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SOME PROPERTIES OF (L, M)-DOUBLE FUZZY GRILL BASES

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Abstract: We introduce the notions of (L, M)-double fuzzy grill bases where L and M are strictly two-sided, commutative quantales. We investigate the relationships between (L, M)-double fuzzy grill and (L, M)-double fuzzy grill bases. Furthermore, we investigate the image of (L, M)-double fuzzy grills.

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1. Introduction

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [3,4]. Working under the name "intuitionistic" did not continue because doubts were thrown about the suitability of this term, especially when working in the case of complete lattice L. These doubts were quickly ended in 2005 by Garcia and Rodabaugh [10]. They proved that this term is unsuitable in mathematics and applications. They concluded that they

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work under the name ”double”. Under this concept many works have been launched [6,7]. Abd El-latif [2] introduced the notions of (L, M)-Double fuzzy grills.

In this paper, we introduce the notions of (L, M)-double fuzzy grill bases where L and M are strictly two-sided, commutative quantales. We investigate the relationships between (L, M)-double fuzzy grill and (L, M)-double fuzzy grill bases. Furthermore, we investigate the image of (L, M)-double fuzzy grills.

2. Preliminaries

Throughout this paper, let L, M be complete lattices with an order-reversing involution $'$ and 0_L (1_L) and 0_M (1_M) are the smallest (largest) elements of L and M respectively. Let X be a non-empty set. The family of all L-fuzzy sets on X will be denoted by L^X. The smallest element and the largest one of L^X will be denoted by 0_X and 1_X respectively. For each $\alpha \in L$, $\underline{\alpha}(x) = \alpha$, for all $x \in X$. Let $f : X \to Y$ be a crisp map. The Zadeh image and preimage operators $f^{\to} : L^X \to L^Y$ and $f^{\leftarrow} : L^Y \to L^X$ are defined by:

$$f^{\to}(\lambda)(y) = \bigvee\{\lambda(x) : x \in X, f(x) = y\},$$

$$f^{\leftarrow}(\mu) = \mu \circ f,$$

for each $\lambda \in L^X$, $\mu \in L^Y$.

Definition 2.1. [13,15,16]. A triple $(L, \leq, \odot)$ is called a strictly two-sided, commutative quantale (briefly, stsc-quantale) if it satisfies the following properties:

(L1) $L = (L, \leq, 0_L, 1_L)$ is a complete lattice,

(L2) $(L, \odot)$ is a commutative semigroup,

(L3) $a = a \odot 1_L$, for each $a \in L$,

(L4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma}(a_i \odot b).$$

A stsc-quantale $(L, \leq, \odot)$ is called a stsc-biquantale if $\odot$ is distributive over arbitrary meets, i.e.

(L) $(\bigwedge_{i\in\Gamma} a_i) \odot b = \bigwedge_{i\in\Gamma}(a_i \odot b)$.

Example 2.2. [13,15,16]. (i) Each frame is a stsc-quantatle. In particular, the unit interval $([0, 1], \leq, \vee, \wedge, 0, 1)$ is a stsc-quantale.

(ii) The unit interval with a left-continuous t-norm t, $([0,1], \leq, t)$, is a stsc-quantatle.

(iii) Every GL-monoid is a stsc-quantale.

(iv) Define a binary operation $\odot$ on $[0,1]$ by $x \odot y = max\{0, x + y - 1\}$. Then $([0,1], \leq, \odot)$ is a stsc-quantale.

Definition 2.3. [13,15,16]. Let $(L, \leq, \odot)$ be a stsc-quantale. A mapping $' : L \to L$ is called an order-reversing involution, if it satisfies the following conditions:

(i) $x'' = x$, for each $x \in L$.

(ii) If $x \leq y$ then, $y' \leq x'$, for each $x, y \in L$.

In this paper, we always assume that $(L, \leq, \odot, \oplus, ')$ (respectively, $(M, \leq, \odot, \oplus, '))$ is a stsc-quantale with an order-reversing involution $'$ and the binary operation $\oplus$ is defined by:

$$x \oplus y = (x' \odot y')'$$

unless otherwise specified. For each $x, y, z \in L$ the following properties satisfies

(i) if $y \leq z$, then $(x \odot y) \leq (x \odot z)$ and $(x \oplus y) \leq (x \oplus z)$,

(ii) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$ [12].

The following equalities can be obtained easily,

(L3$'$) $a = a \oplus 0_L$, for each $a \in L$.

(L4$'$) $(\bigwedge_{i\in\Gamma} a_i) \oplus b = \bigwedge_{i\in\Gamma}(a_i \oplus b)$.

(L$'$) $(\bigvee_{i\in\Gamma} a_i) \oplus b = \bigvee_{i\in\Gamma}(a_i \oplus b)$, if $(L, \leq, \odot, \oplus, ')$ is a stsc-biquantale.

All algebraic operations on L can be extended to the set L^X and M^{L^X} as follows: for all $x \in X$, $\lambda, \mu \in L^X$, $\mathcal{U}, \mathcal{V} \in \mathcal{M}^{\mathcal{L}^{\mathbf{X}}}$;

(i) $\lambda \leq \mu$ iff $\lambda(x) \leq \mu(x)$,

(ii) For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$, for each $x \in X$,

(iii) $(\lambda \oplus \mu)(x) = \lambda(x) \oplus \mu(x)$,

(iv) $\mathcal{U} \leq \mathcal{V}$ iff $\mathcal{U}(\lambda) \leq \mathcal{V}(\lambda)$ for all $\lambda \in L^X$.

Lemma 2.4. [13]. Let $(L, \leq, \odot, \oplus, ')$ be a stsc-quantale with an order-reveresing involution $'$ and $f : X \to Y$ be a map. For each $\{\lambda_i : i \in \Gamma\} \subseteq L^X$ and $\{\mu_j : j \in J\} \subseteq L^Y$, we have the following properties:

(i) $f^{\to}(\oplus_{i\in\Gamma}\lambda_i) = \oplus_{i\in\Gamma} f^{\to}(\lambda_i)$.

(ii) $f^{\leftarrow}(\oplus_{j\in J}\mu_j) = \oplus_{j\in J} f^{\leftarrow}(\mu_j)$.

Definition 2.5. [2] The pair $(\mathcal{G}, \mathcal{G}^*)$ of maps $\mathcal{G}, \mathcal{G}^* : L^X \to M$ is called an (L, M)-double fuzzy grill on X if it satisfies the following conditions:

(DFG1) $\mathcal{G}(\lambda) \leq (\mathcal{G}^*(\lambda))'$, $\forall \lambda \in L^X$,

(DFG2) $\mathcal{G}(0_X) = 0_M$, $\mathcal{G}(1_X) = 1_M$ and, $\mathcal{G}^*(0_X) = 1_M$, $\mathcal{G}^*(1_X) = 0_M$,

(DFG3) $\mathcal{G}(\lambda \oplus \mu) \leq \mathcal{G}(\lambda) \oplus \mathcal{G}(\mu)$,and $\mathcal{G}^*(\lambda \oplus \mu) \geq \mathcal{G}^*(\lambda) \odot \mathcal{G}^*(\mu)$ for each $\lambda, \mu \in L^X$,

(DFG4) If $\lambda \leq \mu$, then $\mathcal{G}(\lambda) \leq \mathcal{G}(\mu)$ and $\mathcal{G}^*(\lambda) \geq \mathcal{G}^*(\mu)$.

The triplet $(X, \mathcal{G}, \mathcal{G}^*)$ is called an (L, M)-double fuzzy grill space. If $(\mathcal{G}_1, \mathcal{G}_1^*)$ and $(\mathcal{G}_2, \mathcal{G}_2^*)$ are (L, M)-double fuzzy grills on X, we say $(\mathcal{G}_1, \mathcal{G}_1^*)$ is finer than $(\mathcal{G}_2, \mathcal{G}_2^*)$ (or $(\mathcal{G}_2, \mathcal{G}_2^*)$ is coarser than $(\mathcal{G}_1, \mathcal{G}_1^*)$) denoted by $(\mathcal{G}_2, \mathcal{G}_2^*) \leq (\mathcal{G}_1, \mathcal{G}_1^*)$ iff $\mathcal{G}_2(\lambda) \leq \mathcal{G}_1(\lambda)$ and $\mathcal{G}_2^*(\lambda) \geq \mathcal{G}_1^*(\lambda)$, $\forall \lambda \in L^X$.

Definition 2.6. [2] Let $(X, \mathcal{G}_1, \mathcal{G}_1^*)$ and $(Y, \mathcal{G}_2, \mathcal{G}_2^*)$ be two (L, M)-double fuzzy grill spaces. Then a map $f : X \to Y$ is said to be:

(i) double g-map if $\mathcal{G}_1(f^{\leftarrow}(\mu)) \leq \mathcal{G}_2(\mu)$ and $\mathcal{G}_1^*(f^{\leftarrow}(\mu)) \geq \mathcal{G}_2^*(\mu)$, $\forall \mu \in L^Y$.

