ exist. The latter is true because "+" and "-" are well-defined. The inverse recursion will imply that $n_1 = 1 = n_1'$, $n_1 \neq n_1'$ and $n_0 = 0 = n_0'$, $n_0 \neq n_0'$ which is impossible. Therefore, $n_i = n_i'$. Similarly it can be shown that $\mathfrak{N}$ is well-defined.

Generally the notion of well-defineness is neglected in the literature on graph theory. Occasionally it is reasoned to show that an algorithm is well-defined and converges. In [3] a graph is defined to be an ordered triple $G = (v(G), E(G), \psi_G)$ with $V(G)$ a nonempty set of vertices, a set $E(G)$ (disjoint from $V(G)$) of edges, and an incidence function ψ_G that associates each edge $e \in E(G)$ with an unordered pair of vertices (not necessary distinct) of G.

Definition 3.1. The simple connected finite graph $(V(G), E(G), \psi_G)$ with $V(G) = \{v_i : i = 1, 2, 3\ldots, n\}$, $E(G) = \{e_j : j = 1, 2, 3, \ldots, n-1\}$ and $\psi_G(e_k) \mapsto v_k v_{k+1}$, $k = 1, 2, 3, \ldots, n-1$ is called a path denoted P_n, $n \geq 1$, $n \in \mathbb{N}$.

Lemma 3.1. *A path P_n, $n \geq 1$, $n \in \mathbb{N}$ is up to isomorphism, well-defined.*

Proof. Assume there exist two non-isomorphic paths both of order n. Denote the paths P_n, P'_n. Clearly $n = n$ unambigiously hence, $|V(P_n)| = |V(P'_n)|$ so from amongst the finite $n!$ ways in labeling the vertices v_i, $i = 1, 2, 3, \ldots, n$ both sets of vertices can be labeled identically. Clearly it is possible to define $\psi_{P_n}(e_k) = v_k v_{k+1}$, $k = 1, 2, 3, \ldots, n-1$. Assume such mapping is not possible for P'_n. Then since $|E(P'_n)| = n - 1$ unambigiously, it implies that at least one ordered pair of vertices v_j, v_m exists such that $\psi_{P'_n}(e_k) = v_j v_m$ for some $k \in \{1, 2, 3, \ldots, n-1\}$. If $m \neq j + 1$ the existence of the edge contradicts the definition of $\psi_{P'_n}$. Therefore, there exists at least one ordered pair of vertices (v_j, v_{j+1}) with at least two distinct edges joining the vertices v_j, v_{j+1}. The latter implies that P'_n is non-simple and disconnected, a contradiction. Therefore, P_n and P'_n are up to isomorphism, identical. This implies that a path is well-defined. □

Other graphical embodiments such as cycles, star graphs, complete graphs and others and the thorny variations there of can be proven to be well-defined. The property of *well-defineness* makes it valid to utilise immediate induction to shorten the proofs of Case (iii) found in Propositions 2.1,2.2,2.3. We recall that if the context of the graph G is clear we may write, $deg_G(v) = deg(v)$.

Illustration 1: (See Proposition 2.1(iii)). For a thorny cycle $C^\star_n \in \mathbb{C}$ the First Zagreb index $M_1(C^\star_n) = n(m+4)(m+1)$, if $n \geq 3$.

Proof. Consider any thorny cycle $C^\star_n \in \mathbb{C}$, $n \geq 3$, $m \in \mathbb{N}_0$. In C_n, $deg_{C_n}(v_i) = 2, \forall i$. Therefore, in $C^\star_n$, $deg_{C^\star_n}(v_i) = (m+2), \forall i$. From the definition of $M_1(G)$ it follows that:

$$\begin{aligned} M_1(C^\star_n) = n(m+2)^2 + nm \cdot 1^2 = n((m+2)^2 + m) &= n(m^2 + 5m + 4) \\ &= n(m+4)(m+1), \end{aligned}$$

$n \geq 3$, $m \in \mathbb{N}_0$. Because both $\mathbb{N}_0$ and $C^\star_n$, $n \geq 3$, $n \in \mathbb{N}$ are well-defined the result follows through immediate induction. □

3.1. Zagreb Indices for Thorny Star Graphs

Let S_{n+1} be a star graph with $n \geq 1$ pendant (external) vertices and a central vertex. A formal definition of a star graph is given below.

