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OPERATIONAL PARAMETERIZATIONS OF MRA FRAME MULTIWAVELETS

Zhongyan Li
Department of Mathematics and Physics
North China Electric Power University
Beijing, 102206, P.R. CHINA

Abstract: The set of all orthogonal multiwavelets with multiresolution analysis of multiplicity r in $L^2(\mathbb{R})$ is parameterized by a set of unitary operators which satisfies certain commutative properties. The parameterization of frame multiwavelets with multiresolution analysis of multiplicity r is discussed by a set of co-isometry operators, and the Riesz multiwavelets with multiresolution analysis is obtained by a set of invertible operators.

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Key Words: multiresolution analysis of multiplicity r, r-MSW-triple, r-FMSW-triple, orthogonal multiwavelet of multiplicity r, r-scaling function

1. Introduction

Let H be a separable complex Hilbert space and $B(H)$ deonote the algebra of all bounded linear operators on H. A frame for H is a sequence of vectors $\{x_j : j \in J\}$ in H for which there are constants $A, B > 0$ satisfying:

$$A\|x\|^2 \leq \Sigma_j |\langle x, x_j\rangle|^2 \leq B\|x\|^2$$

for all $x \in H$. We call A, B the lower and upper frame bounds for the frame, respectively. The frame is called a tight frame if $A = B$, and is called a normalized frame (or Parseval frame) if $A = B = 1$. A sequence x_j is called a Riesz basis if it a frame and is slao a basis for H in the sence that for each

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$x \in H$ there is a unique sequence $\{\alpha_j\}$ in $\mathbb{C}$ such that $x = \sum \alpha_j x_j$ with the convergence being in norm.

Let $H = L^2(\mathbb{R})$. Let T and D be the translation and dilation unitary operators defined by $(Tf)(t) = f(t-1)$ and $(Df)(t) = \sqrt{2}f(2t)$ for all $f \in L^2(\mathbb{R})$. It is obvious that $TD = DT^2$. A function $\psi \in L^2(\mathbb{R})$ is called an (or a single) orthonormal wavelet (Parseval frame wavelet, Riesz wavelet) if $\{D^n T^\ell \psi : n, \ell \in \mathbb{Z}\}$ forms an orthonormal basis (Parseval frame, Riesz basis) for $L^2(\mathbb{R})$. The theory of single wavelets was developed extensively in literature.

An orthonormal (a Parseval frame, Riesz) multiwavelet with multiplicity r is a r-tuple of vectors $\Psi = (\psi_1(t), \cdots, \psi_r(t))$ with each $\psi_i(t)$ in $L^2(\mathbb{R})$, such that $\{D^n T^\ell \psi_i : n, \ell \in \mathbb{Z}, i = 1, 2, \cdots, r\}$ constitutes an orthonormal basis (a Parseval frame, Riesz basis) for $L^2(\mathbb{R})$.

The study of multiwavelets (for general, not necessarily orthonormal) becomes hot topic, which was initiated by Goodman, Lee and Tang [11]. Much effort has been devoted the study of various aspects of multiwavelet systems. For a few examples of recent studies related to multiwavelets, see [3, 4, 5, 6]. The study of orthonormal multiwavelets was initiated by Geronimo, Hardin, and Massopust [18], and there are detailed discussion on constructing orthonormal multi-scaling functions and multi-wavelets that are symmetric or differentiable in [19, 2]. The theory of multiresolution analysis (MRA) of multiplicity r was given in [1], and multiwavelets with MRA have received much attention from mathematicians in recent years [12, 13, 14, 15, 16, 17].

In this note, we will observe multiwavelets with MRAs with operator theoretic approach which introduced by the authors in [20] and [9].

Let Ψ be an r-multiwavelet (or Parseval frame multiwavelet, Riesz multiwavelet). We will use the following notations:

$$\begin{aligned} C_\Psi(D,T) &= \{A \in B(H) : AD^nT^\ell\Psi = D^nT^\ell A\Psi \text{ for all } n, \ell \in \mathbb{Z}\} \\ &= \{A \in B(H) : AD^nT^\ell\psi_i = D^nT^\ell A\psi_i, i = 1, 2 \cdots, r, \text{ for all } n, \ell \in \mathbb{Z}\}. \end{aligned} \tag{1.1}$$

The set $C_\Psi(D,T)$ was called *local commutant* or *point commutant* of $\{D, T\}$ at Ψ ([20, 7]).