(ii) double g-preserving map if $\mathcal{G}_1(\lambda) \geq \mathcal{G}_2(f^{\to}(\lambda))$ and $\mathcal{G}_1^*(\lambda) \leq \mathcal{G}_2^*(f^{\to}(\lambda))$, $\forall \lambda \in L^X$.

Naturally, the composition of double g-maps (resp. double g-preserving maps) is a double g-map(resp. double g-preserving map).

3. (L, M)-Double Fuzzy Grill Bases

Notation 3.1. Let $\mathcal{B}, \mathcal{B}^* : L^X \to M$ be two maps and $\lambda \in L^X$. Then $\langle \mathcal{B} \rangle$ and $\langle \mathcal{B}^* \rangle$ are defined as follows:

$$\langle \mathcal{B} \rangle(\lambda) = \bigwedge_{\mu \geq \lambda} \mathcal{B}(\mu) \text{ and } \langle \mathcal{B}^* \rangle(\lambda) = \bigvee_{\mu \geq \lambda} \mathcal{B}^*(\mu).$$

Definition 3.2. The pair $(\mathcal{B}, \mathcal{B}^*)$ of maps $\mathcal{B}, \mathcal{B}^* : L^X \to M$ is called an (L, M)-double fuzzy grill base on X if it satisfies the following conditions:

(DFGB1) $\mathcal{B}(\lambda) \leq (\mathcal{B}^*(\lambda))'$, $\forall \lambda \in L^X$;

(DFGB2) $\mathcal{B}(0_X) = 0_M$, $\mathcal{B}(1_X) = 1_M$ and $\mathcal{B}^*(0_X) = 1_M$, $\mathcal{B}^*(1_X) = 0_M$;

(DFGB3) $\langle \mathcal{B} \rangle(\lambda \oplus \mu) \leq \mathcal{B}(\lambda) \oplus \mathcal{B}(\mu)$ and $\langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \geq \mathcal{B}^*(\lambda) \odot \mathcal{B}^*(\mu)$, $\forall \lambda, \mu \in L^X$.

If $(\mathcal{B}_1, \mathcal{B}_1^*)$ and $(\mathcal{B}_2, \mathcal{B}_2^*)$ are two (L, M)-double fuzzy grill bases on X, we say $(\mathcal{B}_1, \mathcal{B}_1^*)$ is finer than $(\mathcal{B}_2, \mathcal{B}_2^*)$(or $(\mathcal{B}_2, \mathcal{B}_2^*)$ is coarser than $(\mathcal{B}_1, \mathcal{B}_1^*)$) denoted by $(\mathcal{B}_2, \mathcal{B}_2^*) \leq (\mathcal{B}_1, \mathcal{B}_1^*)$ iff $\mathcal{B}_2(\lambda) \leq \mathcal{B}_1(\lambda)$ and $\mathcal{B}_2^*(\lambda) \geq \mathcal{B}_1^*(\lambda)$, $\forall \lambda \in L^X$.

Theorem 3.3. If $(\mathcal{B}, \mathcal{B}^*)$ is an (L, M)-double fuzzy grill base, then $(\langle \mathcal{B} \rangle, \langle \mathcal{B}^* \rangle)$ is the finest (L, M)-double fuzzy grill satisfies $\langle \mathcal{B} \rangle(\lambda) \leq \mathcal{B}(\lambda)$ and $\langle \mathcal{B}^* \rangle(\lambda) \geq \mathcal{B}^*(\lambda)$, $\forall \lambda \in L^X$.

Proof. (DFG1)For each $\lambda \in L^X$,

$$\begin{aligned} \langle \mathcal{B} \rangle(\lambda) &= \textstyle\bigwedge_{\mu \geq \lambda} \mathcal{B}(\mu) \leq \bigwedge_{\mu \geq \lambda} (\mathcal{B}^*(\mu))' \\ &= (\textstyle\bigvee_{\mu \geq \lambda} \mathcal{B}^*(\mu))' = (\langle \mathcal{B}^* \rangle(\lambda))'. \end{aligned}$$

(DFG2) and (DFG4) are easily checked.

(DFG3) Suppose that there exist $\lambda, \mu \in L^X$ such that

$$\langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \not\geq \langle \mathcal{B}^* \rangle(\lambda) \odot \langle \mathcal{B}^* \rangle(\mu).$$

By the definition of $\langle \mathcal{B}^* \rangle$ and (L4$'$), there exist $\lambda_1, \mu_1 \in L^X$ with $\lambda_1 \geq \lambda$ and $\mu_1 \geq \mu$ such that:

$$\langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \not\geq \mathcal{B}^*(\lambda_1) \odot \mathcal{B}^*(\mu_1).$$

Since $(\mathcal{B}, \mathcal{B}^*)$ is an (L, M)-double fuzzy grill base,

$$\langle \mathcal{B}^* \rangle(\lambda_1 \oplus \mu_1) \geq \mathcal{B}^*(\lambda_1) \odot \mathcal{B}^*(\mu_1).$$

Since $\lambda \oplus \mu \leq \lambda_1 \oplus \mu_1$, we have

$$\langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \geq \langle \mathcal{B}^* \rangle(\lambda_1 \oplus \mu_1) \geq \mathcal{B}^*(\lambda_1) \odot \mathcal{B}^*(\mu_1).$$

It is a contradiction. Thus, $\langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \geq \langle \mathcal{B}^* \rangle(\lambda) \odot \langle \mathcal{B}^* \rangle(\mu)$, for each $\lambda, \mu \in L^X$. Similarly, $\langle \mathcal{B} \rangle(\lambda \oplus \mu) \leq \langle \mathcal{B} \rangle(\lambda) \oplus \langle \mathcal{B} \rangle(\mu)$, for each $\lambda, \mu \in L^X$.

Let $(\mathcal{G}, \mathcal{G}^*)$ be another (L, M)-double fuzzy grill satisfies $\mathcal{G}(\lambda) \leq \mathcal{B}(\lambda)$ and $\mathcal{G}^*(\lambda) \geq \mathcal{B}^*(\lambda)$, $\forall \lambda \in L^X$. Then, we have

$$\langle \mathcal{B} \rangle(\lambda) = \bigwedge_{\mu \geq \lambda} \mathcal{B}(\mu) \geq \bigwedge_{\mu \geq \lambda} \mathcal{G}(\mu) = \mathcal{G}(\lambda),$$

and

$$\langle \mathcal{B}^* \rangle(\lambda) = \bigvee_{\mu \geq \lambda} \mathcal{B}^*(\mu) \leq \bigvee_{\mu \geq \lambda} \mathcal{G}^*(\mu) = \mathcal{G}^*(\lambda).$$

Theorem 3.4. Let $(\mathcal{B}_1, \mathcal{B}_1^*)$ and $(\mathcal{B}_2, \mathcal{B}_2^*)$ be two (L, M)-double fuzzy grill bases on X and Y respectively, and $f : X \to Y$ be a map. Then we have the following properties:

(i) $f : (X, \langle \mathcal{B}_1 \rangle, \langle \mathcal{B}_1^* \rangle) \to (Y, \langle \mathcal{B}_2 \rangle, \langle \mathcal{B}_2^* \rangle)$ is a double g-map if and only if, $\langle \mathcal{B}_1 \rangle(f^{\leftarrow}(\mu)) \leq \mathcal{B}_2(\mu)$ and $\langle \mathcal{B}_1^* \rangle(f^{\leftarrow}(\mu)) \geq \mathcal{B}_2^*(\mu)$, $\forall \mu \in L^Y$.

(ii) $f : (X, \langle \mathcal{B}_1 \rangle, \langle \mathcal{B}_1^* \rangle) \to (Y, \langle \mathcal{B}_2 \rangle, \langle \mathcal{B}_2^* \rangle)$ is a double g-preserving map if and only if, $\langle \mathcal{B}_2 \rangle(f^{\rightarrow}(\lambda)) \leq \mathcal{B}_1(\lambda)$ and $\langle \mathcal{B}_2^* \rangle(f^{\rightarrow}(\lambda)) \geq \mathcal{B}_1^*(\lambda)$, $\forall \lambda \in L^X$.

(iii) If $\mathcal{B}_1(f^{\leftarrow}(\mu)) \leq \mathcal{B}_2(\mu)$ and $\mathcal{B}_1^*(f^{\leftarrow}(\mu)) \geq \mathcal{B}_2^*(\mu)$, $\forall \mu \in L^Y$, then $f : (X, \langle \mathcal{B}_1 \rangle, \langle \mathcal{B}_1^* \rangle) \to (Y, \langle \mathcal{B}_2 \rangle, \langle \mathcal{B}_2^* \rangle)$ is a double g-map.