Definition 3.2. A simple, connected, finite graph $(V(G), E(G), \psi_G)$ with $V(G) = \{v_i : i = 1, 2, 3, \ldots, n\} \cup \{u\}$, $E(G) = \{e_j : j = 1, 2, 3, \ldots, n\}$ and $\psi_G(e_k) \mapsto uv_k$, $k = 1, 2, 3, \ldots, n$ is called a star graph. A star graph is denoted, S_{n+1}.

Similar to the proof of the well-defineness of a path, it can be proved that star graphs are well-defined. Therefore where applicable, immediate induction will be utilised in proofs. As a special case we consider a thorny star graph $S^\star_{n+1}$ with $m \geq 0$ pendant vertices attached to v_i, $i = 1, 2, 3, \ldots, n$ and zero to u.

Proposition 3.2. *For a thorny star graph $S^\star_{n+1}$ the Zagreb indices are:*

(i) $M_1(S^\star_{n+1}) = n(n + m^2 + 3m + 1)$.

(ii) $M_2(S^\star_{n+1}) = n(m + 1)(n + m)$.

(iii) $M_3(S^\star_{n+1}) = n(|n - (m + 1)| + m^2)$.

Proof. Consider any thorny star graph $S^\star_{n+1}$, $n \geq 1$, $m \in \mathbb{N}_0$.

(i) From the definition of $M_1(G)$ it follows that, $M_1(S^\star_{n+1}) = deg(u)^2 + \sum\limits_{i=1}^{n} deg(v_i)^2 + nm = n^2 + n(m+1)^2 + nm = n(n+m^2+3m+1)$, $n \geq 1$, $m \in \mathbb{N}_0$. Because both $\mathbb{N}_0$ and $S^\star_{n+1}$, $n \geq 1$, $n \in \mathbb{N}$ are well-defined the result follows through immediate induction.

(ii) From the definition of $M_2(G)$ it follows that, $M_2(S^\star_{n+1}) = n^2(m + 1) + nm(m + 1) = n(m + 1)(n + m)$, $n \geq 1$, $m \in \mathbb{N}_0$. Because both $\mathbb{N}_0$ and $S^\star_{n+1}$, $n \geq 1$, $n \in \mathbb{N}$ are well-defined the result follows through immediate induction.

(iii) From the definition of $M_3(G)$ it follows that, $M_3(S^\star_{n+1}) = n|n - (m + 1)| + nm|(m + 1) - 1| = n|n - (m + 1)| + nm^2 = n(|n - (m + 1)| + m^2)$, $n \geq 1$, $m \in \mathbb{N}_0$. Because both $\mathbb{N}_0$ and $S^\star_{n+1}$, $n \geq 1$, $n \in \mathbb{N}$ are well-defined the result follows through immediate induction. □

3.2. Zagreb Indices for Thorny Complete Graphs

We recall that a complete graph K_n of order n has vertices say, $v_1, v_2, v_3, \ldots, v_n$ and edges v_iv_j, $i \neq j$, $\forall i, j = 1, 2, 3, \ldots, n$. Similar to the proof of Lemma 3.1

it follows that a complete graph is well-defined. This property is very useful in many induction proofs related to complete graphs. Complete graphs has the added complexity that when K_n is extended to K_{n+1} the degree of the preceding vertices all change. Hence, all previous terms or products must be revisited. Well-defineness solve this in a very elegant way. It also allows for the next result to hold for K_n, $n \geq 1$.

Proposition 3.3. *For a thorny complete graph $K_n^\star$, $n \geq 1$ the Zagreb indices are:*

(i) $M_1(K_n^\star) = n[(n-1)+m]^2 + m.$

(ii) $M_2(K_n^\star) = n[n+(m-1)]\frac{n-1}{2}[n+(m-1)].$

(iii) $M_3(K_n^\star) = mn|n+m-2|.$

Proof. Let the vertices of K_n be denoted as v_i, $i = 1, 2, 3, ..., n$. Consider any thorny complete graph $K_n^\star$, $n \geq 1$, $m \in N_0$ and denote the newly attached pendant vertices as $u_{i,j}$, $i = 1, 2, 3, ..., n$ and $j = 1, 2, 3, ..., m$.