Let ψ_0 be a given single orthonormal wavelet and let $\mathcal{W}$ be the set of all single orthonormal wavelets in $L^2(\mathbb{R})$. A useful result is the one to one correspondence between $\mathcal{W}$ and the unitary operators in $C_{\psi_0}(D,T)$. Otherwords, the authors of [20, 21, 22] parametrized the set of single orthonormal wavelets by unitary operators in $C_{\psi_0}(D,T)$. The parametrization was extended to a general frame wavelet in [9] by the authors which was the key lemma in [9].

We will study the local commutant theory of multiwavelet (Parseval frame multiwavelet, Riesz multiwavelet) with multiplicity r with MRA.

Throughout, the commutant of $S \subset B(H)$, denoted by S' is the set of all operators A in $B(H)$ satisfies the condition $AB = BA$ for each $B \in S$. The standard notation $\{T\}'$ is for the commutant of T.

The following definition is from [1, 2].

Definition 1.1. An orthonormal (or a Parseval frame, a Riesz) multiresolution analysis of multiplicity r (r-MRA) (or a r-FMRA, a r-RMRA) in H is a set $\{V_n : n \in \mathbb{Z}\}$ of closed subspaces in H satisfying the following properties:

(i) $V_n \subset V_{n+1}$, $\forall n \in \mathbb{Z}$.

(ii) $\bigcup_{n\in\mathbb{Z}} V_n = H$.

(iii) $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$.

(iv) $TV_0 = V_0$.

(v) $D^n V_0 = V_n$, $\forall n \in \mathbb{Z}$.

(vi) There exists a r-tuple of $\Phi = (\varphi_1, \cdots, \varphi_r) \in (L^2(\mathbb{R}))^r$ such that $\{T^n\varphi_j, j = 1, \cdots, r, n \in \mathbb{Z}\}$ is an orthonormal basis (or a Parseval frame, a Riesz basis) for V_0.

Where $\Phi = (\varphi_1, \cdots, \varphi_r) \in (L^2(\mathbb{R}))^r$ is called scaling function (or frame scaling function, Riesz scaling function) of multiplicity r (for short r-scaling function).

An orthonormal multiwavelet (or a Parseval frame multiwavelet, a Riesz multiwavelet) with multiplicity r $\Psi = (\psi_1(t), \cdots, \psi_r(t))$ with $\psi_i \in W_0 = V_1 \ominus V_0, i = 1, \cdots, r$ can be from a r-MRA (or a r-FMRA, a r-RMRA), see [1, 2].

Let $\mathcal{M} = \{V_n : n \in \mathbb{Z}\}$ be a r-MRA. Let $\Phi = (\varphi_1, \cdots, \varphi_r)$ be a r-scaling function related to $\mathcal{M}$, let $\Psi = (\psi_1, \cdots, \psi_r)$ be an orthonormal multiwavelet (denote r-multiwavelet) related to this r-MRA. Let $W_n = \overline{span}\{D^n T^\ell \psi_i : \ell \in \mathbb{Z}, i = 1, \cdots, r\}$. Then from [1, 2], we have

$W_n \oplus V_n = V_{n+1}$, $n \in \mathbb{Z}$,

and $V_0 = \bigoplus_{-\infty}^{-1} W_n = (\bigoplus_{n=0}^{\infty} W_n)^\perp$.

Therefore, the set $\{T^\ell\varphi_i, D^n T^\ell \psi_i : n \geq 0, \ell \in \mathbb{Z}, i = 1, \cdots, r\}$ is an orthonormal basis for H. We will use P_Φ for the orthogonal projection onto the space V_0. So we have $P_\Phi \Psi = (P_\Phi\psi_1, \cdots, P_\Phi\psi_r) = (0, \cdots, 0) = 0$ and $P_\Phi^\perp \Psi = \Psi$.