(iv) If $\mathcal{B}_2(f^{\rightarrow}(\lambda)) \leq \mathcal{B}_1(\lambda)$ and $\mathcal{B}_2^*(f^{\rightarrow}(\lambda)) \geq \mathcal{B}_1^*(\lambda)$, $\forall \lambda \in L^X$, then $f : (X, \langle \mathcal{B}_1 \rangle, \langle \mathcal{B}_1^* \rangle) \to (Y, \langle \mathcal{B}_2 \rangle, \langle \mathcal{B}_2^* \rangle)$ is a double g-preserving map.

Proof. It sufficient to prove only condition (i), because the other conditions can be obtained similarly.

(i)($\Rightarrow$:) Since $\langle \mathcal{B}_2 \rangle(\mu) \leq \mathcal{B}_2(\mu)$ and $\langle \mathcal{B}_2^* \rangle(\mu) \geq \mathcal{B}_2^*(\mu)$, $\forall \mu \in L^Y$, it is trivial.

($\Leftarrow$:) Let $\langle\mathcal{B}_1\rangle(f^{\leftarrow}(\mu)) \leq \mathcal{B}_2(\mu)$ and $\langle\mathcal{B}_1^*\rangle(f^{\leftarrow}(\mu)) \geq \mathcal{B}_2^*(\mu)$, $\forall\mu \in L^Y$. We will show that f is a double g-map. For arbitrary $\mu \in L^Y$, we have:

$$\begin{aligned}\langle\mathcal{B}_2\rangle(\mu) &= \bigwedge_{\nu\geq\mu}\mathcal{B}_2(\nu)\\ &\geq \bigwedge_{f^{\leftarrow}(\nu)\geq f^{\leftarrow}(\mu)}\langle\mathcal{B}_1\rangle(f^{\leftarrow}(\nu)) \geq \langle\mathcal{B}_1\rangle(f^{\leftarrow}(\mu)),\end{aligned}$$

$$\begin{aligned}\langle\mathcal{B}_2^*\rangle(\mu) &= \bigvee_{\nu\geq\mu}\mathcal{B}_2^*(\nu)\\ &\leq \bigvee_{f^{\leftarrow}(\nu)\geq f^{\leftarrow}(\mu)}\langle\mathcal{B}_1^*\rangle(f^{\leftarrow}(\nu)) \leq \langle\mathcal{B}_1^*\rangle(f^{\leftarrow}(\mu)).\end{aligned}$$

Therefore, f is a double g-map.

The converse of Theorem 3.4(iii, iv) need not be true.

Example 3.5. Let $L = M = [0, 1]$, $\oplus = \vee$ and $\odot = \wedge$. Let $X = \{a, b, c, d\}$ be a set. Define the maps $\mathcal{B}_1, \mathcal{B}_1^*, \mathcal{B}_2, \mathcal{B}_2^* : L^X \to M$ as follows:

$$\mathcal{B}_1(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0} \\ 0.2, & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{a,b,c\}}\} \\ 1, & \text{otherwise,} \end{cases}$$

$$\mathcal{B}_1^*(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0} \\ 0.5, & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{a,b,c\}}\} \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{B}_2(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0} \\ 0.5, & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{a,b\}}\} \\ 1, & \text{otherwise,} \end{cases}$$

$$\mathcal{B}_2^*(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0} \\ 0.4, & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{a,b\}}\} \\ 0, & \text{otherwise.} \end{cases}$$

Then we obtain

$$\langle\mathcal{B}_1\rangle(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0} \\ 0.2, & \text{if } \underline{0} \neq \lambda \leq \chi_{\{a,b,c\}} \\ 1, & \text{otherwise,} \end{cases} \quad \langle\mathcal{B}_1^*\rangle(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0} \\ 0.5, & \text{if } \underline{0} \neq \lambda \leq \chi_{\{a,b,c\}} \\ 0, & \text{otherwise,} \end{cases}$$

$$\langle\mathcal{B}_2\rangle(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0} \\ 0.5, & \text{if } \underline{0} \neq \lambda \leq \chi_{\{a,b\}} \\ 1, & \text{otherwise,} \end{cases} \quad \langle\mathcal{B}_2^*\rangle(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0} \\ 0.4, & \text{if } \underline{0} \neq \lambda \leq \chi_{\{a,b\}} \\ 0, & \text{otherwise,} \end{cases}$$

Thus $(\mathcal{B}_1, \mathcal{B}_1^*)$ and $(\mathcal{B}_2, \mathcal{B}_2^*)$ are (L, M)-double fuzzy grill bases. Clearly, the identity map $id_X : (X, \langle\mathcal{B}_1\rangle, \langle\mathcal{B}_1^*\rangle) \to (X, \langle\mathcal{B}_2\rangle, \langle\mathcal{B}_2^*\rangle)$ is a double g-map and the identity map $id_X : (X, \langle \mathcal{B}_2\rangle, \langle \mathcal{B}_2^*\rangle) \to (X, \langle \mathcal{B}_1\rangle, \langle \mathcal{B}_1^*\rangle)$ is a double g-preserving map. But

$$\mathcal{B}_1(\chi_{\{a,b\}}) = 1 > \mathcal{B}_2(\chi_{\{a,b\}}) = 0.5,$$

$$\mathcal{B}_1^*(\chi_{\{a,b\}}) = 0 < \mathcal{B}_2^*(\chi_{\{a,b\}}) = 0.4.$$

Theorem 3.6. Let $f_i : X \to X_i$ be a map, for all $i \in \Gamma$ and $\{(\mathcal{B}_i, \mathcal{B}_i^*)\}_{i\in\Gamma}$ be a family of (L, M)-double fuzzy grill base on X_i satisfying the following condition:

(C) If $\mathcal{B}_i(\lambda_i) \neq 1_M$ and $\mathcal{B}_i^*(\lambda_i) \neq 0_M$, $\forall i \in \Gamma$, then we have $\oplus_{i\in K} f_i^{\leftarrow}(\lambda_i) \neq 1_X$, for every finite subset K of Γ. We define the maps $\bigwedge_{i\in\Gamma} f_i^{\leftarrow}(\mathcal{B}_i), \bigvee_{i\in\Gamma} f_i^{\leftarrow}(\mathcal{B}_i^*) : L^X \to M$ as:

$$\bigwedge_{i\in\Gamma} f_i^{\leftarrow}(\mathcal{B}_i)(\lambda) = \begin{cases} \bigwedge\{\oplus_{i\in K}\mathcal{B}_i(\lambda_i), & \text{if} \quad \lambda = \oplus_{i\in K} f_i^{\leftarrow}(\lambda_i)\} \\ 1_M, & \text{otherwise}, \end{cases}$$

$$\bigvee_{i\in\Gamma} f_i^{\leftarrow}(\mathcal{B}_i^*)(\lambda) = \begin{cases} \bigvee\{\odot_{i\in K}\mathcal{B}_i^*(\lambda_i), & \text{if} \quad \lambda = \oplus_{i\in K} f_i^{\leftarrow}(\lambda_i)\} \\ 0_M, & \text{otherwise}, \end{cases}$$

where $\bigwedge$ and $\bigvee$ are taken for every finite subset K of Γ such that $\lambda = \oplus_{i\in K} f_i^{\leftarrow}(\lambda_i)$.

Let $\mathcal{B} = \bigwedge_{i\in\Gamma} f_i^{\leftarrow}(\mathcal{B}_i)$ and $\mathcal{B}^* = \bigvee_{i\in\Gamma} f_i^{\leftarrow}(\mathcal{B}_i^*)$ be given. Then:

(i) $(\mathcal{B}, \mathcal{B}^*)$ is an (L, M)-double fuzzy grill base on X,

(ii) $(\langle\mathcal{B}\rangle, \langle\mathcal{B}^*\rangle)$ is the finest (L, M)-double fuzzy grill for which,
$f_i : (X, \langle\mathcal{B}\rangle, \langle\mathcal{B}^*\rangle) \to (X_i, \langle\mathcal{B}_i\rangle, \langle\mathcal{B}_i^*\rangle)$ is a double g-map for all $i \in \Gamma$,

(iii) A map $f : (Y, \mathcal{G}, \mathcal{G}^*) \to (X, \langle\mathcal{B}\rangle, \langle\mathcal{B}^*\rangle)$ is a double g-map if and only if, for each $i \in \Gamma$, $f_i \circ f : (Y, \mathcal{G}, \mathcal{G}^*) \to (X_i, \langle\mathcal{B}_i\rangle, \langle\mathcal{B}_i^*\rangle)$ is a double g-map,

(iv) $\langle\bigwedge_{i\in\Gamma} f_i^{\leftarrow}(\mathcal{B}_i)\rangle = \langle\bigwedge_{i\in\Gamma} f_i^{\leftarrow}(\langle\mathcal{B}_i\rangle)\rangle$ and
$\langle\bigvee_{i\in\Gamma} f_i^{\leftarrow}(\mathcal{B}_i^*)\rangle = \langle\bigvee_{i\in\Gamma} f_i^{\leftarrow}(\langle\mathcal{B}_i^*\rangle)\rangle$.