(i) From the definition of $M_1(G)$ it follows that

$$M_1(K_n^\star) = \sum_{w \in V(G_n^\star)} deg(w)^2 = \sum_{i=1}^{n} deg(v_i)^2 + \sum_{j=1}^{m} deg(u_{i,j})^2$$
$$= n[(n-1)+m]^2 + mn = n[(n-1)+m]^2 + m,$$

$n \geq 1$, $m \in N_0$. Because both $\mathbb{N}_0$ and $K_n^\star$ $n \geq 1$, $n \in \mathbb{N}$ are well-defined the result follows through immediate induction.

(ii) From the definition of $M_2(G)$ it follows that,

$$M_2(K_n^\star) = \frac{n(n-1)}{2}[n+(m-1)]^2 + mn[n+(m-1)]$$
$$= n[n+(m-1)]\frac{n-1}{2}[n+(m-1)],$$

$n \geq 1$, $m \in \mathbb{N}_0$. As both $\mathbb{N}_0$ and $K_n^\star$, $n \geq 1$, $n \in \mathbb{N}$ are well-defined the result follows through immediate induction.

(iii) From the definition of $M_3(G)$ it follows that,

$$M_3(K_n^*) = \frac{n(n-1)}{2}|[n+(m-1)] - [n+(m-1)]| + mn|n+(m-1)-1|$$
$$= mn|n+m-2|,$$

$n \geq 1$, $m \in \mathbb{N}_0$. Since both $\mathbb{N}_0$ and K_n^*, $n \geq 1$, $n \in \mathbb{N}$ are well-defined the result follows through immediate induction. □

4. The Eccentric Connectivity Index of Thorny Cycle, Thorny Star and Thorny Complete Graphs

The eccentric connectivity index of a graph is defined in [6,9].

4.1. Eccentric Connectivity Index

Recall that the eccentric connectivity index of a graph G, denoted $\xi^c(G)$ is defined to be, $\xi^c(G) = \sum\limits_{v \in V(G)} deg(v)\xi_G(v)$, with $\xi_G(v)$ the eccentricity of vertex v. When the context of G is clear we may abbreviate $\xi_G(v)$ to $\xi(v)$.

The eccentric connectivity index of thorny cycles, stars and complete graphs are computed in this section.

Proposition 4.1. *The eccentric connectivity index of $C_n^\star$, $n \geq 3$ is given by,*

$$\xi^c(C_n^\star) = \begin{cases} \frac{1}{2}(n(m+2)(n+2) + mn(n+4)), & \text{if } n \text{ is even}, \\ \frac{1}{2}(n(m+2)(n+1) + mn(n+3)), & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Case (i): Let $n \geq 4$ and even. For $C_4^\star$ let the cycle vertices be v_1, v_2, v_3, v_4 and let the respective m end-vertices be $v_{i,j}$, $1 \leq i \leq 4$, $1 \leq j \leq m$. Consider $m \geq 1$, a constant. Clearly, $deg(v_i) = m+2$, $\xi(v_i) = 3$, $deg(v_{i,j}) = 1$, $\xi(v_{i,j}) = 4$. From the definition it follows that $\xi^c(C_4^\star) = 4 \cdot (m+2) \cdot 3 + 4m \cdot 4 = 4 \cdot (m+2) \cdot \frac{1}{2}(4+2) + \frac{1}{2} \cdot 4m \cdot (4+4) = \frac{1}{2} \cdot (n(m+2)(n+2) + nm(n+4))_{n=4}$. Hence, the result holds for $C_4^\star$.

Assume the results holds for $C_k^\star$, $k \geq 4$ and even. So we have $\xi^c(C_k^\star) = \frac{1}{2}(k(m+2)(k+2) + mk(k+4))$. Consider $C_{k+2}^\star$. Clearly, $k+2$ is even, $deg(v_i)$, $1 \leq i \leq k+2$ remain $(m+2)$ and all $deg(v_{i,j})$, $1 \leq i \leq k+2$, $1 \leq j \leq m$ remain 1. However, $\xi_{C_{k+2}^\star}(v_i) = \xi_{C_k^\star}(v_i) + 1$ and $\xi_{C_{k+2}^\star}(v_{i,j}) = \xi_{C_k^\star}(v_{i,j}) + 1$, $1 \leq j \leq m$. From the definition of ξ^c and substituting the new values, we derive the result, $\xi^c(C_{k+2}^\star) = \frac{1}{2}(((k+2)+2)(m+2)((k+2)+2) + m(k+2)((k+2)+4))$. Thus, through induction the result holds $\forall k \geq 4$, k even.