2. The Operational Parameterization of r-MSW-triples

Let $M = \{V_n : n \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity r. Let $\Phi = (\varphi_1, \cdots, \varphi_r)$ be the r-scaling function related to M (i.e. $\{T^\ell\varphi_i : \ell \in \mathbb{Z}, i = 1, 2, \cdots, r\}$ ia an orthonormal basis for V_0). Let $\Psi = (\psi_1, \cdots, \psi_r)$ be an orthonormal multiwavelet of multiplicity r related to M (i.e., $\psi_i \in W_0, i = 1, \cdots, r$). We call $< M, \Phi, \Psi >$ a r-MSW-triple. Let $\mathcal{MSW}$ be the set of all such r-MSW-triples. Let $M_0 = \{V_n\}$, we say that a unitary operator U maps a r-MSW-triple $< M_0, \Phi_0, \Psi_0 >$ into a r-MSW-triple $< M, \Phi, \Psi >$ if

$$\begin{aligned}
&\Phi = U\Phi_0, \text{ i.e., } \varphi_i = U\varphi_{0i},\ i = 1, 2, \cdots, r,\\
&\Psi = U\Psi_0, \text{ i.e., } \psi_i = U\psi_{0i},\ i = 1, 2, \cdots, r,\\
&V_0' = UV_0,\\
&V_n' = D^n V_0',\ n \in \mathbb{Z},\\
&M = \{V_n' : n \in \mathbb{Z}\}.
\end{aligned}$$

We write $< M, \Phi, \Psi >= U < M_0, \Phi_0, \Psi_0 >$.

We know that for arbitrary unitary operator U, $U < M_0, \Phi_0, \Psi_0 >$ is not necessarily be a r-MSW-triple.

For a r-MSW-triple $< M_0, \Phi_0, \Psi_0 >$, define

$$\begin{aligned}
C(M_0, \Phi_0, \Psi_0) &:= \{T\}' \bigcap \{B(H)P_{\Phi_0} + \mathcal{U}_{\Psi_0}(D, T)P^{\perp}_{\Phi_0}\}\\
\mathcal{U}(M_0, \Phi_0, \Psi_0) &:= \text{unitary operators in}\, C(M_0, \Phi_0, \Psi_0).
\end{aligned}$$

Where $\mathcal{U}_{\Psi_0}(D, T)$ denotes the set of unitary operators in $C_{\Psi_0}(D, T)$.

It is clear that the set of unitary operators in $\{D, T\}'$ is a subset of $\mathcal{U}(M, \Phi, \Psi)$ for every r-MSW-triple $< M, \Phi, \Psi >$

Let $W^r(D, T)$ be the set of all r-multiwavelets in H.

Lemma 2.1. ([7]) Let $\Psi_0 = (\psi_{01}, \cdots, \psi_{0r})$ be an arbitrary fixed element in $W^r(D, T)$. Then $W^r(D, T) = \mathcal{U}_{\Psi_0}(D, T)\Psi_0$. Furthermore, the mapping $\theta : \mathcal{U}_{\Psi_0}(D, T) \to W^r(D, T)$ given by $\theta(U) = U\Psi_0$ is a bijection.

Lemma 2.2. For any $U \in \mathcal{U}(M_0, \Phi_0, \Psi_0)$, then $UD^nT^\ell\Psi_0 = D^nT^\ell U\Psi_0$, $\forall n \geq 0, \ell \in \mathbb{Z}$.

Proof. Since $U \in \{T\}' \bigcap \{B(H)P_{\Phi_0} + \mathcal{U}_{\Psi_0}(D, T)P^{\perp}_{\Phi_0}\}$, $U = AP_{\Phi_0} + U_1P^{\perp}_{\Phi_0}$, for some $A \in B(H)$ and $U_1 \in \mathcal{U}_{\Psi_0}(D, T)$. Since $W_n = \overline{span}\{D^nT^\ell\Psi_0, \ell \in \mathbb{Z}\}$, when $n \geq 0$, $W_n \perp V_0$, we have $P_{\Phi_0}D^nT^\ell\Psi_0 = 0$, $P^{\perp}_{\Phi_0}(D^nT^\ell\Psi_0) = D^nT^\ell\Psi_0$. For any $n \geq 0, \ell \in \mathbb{Z}$,

$$\begin{aligned}
UD^nT^\ell\Psi_0 &= (AP_{\Phi_0} + U_1P^{\perp}_{\Phi_0})D^nT^\ell\Psi_0\\
&= AP_{\Phi_0}D^nT^\ell\Psi_0 + U_1P^{\perp}_{\Phi_0}D^nT^\ell\Psi_0\\
&= 0 + U_1D^nT^\ell\Psi_0 = D^nT^\ell U_1\Psi_0\\
&= D^nT^\ell(AP_{\Phi_0}\Psi_0 + U_1P^{\perp}_{\Phi_0}\Psi_0)\\
&= D^nT^\ell U\Psi_0.
\end{aligned}$$

□

Theorem 2.3. Let $< M_0, \Phi_0, \Psi_0 >$ be a r-MSW-triple. The mapping

$$T : \mathcal{U}(M_0, \Phi_0, \Psi_0) \to \mathcal{MSW}$$

given by $T(U) = U < M_0, \Phi_0, \Psi_0 >$ is a bijection.