Proof. (i) (DFGB1) For every finite subset K of Γ such that $\lambda = \oplus_{i\in K} f_i^{\leftarrow}(\lambda_i)$ we have,

$$\begin{aligned} \mathcal{B}(\lambda) &= \textstyle\bigwedge_{i\in\Gamma} f_i^{\leftarrow}(\mathcal{B}_i)(\lambda) = \bigwedge \oplus_{i\in K}\mathcal{B}_i(\lambda_i) \\ &\leq \textstyle\bigwedge \oplus_{i\in K}(\mathcal{B}_i^*(\lambda_i))' = (\bigvee \odot_{i\in K}(\mathcal{B}_i^*(\lambda_i)))' = (\mathcal{B}^*(\lambda))'. \end{aligned}$$

(DFGB2) It is easy.

(DFGB3) Suppose that there exist $\lambda, \mu \in L^X$ such that

$$\langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \ngeq \mathcal{B}^*(\lambda) \odot \mathcal{B}^*(\mu).$$

By the definition of $\mathcal{B}^*(\lambda)$ and (L4$'$), there exists a finite subset K of Γ with $\lambda = \oplus_{k \in K} f_k^{\leftarrow}(\lambda_k)$ such that

$$\langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \ngeq (\odot_{k \in K} \mathcal{B}_k^*(\lambda_k)) \odot \mathcal{B}^*(\mu).$$

Again, by the definition of $\mathcal{B}^*(\mu)$ and (L4$'$), there exists a finite subset J of Γ with $\mu = \oplus_{j \in J} f_j^{\leftarrow}(\mu_j)$ such that

$$\langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \ngeq (\odot_{k \in K} \mathcal{B}_k^*(\lambda_k)) \odot (\odot_{j \in J} \mathcal{B}_j^*(\mu_j)).$$

Put $m \in (K \cup J)$ such that

$$\rho_m = \begin{cases} \lambda_m, & \text{if} \quad m \in K - (K \cap J) \\ \mu_m, & \text{if} \quad m \in J - (K \cap J) \\ \lambda_m \oplus \mu_m, & \text{if} \quad m \in K \cap J. \end{cases}$$

Since for each $m \in K \cap J$, $\langle \mathcal{B}_m^* \rangle(\lambda_m \oplus \mu_m) \geq \mathcal{B}_m^*(\lambda_m) \odot \mathcal{B}_m^*(\mu_m)$, we have

$$\begin{aligned} \langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \quad & \ngeq (\odot_{m \in (K \cup J)-(K \cap J)} \mathcal{B}_m(\rho_m)) \\ & \odot (\odot_{m \in (K \cap J)} \langle \mathcal{B}_m^* \rangle(\lambda_m \oplus \mu_m)). \end{aligned}$$

From the definition of $\langle \mathcal{B}_m^* \rangle$, there exists $\nu_m \in L^{X_m}$ with $\nu_m \geq \lambda_m \oplus \mu_m$ such that:

$$\begin{aligned} \langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \quad & \ngeq (\odot_{m \in (K \cup J)-(K \cap J)} \mathcal{B}_m^*(\rho_m)) \\ & \odot (\odot_{m \in (K \cap J)} \mathcal{B}_m^*(\nu_m)). \end{aligned}$$

On the other hand, since

$$\begin{aligned} \lambda \oplus \mu \quad & = (\oplus_{k \in K} f_k^{\leftarrow}(\lambda_j)) \oplus (\oplus_{j \in J} f_j^{\leftarrow}(\mu_j)) \\ & \leq (\oplus_{m \in (K \cup J)-(K \cap J)} f_m^{\leftarrow}(\rho_m)) \\ & \oplus (\oplus_{m \in (K \cap J)} f_m^{\leftarrow}(\nu_m)), \end{aligned}$$

and since $K \cup J$ is finite we have,

$$\begin{aligned} \langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \quad & \geq (\odot_{m \in (K \cup J)-(K \cap J)} \mathcal{B}_m^*(\rho_m)) \\ & \odot (\odot_{m \in (K \cap J)} \mathcal{B}_m^*(\nu_m)). \end{aligned}$$

It is a contradiction. Then, $\langle \mathcal{B}^* \rangle(\lambda \oplus \mu) \geq \mathcal{B}^*(\lambda) \odot \mathcal{B}^*(\mu), \forall \lambda, \mu \in L^X$. Similarly, $\langle \mathcal{B} \rangle(\lambda \oplus \mu) \leq \mathcal{B}(\lambda) \oplus \mathcal{B}(\mu), \forall \lambda, \mu \in L^X$. Hence $(\mathcal{B}, \mathcal{B}^*)$ is an (L, M)-double fuzzy grill base on X.

(ii) Since $\mathcal{B}(f_i^{\leftarrow}(\lambda_i)) \leq \mathcal{B}_i(\lambda_i)$ and $\mathcal{B}^*(f_i^{\leftarrow}(\lambda_i)) \geq \mathcal{B}_i^*(\lambda_i)$, for each $i \in \Gamma$ and by Theorem 3.4(iii), we have $f_i : (X, \langle\mathcal{B}\rangle, \langle\mathcal{B}^*\rangle) \to (X_i, \langle\mathcal{B}_i\rangle, \langle\mathcal{B}_i^*\rangle)$ is a double g-map.

Let $(\mathcal{G}, \mathcal{G}^*)$ be another (L, M)-double fuzzy grill on X such that for each $i \in \Gamma$, the map $f_i : (X, \mathcal{G}, \mathcal{G}^*) \to (X_i, \langle\mathcal{B}_i\rangle, \langle\mathcal{B}_i^*\rangle)$ is a double g-map. Then, for each $i \in \Gamma$,

$$\mathcal{G}(f_i^{\leftarrow}(\lambda_i)) \leq \langle\mathcal{B}_i\rangle(\lambda_i), \mathcal{G}^*(f_i^{\leftarrow}(\lambda_i)) \geq \langle\mathcal{B}_i^*\rangle(\lambda_i), \tag{3.1}$$

For each finite subset K of Γ with $\lambda \leq \oplus_{k\in K}(f_k^{\leftarrow}(\lambda_k))$, we have

$$\begin{aligned}
\mathcal{G}(\lambda) &\leq \mathcal{G}(\oplus_{k\in K}(f_k^{\leftarrow}(\lambda_k))) && \text{(by (DFG4))}\\
&\leq \oplus_{k\in K}\mathcal{G}(f_k^{\leftarrow}(\lambda_k)) && \text{(by (DFG3))}\\
&\leq \oplus_{k\in K}\langle\mathcal{B}_k\rangle(\lambda_k) && \text{(by (3.1))}\\
&\leq \oplus_{k\in K}\mathcal{B}_k(\lambda_k), && \text{(by Theorem 3.3)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}^*(\lambda) &\geq \mathcal{G}^*(\oplus_{k\in K}(f_k^{\leftarrow}(\lambda_k))) && \text{(by (DFG4))}\\
&\geq \odot_{k\in K}\mathcal{G}(f_k^{\leftarrow}(\lambda_k)) && \text{(by (DFG3))}\\
&\geq \odot_{k\in K}\langle\mathcal{B}_k\rangle(\lambda_k) && \text{(by (3.1))}\\
&\geq \odot_{k\in K}\mathcal{B}_k(\lambda_k). && \text{(by Theorem 3.3)}
\end{aligned}$$

If $\bigwedge$ and $\bigvee$ are taken over all families $\{\lambda_k \in L^{X_k} : \lambda \leq \oplus_{k\in K}(f_k^{\leftarrow}(\lambda_k))\}$, we have

$\mathcal{G}(\lambda) \leq \langle\mathcal{B}\rangle(\lambda)$ and $\mathcal{G}^*(\lambda) \geq \langle\mathcal{B}^*\rangle(\lambda)$.

(iii) Necessity of the composition condition is clear since the composition of double g-maps is a double g-map.