Case (ii): Let $n \geq 3$ and odd. For $C_3^\star$ let the cycle vertices be v_1, v_2, v_3 and let the respective m end-vertices be $v_{i,j}$, $1 \leq i \leq 3$, $1 \leq j \leq m$. Consider $m \geq 1$, a constant. Clearly, $deg(v_i) = m+2$, $\xi(v_i) = 2$, $deg(v_{i,j}) = 1$,

$\xi(v_{i,j}) = 3$. From the definition it follows that $\xi^c(C_3^\star) = 3\cdot(m+2)\cdot 2+3m\cdot 3 = 3\cdot(m+2)\cdot\frac{1}{2}(3+1)+\frac{1}{2}\cdot 3n\cdot(3+3) = \frac{1}{2}\cdot(n(m+2)(n+1)+nm(n+3))_{n=3}$. Hence, the result holds for $C_3^\star$.

For the induction assumption $n = k$, k odd, and the induction step $n = k+2$ the reasoning is similar to that in Case (i). That settles the result for $k \geq 3$, k odd. □

Proposition 4.2. *The eccentric connectivity index of $S_{n+1}^\star$, $n \geq 2$, $m \geq 1$ is given by, $\xi^c(S_{n+1}^\star) = n(5+7m)$.*

Proof. Consider the thorn star graph $S_{n+1}^\star$ for any $n \geq 2$ and any $m \geq 1$, m a constant. The labeling of the vertices of the star graph S_{n+1} follows from Definition 3.2. For the thorn star graph label the end-vertices, $v_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq m$, respectively. Clearly $deg(u) = n$, $deg(v_i) = m+1$, $deg(v_{i,j}) = 1$ and $\xi(u) = 2$, $\xi(v_i) = 3$, $\xi(v_{i,j}) = 4$. From the definition of ξ^c we have $\xi^c(S_{n+1}^\star) = n\cdot 2+n\cdot(m+1)\cdot 3+nm\cdot 1\cdot 4 = n(5+7m)$. Because both S_{n+1}, $n \geq 2$, $n \in \mathbb{N}$ and $S_{n+1}^\star$, $m \in \mathbb{N}$ are well-defined the result follows through immediate induction. □

Proposition 4.3. *The eccentric connectivity index of $K_n^\star$, $n \geq 1$ and $m \geq 0$ is given by, $\xi^c(K_n^\star) = n(2n+5m-2)$.*

Proof. Let the vertices of K_n be denoted as v_i, $i = 1,2,3,...,n$ and the newly attached pendant vertices be denoted as $u_{i,j}$, $i = 1,2,3,...,n$ and $j = 1,2,3,...,m$.

The degree and eccentricity of the vertices of K_n^* are given by,

$$\begin{aligned} deg(v_i) &= (n-1)+m,\\ deg(u_{i,j}) &= 1,\\ \xi(v_i) &= 2,\\ \xi(u_{i,j}) &= 3. \end{aligned}$$

Thus, the eccentric connectivity index of $K_n^\star$ is computed as:

$$\begin{aligned} \xi^c(K_n^\star) &= \sum_{u\in V} deg(u)\xi(u)\\ &= \sum_{i=1}^{n} deg(v_i)\xi(v_i) + \sum_{i=1}^{n}\sum_{j=1}^{m} deg(u_{i,j})\xi(u_{i,j})\\ &= \sum_{i=1}^{n} 2[(n-1)+m] + \sum_{i=1}^{n}\sum_{j=1}^{m} 1\cdot 3\\ &= n(2n+5m-2). \end{aligned}$$

Thus,

$$\xi^c(K_n^\star) = n(2n+5m-2), \text{ for } n \geq 1 \text{ and } m \geq 0.$$ □

4.2. General Thorny Graph

The general thorn or thorny graph was defined by Gutman [6]. See Section 2. Consider a connected simple graph G of order $n \geq 2$. For $n = 2$ consider one vertex u, non-pendant and the other vertex pendant to u. For $n \geq 3$ write $V(G) = V_1(G) \cup V_2(G) = \{u \in V(G) : \text{non-pendant vertices}\} \cup \{v \in V(G) : \text{pendant vertices}\}$.The *thorn number* of a vertex $u \in V_1(G)$ is the number of pendant vertices (end-vertices or thorns) attached to u. The thorn number is denoted $t(u)$. In the context of this paper a m-regular thorny graph, denoted $G^{\star}_{\epsilon_1,m}$, $\epsilon_1 = |V_1(G)|$ has $t(u)_{\forall u \in V_1(G)} = m \geq 1$. From the aforesaid we can derive a recursive result in general.