Conversely, for each finite index subset K of Γ with $\lambda \leq \oplus_{k\in K}(f_k^{\leftarrow}(\lambda_k))$, since for each $i \in \Gamma$, $f_i \circ f : (Y, \mathcal{G}, \mathcal{G}^*) \to (X_i, \langle\mathcal{B}_i\rangle, \langle\mathcal{B}_i^*\rangle)$ is a double g-map, we have, for each $i \in \Gamma$,

$$\mathcal{G}((f_i \circ f)^{\leftarrow}(\lambda_i)) \leq \langle\mathcal{B}_i\rangle(\lambda_i), \mathcal{G}^*((f_i \circ f)^{\leftarrow}(\lambda_i)) \geq \langle\mathcal{B}_i^*\rangle(\lambda_i).$$

It follows, for each $k \in K$,

$$\mathcal{G}((f_k \circ f)^{\leftarrow}(\lambda_k)) \leq \langle\mathcal{B}_k\rangle(\lambda_k), \mathcal{G}^*((f_k \circ f)^{\leftarrow}(\lambda_k)) \geq \langle\mathcal{B}_k^*\rangle(\lambda_k).$$

Since $f^{\leftarrow}(\lambda) \leq \oplus_{k\in K}(f_k \circ f)^{\leftarrow}(\lambda_k)$, we have

$$\begin{aligned}
\mathcal{G}(f^{\leftarrow}(\lambda)) &\leq \mathcal{G}(\oplus_{k\in K}(f_k \circ f)^{\leftarrow}(\lambda_k)) && \text{(by (DFG4))}\\
&\leq \oplus_{k\in K}\mathcal{G}((f_k \circ f)^{\leftarrow}(\lambda_k)) && \text{(by (DFG3))}\\
&\leq \oplus_{k\in K}\langle\mathcal{B}_k\rangle(\lambda_k) \leq \oplus_{k\in K}\mathcal{B}_k(\lambda_k),
\end{aligned}$$

$$\begin{aligned}\mathcal{G}^*(f^{\leftarrow}(\lambda)) &\geq \mathcal{G}^*(\oplus_{k\in K}(f_k\circ f)^{\leftarrow}(\lambda_k)) \quad \text{(by (DFG4))}\\ &\geq \odot_{k\in K}\mathcal{G}^*((f_k\circ f)^{\leftarrow}(\lambda_k)) \quad \text{(by (DFG3))}\\ &\geq \odot_{k\in K}\langle\mathcal{B}_k^*\rangle(\lambda_k)\geq \odot_{k\in K}\mathcal{B}_k^*(\lambda_k).\end{aligned}$$

This implies that $\mathcal{G}(f^{\leftarrow}(\lambda))\leq\langle\mathcal{B}\rangle(\lambda)$ and $\mathcal{G}^*(f^{\leftarrow}(\lambda))\geq\langle\mathcal{B}^*\rangle(\lambda)$.

(iv) Let $\mathcal{G}=\bigwedge_{i\in\Gamma}f_i^{\leftarrow}(\langle\mathcal{B}_i\rangle)$ and $\mathcal{G}^*=\bigvee_{i\in\Gamma}f_i^{\leftarrow}(\langle\mathcal{B}_i^*\rangle)$. Then, for each $\lambda_i\in L^{X_i}$ we have

$$\langle\mathcal{G}\rangle(f_i^{\leftarrow}(\lambda_i))\leq\mathcal{G}(f_i^{\leftarrow}(\lambda_i))\leq\langle\mathcal{B}_i\rangle(\lambda_i),$$

$$\langle\mathcal{G}^*\rangle(f_i^{\leftarrow}(\lambda_i))\geq\mathcal{G}^*(f_i^{\leftarrow}(\lambda_i))\geq\langle\mathcal{B}_i\rangle(\lambda_i).$$

Thus, $f_i:(X,\langle\mathcal{G}\rangle,\langle\mathcal{G}^*\rangle)\to(X_i,\langle\mathcal{B}_i\rangle,\langle\mathcal{B}_i^*\rangle)$ is a double g-map. By (ii) we have

$$\langle\mathcal{B}\rangle\geq\langle\mathcal{G}\rangle,\ \langle\mathcal{B}^*\rangle\leq\langle\mathcal{G}^*\rangle. \tag{3.2}$$

Conversely, $id_X:(X,\langle\mathcal{B}\rangle,\langle\mathcal{B}^*\rangle)\to(X_i,\langle\mathcal{G}\rangle,\langle\mathcal{G}^*\rangle)$ is a double g-map. In fact, for each family $\{\lambda_k:\lambda\leq\oplus_{k\in K}(f_k^{\leftarrow}(\lambda_k))\}$ we have:

$$\begin{aligned}\langle\mathcal{B}\rangle(\lambda) &\leq\langle\mathcal{B}\rangle(\oplus_{k\in K}(f_k^{\leftarrow}(\lambda_k)))\\ &\leq\oplus_{k\in K}\langle\mathcal{B}\rangle((f_k^{\leftarrow}(\lambda_k)))\leq\oplus_{k\in K}\langle\mathcal{B}_k\rangle(\lambda_k),\end{aligned}$$

$$\begin{aligned}\langle\mathcal{B}^*\rangle(\lambda) &\geq\langle\mathcal{B}^*\rangle(\oplus_{k\in K}(f_k^{\leftarrow}(\lambda_k)))\\ &\geq\odot_{k\in K}\langle\mathcal{B}^*\rangle((f_k^{\leftarrow}(\lambda_k)))\geq\odot_{k\in K}\langle\mathcal{B}_k^*\rangle(\lambda_k).\end{aligned}$$

By (L4′), (L4) and the definition of $\mathcal{G}$, $\mathcal{G}^*$ we obtain

$$\langle\mathcal{B}\rangle\leq\langle\mathcal{G}\rangle \text{ and } \langle\mathcal{B}^*\rangle\geq\langle\mathcal{G}^*\rangle. \tag{3.3}$$

By (3.2) and (3.3), the proof is completed.

Corollary 3.7. Let $\{(\mathcal{B}_i,\mathcal{B}_i^*)\}_{i\in\Gamma}$ be a family of (L,M)-double fuzzy grill bases on X_i. Let $X=\Pi_{i\in\Gamma}X_i$ be a product set and $\pi_i:X\to X_i$ be a projection map, for each $i\in\Gamma$. We define the maps $\bigwedge_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{B}_i),\bigvee_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{B}_i^*):L^X\to M$ as:

$$\bigwedge_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{B}_i)(\lambda)=\begin{cases}\bigwedge\{\oplus_{i\in K}\mathcal{B}_i(\lambda_i), & \text{if}\quad \lambda=\oplus_{i\in K}\pi_i^{\leftarrow}(\lambda_i)\}\\ 1_M, & \text{otherwise,}\end{cases}$$

$$\bigvee_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{B}_i^*)(\lambda)=\begin{cases}\bigvee\{\odot_{i\in K}\mathcal{B}_i^*(\lambda_i), & \text{if}\quad \lambda=\oplus_{i\in K}\pi_i^{\leftarrow}(\lambda_i)\}\\ 0_M, & \text{otherwise,}\end{cases}$$

where $\bigwedge$ and $\bigvee$ are taken for every finite subset K of Γ such that $\lambda=\oplus_{i\in K}\pi_i^{\leftarrow}(\lambda_i)$.

Let $\mathcal{B}=\bigwedge_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{B}_i)$ and $\mathcal{B}^*=\bigvee_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{B}_i^*)$ be given. Then:

(i) $(\langle\mathcal{B}\rangle, \langle\mathcal{B}^*\rangle)$ is the finest (L, M)-double fuzzy grill on X for which each projection map $\pi_i : (X, \langle\mathcal{B}\rangle, \langle\mathcal{B}^*\rangle) \to (X_i, \langle\mathcal{B}_i\rangle, \langle\mathcal{B}_i^*\rangle)$ is a double g-map, for each $i \in \Gamma$.

(ii) A map $f : (Y, \mathcal{G}, \mathcal{G}^*) \to (X, \langle\mathcal{B}\rangle, \langle\mathcal{B}^*\rangle)$ is a double g-map if and only if, $\pi_i \circ f : (Y, \mathcal{G}, \mathcal{G}^*) \to (X_i, \langle\mathcal{B}_i\rangle, \langle\mathcal{B}_i^*\rangle)$ is a double g-map, for each $i \in \Gamma$.

Proof. From Theorem 3.6, we only show that a family $\{(\mathcal{B}_i, \mathcal{B}_i^*)\}_{i\in\Gamma}$ satisfies the condition (C). Let $\mathcal{B}_i(\lambda_i) \neq 1_M$ and $\mathcal{B}_i^*(\lambda_i) \neq 0_M$, $\forall i \in \Gamma$. Then for every finite subset K of Γ, we have $\mathcal{B}_k(\lambda_k) \neq 1_M$ and $\mathcal{B}_k^*(\lambda_k) \neq 0_M$, $\forall k \in K$. Therefore $\lambda_k \neq 1_{X_k}$, $\forall k \in K$. So, for each $k \in K$, there exists $x_k \in X_k$ with $\lambda_k(x_k) \neq 1_L$. Pick up $x \in X$ such that $\pi_k(x) = x_k$ for all $k \in K$. Then,

$$\oplus_{k\in K}\pi_k^{\leftarrow}(\lambda_k)(x) = \oplus_{k\in K}\lambda_k(\pi_k(x)) \neq 1_X.$$

By Definition 2.11, $(\langle\bigwedge_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{B}_i)\rangle, \langle\bigvee_{i\in\Gamma}\pi_i^{\Leftarrow}(\mathcal{B}_i^*)\rangle)$ in Corollary 3.7, is the product of (L, M)-fuzzy grills $\{(\langle\mathcal{B}_i\rangle, \langle\mathcal{B}_i^*\rangle)\}_{i\in\Gamma}$.

4. The Image of (L, M)-Double Fuzzy Grills

Theorem 4.1. Let $f_i : X_i \to X$ be a map, for all $i \in \Gamma$ and $\{(\mathcal{G}_i, \mathcal{G}_i^*)\}_{i\in\Gamma}$ be a family of (L, M)-double fuzzy grills on X_i satisfying the following condition:

(C) If $\mathcal{G}_i(\lambda_i) \neq 1_M$ and $\mathcal{G}_i^*(\lambda_i) \neq 0_M$ $\forall i \in \Gamma$, then we have $\oplus_{i\in K}f_i^{\rightarrow}(\lambda_i) \neq 1_X$, for every finite subset K of Γ. We define the maps $\bigwedge_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{G}_i), \bigvee_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{G}_i^*) : L^X \to M$ as:

$$\bigwedge_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{G}_i)(\lambda) = \begin{cases} \bigwedge\{\oplus_{i\in K}\mathcal{G}_i(\lambda_i), & \text{if} \quad \lambda = \oplus_{i\in K}f_i^{\rightarrow}(\lambda_i)\} \\ 1_M, & \text{otherwise,} \end{cases}$$

$$\bigvee_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{G}_i^*)(\lambda) = \begin{cases} \bigvee\{\odot_{i\in K}\mathcal{G}_i^*(\lambda_i), & \text{if} \quad \lambda = \oplus_{i\in K}f_i^{\rightarrow}(\lambda_i)\} \\ 0_M, & \text{otherwise,} \end{cases}$$

where $\bigwedge$ and $\bigvee$ are taken for every finite subset K of Γ such that $\lambda = \oplus_{i\in K}f_i^{\rightarrow}(\lambda_i)$.

Let $\mathcal{G} = \bigwedge_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{G}_i)$ and $\mathcal{G}^* = \bigvee_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{G}_i^*)$ be given. Then:

(i) $(\mathcal{G}, \mathcal{G}^*)$ is the finest (L, M)-double fuzzy grill on X for which, $f_i : (X_i, \mathcal{G}_i, \mathcal{G}_i^*) \to (X, \mathcal{G}, \mathcal{G}^*)$ is a double g-preserving map for all $i \in \Gamma$.

(ii) A map $f : (X, \mathcal{G}, \mathcal{G}^*) \to (Y, \mathcal{H}, \mathcal{H}^*)$ is a double g-preserving map if and only if for each $i \in \Gamma$, $f \circ f_i : (X_i, \mathcal{G}_i, \mathcal{G}_i^*) \to (Y, \mathcal{H}, \mathcal{H}^*)$ is a double g-preserving map.

Proof. (i) (DFG1) Similar to Theorem 3.6.

(DFG1) It is easy.

(DFG3) For all finite subsets K and J of Γ such that $\lambda = \oplus_{k\in K} f_k^{\to}(\lambda_k)$ and $\mu = \oplus_{j\in J} f_j^{\to}(\mu_j)$ we have

$$\lambda \oplus \mu = (\oplus_{k\in K} f_k^{\to}(\lambda_k)) \oplus (\oplus_{j\in J} f_j^{\to}(\mu_j)).$$

Put $m \in (K \cup J)$ such that

$$\rho_m = \begin{cases} \lambda_m, & \text{if} \quad m \in K - (K \cap J) \\ \mu_m, & \text{if} \quad m \in J - (K \cap J) \\ \lambda_m \oplus \mu_m, & \text{if} \quad m \in K \cap J. \end{cases}$$

since,

$$\begin{aligned} \lambda \oplus \mu \quad &= (\oplus_{k\in K} f_k^{\to}(\lambda_k)) \oplus (\oplus_{j\in J} f_j^{\to}(\mu_j)) \\ &= \oplus_{m\in K\cup J} f_m^{\to}(\rho_m), \end{aligned}$$

there exists a finite subset $K \cup J$ of Γ such that:

$$\mathcal{G}(\lambda \oplus \mu) \leq (\oplus_{k\in K} \mathcal{G}_k(\lambda_k)) \oplus (\oplus_{j\in J} \mathcal{G}_j(\mu_j)),$$

$$\mathcal{G}^*(\lambda \oplus \mu) \geq (\odot_{k\in K} \mathcal{G}_k(\lambda_k)) \odot (\odot_{j\in J} \mathcal{G}_j(\mu_j)).$$

By (L4$'$), (L4) and the definition of $\mathcal{G}$, $\mathcal{G}^*$, we have

$$\mathcal{G}(\lambda \oplus \mu) \leq \mathcal{G}(\lambda) \oplus \mathcal{G}(\mu), \mathcal{G}^*(\lambda \oplus \mu) \geq \mathcal{G}^*(\lambda) \odot \mathcal{G}^*(\mu)$$

By the definition of $\mathcal{G}$ and $\mathcal{G}^*$ we have, $\mathcal{G}(f_i^{\to}(\lambda_i)) \leq \mathcal{G}_i(\lambda_i)$ and $\mathcal{G}^*(f_i^{\to}(\lambda_i)) \geq \mathcal{G}_i^*(\lambda_i)$, for each $i \in \Gamma$. Then, $f_i : (X_i, \mathcal{G}_i, \mathcal{G}_i^*) \to (X, \mathcal{G}, \mathcal{G}^*)$ is a double g-preserving map for all $i \in \Gamma$.

Let $(\mathcal{K}, \mathcal{K}^*)$ be another (L, M)-double fuzzy grill on X such that for each $i \in \Gamma$, the map $f_i : (X_i, \mathcal{G}_i, \mathcal{G}_i^*) \to (X, \mathcal{K}, \mathcal{K}^*)$ is a double g-preserving map. Then, for each $i \in \Gamma$,

$$\mathcal{K}(f_i^{\to}(\lambda_i)) \leq \mathcal{G}_i(\lambda_i), \mathcal{K}^*(f_i^{\to}(\lambda_i)) \geq \mathcal{G}_i^*(\lambda_i), \qquad (4.1)$$

For each finite subset K of Γ with $\lambda = \oplus_{k\in K}(f_k^{\rightarrow}(\lambda_k))$, we have

$$\begin{aligned}\mathcal{K}(\lambda) &= \mathcal{K}(\oplus_{k\in K}(f_k^{\rightarrow}(\lambda_k)))\\ &\leq \oplus_{k\in K}\mathcal{K}(f_k^{\leftarrow}(\lambda_k)) && \text{(by (DFG3))}\\ &\leq \oplus_{k\in K}\mathcal{G}_k(\lambda_k), && \text{(by (4.1))}\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}^*(\lambda) &= \mathcal{K}^*(\oplus_{k\in K}(f_k^{\rightarrow}(\lambda_k)))\\ &\geq \odot_{k\in K}\mathcal{K}^*(f_k^{\leftarrow}(\lambda_k)) && \text{(by (DFG3))}\\ &\geq \odot_{k\in K}\mathcal{G}_k^*(\lambda_k). && \text{(by (4.1))}\end{aligned}$$

By (L4′), (L4) and the definition of $\mathcal{G}$, $\mathcal{G}^*$ we have,

$$\mathcal{K}(\lambda) \leq \mathcal{G}(\lambda) \text{ and } \mathcal{K}^*(\lambda) \geq \mathcal{G}^*(\lambda).$$

(ii) Necessity of the composition condition is clear since the composition of double g-preserving maps is a double g-preserving map.