Theorem 4.4. *For a m-regular thorny graph $G^t_{\epsilon_1,m}$ with $G' = \langle V_1(G)\rangle$, we have:*

$$\xi^c(G^t_{\epsilon_1,m}) = \xi^c(G') + \sum_{i=1}^{\epsilon_1} deg_{G'}(u_i) + 2m \cdot \sum_{i=1}^{\epsilon_1} \xi_{G'}(u_i) + 3m\epsilon_1.$$

Proof. Label the vertices in $V_1(G)$ as u_i, $1 \leq i \leq \epsilon_1$ and the corresponding thorns of each u_i as $v_{i,j}$, $1 \leq j \leq m$. In $G^t_{\epsilon_1,m}$ the degree and eccentricity of the vertices are given by,

$$\begin{aligned} deg(u_i) &= deg_{G'}(u_i) + m, \\ deg(v_{i,j}) &= 1, \forall i, j \\ \xi(u_i) &= \xi_{G'}(u_i) + 1, \\ \xi(v_{i,j}) &= \xi_{G'}(u_i) + 2,\ 1 \leq i \leq \epsilon_1, \forall j \end{aligned}$$

Thus, the eccentric connectivity index of $G^t_{\epsilon_1,m}$ is computed as:

$$\begin{aligned} \xi^c(G^t_{\epsilon_1,m}) &= \sum_{i=1}^{\epsilon_1} deg(u_i)\xi(u_i) + \sum_{i=1}^{\epsilon_1}\sum_{j=1}^{m} deg(v_{i,j})\xi(v_{i,j}) \\ &= \sum_{i=1}^{\epsilon_1}(deg_{G'}(u_i) + m)(\xi_{G'}(u_i) + 1) + m \cdot \sum_{i=1}^{\epsilon_1}(\xi_{G'}(u_i) + 2) \\ &= \sum_{i=1}^{\epsilon_1} deg_{G'}(u_i)\xi_{G'}(u_i) + \sum_{i=1}^{\epsilon_1} deg_{G'}(u_i) + m\sum_{i=1}^{\epsilon_1} \xi_{G'}(u_i) + \end{aligned}$$

$$m\epsilon_1 + m\sum_{i=1}^{\epsilon_1} \xi_{G'}(u_i) + 2m\epsilon_1.$$

Thus,

$$\xi^c(G^t_{\epsilon_1,m}) = \xi^c(G') + \sum_{i=1}^{\epsilon_1} deg_{G'}(u_i) + 2m \cdot \sum_{i=1}^{\epsilon_1} \xi_{G'}(u_i) + 3m\epsilon_1. \qquad \square$$

Note that if the number of edges of G' is known say, $\varepsilon(G')$ then the result can be simplified to $\xi^c(G^t_{\epsilon_1,m}) = \xi^c(G') + 2m \cdot \sum_{i=1}^{\epsilon_1} \xi_{G'}(u_i) + 2\varepsilon(G') + 3m\epsilon_1$.

5. Conclusion

This paper offers a wide scope for replication in respect of many other standard graphs. The families of small graphs is an excellent source for further seminar papers.

The notion of a m-regular thorny graph $G^t_{\epsilon_1,m}$ offers scope for the concept of a *general bushy thorn graph.* If a m_1-regular thorny graph $G^t_{\epsilon_1,m_1}$ is layered with m_2 additional thorns added to all vertices we have $(G^t_{\epsilon_1,m_1})^t_{\epsilon_2,m_2}$, $\epsilon_2 = \epsilon_1(m_1+1)$. For q layers we propose the notation, $G^t_{(\epsilon_i,m_i)}$, $1 \leq i \leq q$. Coding an application to recursively apply Theorem 4.4 to determine $\xi^c(G^t_{(\epsilon_i,m_i)})$ for a $1 \leq i \leq q$ layered bushy thorn graph is an interesting assignment. A complexity analysis is an open study as well.

Formalising general results similar to that of Theorem 4.4 for the Zagreb indices remain interesting open problems.

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