Conversely, for each finite index subset K of Γ with $\lambda = \oplus_{k\in K}(f_k^{\rightarrow}(\lambda_k))$, since for each $i \in \Gamma$, $f \circ f_i : (X_i, \mathcal{G}_i, \mathcal{G}_i^*) \to (Y, \mathcal{H}, \mathcal{H}^*)$ is a double g-preserving map, we have, for each $i \in \Gamma$,

$$\mathcal{H}((f \circ f_i)^{\rightarrow}(\lambda_i)) \leq \mathcal{G}_i(\lambda_i), \mathcal{H}^*((f \circ f_i)^{\rightarrow}(\lambda_i)) \geq \mathcal{G}_i^*(\lambda_i),$$

It follows, for all $k \in K$,

$$\begin{aligned}&\mathcal{H}((f \circ f_k)^{\rightarrow}(\lambda_k)) \leq \mathcal{G}_k(\lambda_k),\\ &\mathcal{H}^*((f \circ f_k)^{\rightarrow}(\lambda_k)) \geq \mathcal{G}_k^*(\lambda_k),\end{aligned} \tag{4.2}$$

Since $f^{\rightarrow}(\lambda) = \oplus_{k\in K}(f \circ f_k)^{\rightarrow}(\lambda_k)$ we have,

$$\begin{aligned}\mathcal{H}(f^{\rightarrow}(\lambda)) &= \mathcal{H}(\oplus_{k\in K}(f \circ f_k)^{\rightarrow}(\lambda_k))\\ &\leq \oplus_{k\in K}\mathcal{H}((f \circ f_k)^{\rightarrow}(\lambda_k)) && \text{(by (DFG3))}\\ &\leq \oplus_{k\in K}\mathcal{G}_k(\lambda_k), && \text{(by (4.2))}\end{aligned}$$

$$\begin{aligned}\mathcal{H}^*(f^{\rightarrow}(\lambda)) &= \mathcal{H}^*(\oplus_{k\in K}(f \circ f_k)^{\rightarrow}(\lambda_k))\\ &\geq \odot_{k\in K}\mathcal{H}^*((f \circ f_k)^{\rightarrow}(\lambda_k)) && \text{(by (DFG3))}\\ &\geq \odot_{k\in K}\mathcal{G}_k^*(\lambda_k). && \text{(by (4.2))}\end{aligned}$$

By (L4′), (L4) and the definition of $\mathcal{G}$, $\mathcal{G}^*$ we have, $\mathcal{H}(f^{\rightarrow}(\lambda)) \leq \mathcal{G}(\lambda)$ and $\mathcal{H}^*(f^{\rightarrow}(\lambda)) \geq \mathcal{G}^*(\lambda)$. Therefore, $f : (X, \mathcal{G}, \mathcal{G}^*) \to (Y, \mathcal{H}, \mathcal{H}^*)$ is a double g-preserving map.

Corollary 4.2. Let $f_i : X_i \to X$ be a map, for all $i \in \Gamma$ and $\{(\mathcal{B}_i, \mathcal{B}_i^*)\}_{i\in\Gamma}$ be a family of (L, M)-double fuzzy grill bases on X satisfying the following condition:

(C) If $\mathcal{B}_i(\lambda_i) \neq 1_M$ and $\mathcal{B}_i^*(\lambda_i) \neq 0_M$, $\forall i \in \Gamma$, then we have $\oplus_{i\in K} f_i^{\rightarrow}(\lambda_i) \neq 1_X$, for every finite subset K of Γ. We define the maps $\bigwedge_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{B}_i), \bigvee_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{B}_i^*) : L^X \to M$ as:

$$\bigwedge_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{B}_i)(\lambda) = \begin{cases} \bigwedge\{\oplus_{i\in K}\mathcal{B}_i(\lambda_i), & \text{if} \quad \lambda = \oplus_{i\in K} f_i^{\rightarrow}(\lambda_i)\} \\ 1_M, & \text{otherwise,} \end{cases}$$

$$\bigvee_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{B}_i^*)(\lambda) = \begin{cases} \bigvee\{\odot_{i\in K}\mathcal{B}_i^*(\lambda_i), & \text{if} \quad \lambda = \oplus_{i\in K} f_i^{\rightarrow}(\lambda_i)\} \\ 0_M, & \text{otherwise,} \end{cases}$$

where $\bigwedge$ and $\bigvee$ are taken for every finite subset K of Γ such that $\lambda = \oplus_{i\in K} f_i^{\rightarrow}(\lambda_i)$.

Let $\mathcal{B} = \bigwedge_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{B}_i)$ and $\mathcal{B}^* = \bigvee_{i\in\Gamma} f_i^{\Rightarrow}(\mathcal{B}_i^*)$ be given. Then:

(i) $(\mathcal{B}, \mathcal{B}^*)$ is an (L, M)-double fuzzy grill base on X for which each map $f_i : X_i \to X$, $\mathcal{B}(f_i^{\rightarrow}(\lambda_i)) \leq \mathcal{B}_i(\lambda_i)$ and $\mathcal{B}^*(f_i^{\rightarrow}(\lambda_i)) \geq \mathcal{B}_i^*(\lambda_i)$ for each $\lambda_i \in L^{X_i}$.

(ii) $\langle\mathcal{B}\rangle = \bigwedge_{i\in\Gamma} f_i^{\Rightarrow}(\langle\mathcal{B}_i\rangle)$ and $\langle\mathcal{B}^*\rangle = \bigvee_{i\in\Gamma} f_i^{\Rightarrow}(\langle\mathcal{B}_i^*\rangle)$

(iii) A map $f : (X, \langle\mathcal{B}\rangle, \langle\mathcal{B}^*\rangle) \to (Y, \mathcal{H}, \mathcal{H}^*)$ is a double g-preserving map if and only if for each $i \in \Gamma$, $f \circ f_i : (X_i, \langle\mathcal{B}_i\rangle, \langle\mathcal{B}_i^*\rangle) \to (Y, \mathcal{H}, \mathcal{H}^*)$ is a double g-preserving map.

Proof. (i) and (iii) are similarly proved from Theorem 4.1.

(ii) Let $\mathcal{G} = \bigwedge_{i\in\Gamma} f_i^{\Rightarrow}(\langle\mathcal{B}_i\rangle)$ and $\mathcal{G}^* = \bigvee_{i\in\Gamma} f_i^{\Rightarrow}(\langle\mathcal{B}_i^*\rangle)$. Then, for each family $\{\lambda_k : \lambda \leq \oplus_{k\in K} f_k^{\rightarrow}(\lambda_k)\}$ we have:

$$\begin{aligned} \langle\mathcal{B}\rangle(\lambda) &\leq \langle\mathcal{B}\rangle(\oplus_{k\in K} f_k^{\rightarrow}(\lambda_k)) \\ &\leq \oplus_{k\in K}\langle\mathcal{B}\rangle(f_k^{\rightarrow}(\lambda_k)) \leq \oplus_{k\in K}\langle\mathcal{B}_k\rangle(\lambda_k), \end{aligned}$$

$$\begin{aligned} \langle\mathcal{B}^*\rangle(\lambda) &\geq \langle\mathcal{B}^*\rangle(\oplus_{k\in K} f_k^{\rightarrow}(\lambda_k)) \\ &\geq \odot_{k\in K}\langle\mathcal{B}^*\rangle(f_k^{\rightarrow}(\lambda_k)) \geq \odot_{k\in K}\langle\mathcal{B}_k^*\rangle(\lambda_k). \end{aligned}$$

By (L4′), (L4) and the definition of $\mathcal{G}$, $\mathcal{G}^*$, we obtain

$$\langle\mathcal{B}\rangle \leq \mathcal{G} \text{ and } \langle\mathcal{B}^*\rangle \geq \mathcal{G}^*. \tag{4.3}$$

Conversely, since

$$\begin{aligned} &\{\lambda_i \in L^{X_i} : \langle\mathcal{B}_i\rangle(\lambda_i) \neq 0_M\} \subseteq \{\mu_i \in L^{X_i} : \mathcal{B}_i(\mu_i) \neq 0_M\}, \\ &\{\lambda_i \in L^{X_i} : \langle\mathcal{B}_i^*\rangle(\lambda_i) \neq 1_M\} \supseteq \{\mu_i \in L^{X_i} : \mathcal{B}_i^*(\mu_i) \neq 1_M\}, \end{aligned}$$

then $\mathcal{G}(\lambda) \leq \mathcal{B}(\lambda)$ and $\mathcal{G}^*(\lambda) \geq \mathcal{B}^*(\lambda)$. By Theorem 3.3, we have

$$\langle \mathcal{B} \rangle \geq \mathcal{G} \text{ and } \langle \mathcal{B}^* \rangle \leq \mathcal{G}^*. \tag{4.4}$$

From (4.3) and (4.4) we have:

$$\langle \mathcal{B} \rangle = \mathcal{G} \text{ and } \langle \mathcal{B}^* \rangle = \mathcal{G}^*.$$

Example 4.3. Let $L = M = [0, 1]$, $\oplus = \vee$ and $\odot = \wedge$. Let $X = \{a, b, c\}$ and $Y = \{x, y\}$ be to sets. Define $\mu \in L^X$ as follows:

$$\mu(a) = 0.5, \qquad \mu(b) = 1, \qquad \mu(c) = 1.$$

Define (L, M)-double fuzzy grill $(\mathcal{G}, \mathcal{G}^*)$ (where $\mathcal{G}, \mathcal{G}^* : L^X \to M$) as follows:

$$\mathcal{G}(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0} \\ 0.4, & \text{if } \underline{0} \neq \lambda \leq \mu \\ 1, & \text{otherwise.} \end{cases}$$

$$\mathcal{G}^*(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0} \\ 0.5, & \text{if } \underline{0} \neq \lambda \leq \mu \\ 0, & \text{otherwise.} \end{cases}$$

Let $f : X \to Y$ be a map defined by

$$f(a) = x, \qquad f(b) = x, \qquad f(c) = y.$$

Since $\mathcal{G}(\mu) = 0.4 \neq 1$, $\mathcal{G}^*(\mu) = 0.5 \neq 0$ and $f^{\rightarrow}(\mu) = 1_Y$, we cannot define an (L, M)-double fuzzy grill $(f^{\Rightarrow}(\mathcal{G}), f^{\Rightarrow}(\mathcal{G}^*))$.

Example 4.4. Let $L = M = [0, 1]$, $\oplus = \vee$ and $\odot = \wedge$. Let $X = \{a, b, c, d\}$ be a set. Define (L, M)-double fuzzy grill bases $(\mathcal{B}_i, \mathcal{B}_i^*)$ (where, $\mathcal{B}_i, \mathcal{B}_i^* : L^X \to M$), $i = 1, 2, 3$ as follows:

$$\mathcal{B}_1(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0} \\ 0.4, & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{a,b.c\}}\} \\ 1, & \text{otherwise,} \end{cases}$$

$$\mathcal{B}_1^*(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0} \\ 0.5, & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{a,b.c\}}\} \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{B}_2(\lambda) = \begin{cases} 0, & \text{if} \quad \lambda = \underline{0} \\ 0.3, & \text{if} \quad \lambda \in \{\chi_{\{c\}}, \chi_{\{b\}}, \chi_{\{a,b.c\}}\} \\ 1, & \text{otherwise,} \end{cases}$$

$$\mathcal{B}^*_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0} \\ 0.6, & \text{if} \quad \lambda \in \{\chi_{\{c\}}, \chi_{\{b\}}, \chi_{\{a,b.c\}}\} \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{B}_3(\lambda) = \begin{cases} 0, & \text{if} \quad \lambda = \underline{0} \\ 0.3, & \text{if} \quad \lambda \in \{\chi_{\{c\}}, \chi_{\{b\}}, \chi_{\{a,b.d\}}\} \\ 1, & \text{otherwise.} \end{cases}$$

$$\mathcal{B}^*_3(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0} \\ 0.6, & \text{if} \quad \lambda \in \{\chi_{\{c\}}, \chi_{\{b\}}, \chi_{\{a,b.d\}}\} \\ 0, & \text{otherwise.} \end{cases}$$

We obtain

$$\langle\mathcal{B}_1\rangle(\lambda) = \begin{cases} 0, & \text{if} \quad \lambda = \underline{0} \\ 0.4, & \text{if} \quad \underline{0} \neq \lambda \leq \chi_{\{a,b.c\}} \\ 1, & \text{otherwise,} \end{cases} \quad \langle\mathcal{B}^*_1\rangle(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0} \\ 0.5, & \text{if} \quad \underline{0} \neq \lambda \leq \chi_{\{a,b.c\}} \\ 0, & \text{otherwise,} \end{cases}$$

$$\langle\mathcal{B}_2\rangle(\lambda) = \begin{cases} 0, & \text{if} \quad \lambda = \underline{0} \\ 0.3, & \text{if} \quad \underline{0} \neq \lambda \leq \chi_{\{a,b.c\}} \\ 1, & \text{otherwise,} \end{cases} \quad \langle\mathcal{B}^*_2\rangle(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0} \\ 0.6, & \text{if} \quad \underline{0} \neq \lambda \leq \chi_{\{a,b.c\}} \\ 0, & \text{otherwise,} \end{cases}$$

$$\langle\mathcal{B}_3\rangle(\lambda) = \begin{cases} 0, & \text{if} \quad \lambda = \underline{0} \\ 0.3, & \text{if} \quad \underline{0} \neq \lambda \leq \chi_{\{a,b.d\}} \\ 1, & \text{otherwise,} \end{cases} \quad \langle\mathcal{B}^*_3\rangle(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0} \\ 0.6, & \text{if} \quad \underline{0} \neq \lambda \leq \chi_{\{a,b.d\}} \\ 0, & \text{otherwise.} \end{cases}$$

Let $id_X : X \to X$ be an identity map. We cannot define $id_X^{\Rightarrow}(\mathcal{B}_1) \wedge id_X^{\Rightarrow}(\mathcal{B}_3)$, $id_X^{\Rightarrow}(\mathcal{B}^*_1) \vee id_X^{\Rightarrow}(\mathcal{B}^*_3)$ because $\mathcal{B}_1(\chi_{\{a,b,c\}}) = 0.4 \neq 1$, $\mathcal{B}^*_1(\chi_{\{a,b,c\}}) = 0.5 \neq 0$ and $\mathcal{B}_3(\chi_{\{a,b,d\}}) = 0.3 \neq 1$, $\mathcal{B}^*_3(\chi_{\{a,b,d\}}) = 0.6 \neq 0$ but $id_X^{\Rightarrow}(\chi_{\{a,b,c\}}) \vee id_X^{\Rightarrow}(\chi_{\{a,b,d\}}) = \chi_{\{a,b,c\}} \vee \chi_{\{a,b,d\}} = 1_X$. (i.e condition (C) of Corollary 4.2 was not satisfied).

Also, we have

$$(id_X^{\Rightarrow}(\mathcal{B}_1) \wedge id_X^{\Rightarrow}(\mathcal{B}_2))(\lambda) = \begin{cases} 0, & \text{if} \quad \lambda = \underline{0} \\ 0.3, & \text{if} \quad \lambda \in \{\chi_{\{b\}}, \chi_{\{c\}}, \chi_{\{a,b.c\}}\} \\ 0.4, & \text{if} \quad \lambda \in \{\chi_{\{a\}}, \chi_{\{a,b\}}, \chi_{\{a,c\}}, \chi_{\{b,c\}}\} \\ 1, & \text{otherwise,} \end{cases}$$

$$(id_X^{\Rightarrow}(\mathcal{B}_1^*) \vee id_X^{\Rightarrow}(\mathcal{B}_2^*))(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0} \\ 0.6, & \text{if} \quad \lambda \in \{\chi_{\{b\}}, \chi_{\{c\}}, \chi_{\{a,b.c\}}\} \\ 0.5, & \text{if} \quad \lambda \in \{\chi_{\{a\}}, \chi_{\{a,b\}}, \chi_{\{a,c\}}, \chi_{\{b,c\}}\} \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\langle id_X^{\Rightarrow}(\mathcal{B}_1) \wedge id_X^{\Rightarrow}(\mathcal{B}_2)\rangle(\lambda) = \begin{cases} 0, & \text{if} \quad \lambda = \underline{0} \\ 0.3, & \text{if} \quad \underline{0} \neq \lambda \leq \chi_{\{a,b,c\}} \\ 1, & \text{otherwise,} \end{cases}$$

$$\langle id_X^{\Rightarrow}(\mathcal{B}_1^*) \vee id_X^{\Rightarrow}(\mathcal{B}_2^*)\rangle(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0} \\ 0.6, & \text{if} \quad \underline{0} \neq \lambda \leq \chi_{\{a,b,c\}} \\ 0, & \text{otherwise.} \end{cases}$$

$$(id_X^{\Rightarrow}(\langle\mathcal{B}_1\rangle) \wedge id_X^{\Rightarrow}(\langle\mathcal{B}_2\rangle))(\lambda) = \begin{cases} 0, & \text{if} \quad \lambda = \underline{0} \\ 0.3, & \text{if} \quad \underline{0} \neq \lambda \leq \chi_{\{a,b,c\}} \\ 1, & \text{otherwise,} \end{cases}$$

$$(id_X^{\Rightarrow}(\langle\mathcal{B}_1^*\rangle) \vee id_X^{\Rightarrow}(\langle\mathcal{B}_2^*\rangle))(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0} \\ 0.6, & \text{if} \quad \underline{0} \neq \lambda \leq \chi_{\{a,b,c\}} \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\langle id_X^{\Rightarrow}(\mathcal{B}_1) \wedge id_X^{\Rightarrow}(\mathcal{B}_2)\rangle = id_X^{\Rightarrow}(\langle\mathcal{B}_1\rangle) \wedge id_X^{\Rightarrow}(\langle\mathcal{B}_2\rangle),$$

$$\langle id_X^{\Rightarrow}(\mathcal{B}_1^*) \vee id_X^{\Rightarrow}(\mathcal{B}_2^*)\rangle = id_X^{\Rightarrow}(\langle\mathcal{B}_1^*\rangle) \vee id_X^{\Rightarrow}(\langle\mathcal{B}_2^*\rangle).$$

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