Proof. Let $< M_0, \Phi_0, \Psi_0 >$ be a r-MSW-triple and let $V \in \mathcal{U}(M_0, \Phi_0, \Psi_0)$. Then $V = AP_{\Phi_0} + V' P_{\Phi_0}^{\perp}$ for some $A \in B(H)$ and a unitary operator $V' \in \mathcal{U}_{\Psi_0}(D, T)$. Since $\psi_{0i} \in V_0^{\perp}, i = 1, 2, \cdots, r$, we have $V\Psi_0 = V'\Psi_0$. Let $\Psi = V\Psi_0$, i.e. $(\psi_1, \cdots, \psi_r) = (V\psi_{01}, \cdots, V\psi_{0r})$. By Lemma 2.1, $\Psi = V\Psi_0$ is an orthonormal multiwavelet with multiplicity r. The wavelet basis is $\{D^n T^\ell \psi_i, i = 1, \cdots, r, n, \ell \in \mathbb{Z}\}$. We write $V_k' = \overline{span}\{D^n T^\ell \Psi : n < k, \ell \in \mathbb{Z}\}$. Especially $V_0' = \overline{span}\{D^n T^\ell \Psi : n < 0, \ell \in \mathbb{Z}\}$. By Lemma 2.2, we have $VD^nT^\ell\Psi_0 = D^nT^\ell V\Psi_0$ for $n \geq 0$. So $V_0'^{\perp} = VV_0^{\perp}$. Since V is a unitary, we have $V_0' = VV_0$. Since D is a unitary operator, $H = D(V_0'^{\perp} \oplus V_0') = D(V_0'^{\perp}) \oplus DV_0'$. We have

$$\begin{aligned} (DV_0')^{\perp} &= D(V_0'^{\perp}) \\ &= \overline{span}\{D^nT^\ell\psi_i : n \geq 1,\ \ell \in \mathbb{Z},\ i = 1, \cdots, r\} \\ &\subseteq \overline{span}\{D^nT^\ell\psi_i : n \geq 0,\ \ell \in \mathbb{Z},\ i = 1, \cdots, r\} \\ &= V_0'^{\perp}. \end{aligned}$$

Thus $DV_0' \supset V_0'$. So for arbitrary $n \in \mathbb{Z}$, we have $D^nV_0' \subset D^{n+1}V_0'$, equivalently, $V_n' \subset V_{n+1}'$. Let $f \in \bigcap_{n\in\mathbb{Z}} D^nV_0'$. Then $f \perp (D^nV_0')^{\perp}$ for each $n \in \mathbb{Z}$. Since $(D^nV_0')^{\perp} = span\{D^mT^\ell\psi_i, i = 1, \cdots, r,\ m \geq n,\ \ell \in \mathbb{Z}\}$. We have $f \perp D^nT^\ell\psi_i, i = 1, \cdots, r, n, \ell \in \mathbb{Z}$. Because $\{D^nT^\ell\psi_i, i = 1, \cdots, r,\ n, \ell \in \mathbb{Z}\}$ is an orthonormal basis, we have $f = 0$. This proves that $\bigcap_{n\in\mathbb{Z}} D^nV_0' = \{0\}$.

Since $\{T^\ell\Phi, D^nT^\ell\Psi : n \geq 0, \ell \in \mathbb{Z}\} = \{T^\ell V\Phi_0, D^nT^\ell V\Psi_0 : n \geq 0, \ell \in \mathbb{Z}\} = \{VT^\ell\Phi_0, VD^nT^\ell\Psi_0 : n \geq 0, \ell \in \mathbb{Z}\}$ and $\{D^nT^\ell\Psi : n, \ell \in \mathbb{Z}\}$ are two orthonormal basis for H, we have

$$\begin{aligned} &\overline{span}\{T^\ell\phi_i,\ i = 1, \cdots, r, \ell \in \mathbb{Z}\} \\ &\qquad = \overline{span}\{D^nT^\ell\psi_i,\ i = 1, \cdots, r, n < 0,\ \ell \in \mathbb{Z}\} = V_0' \end{aligned}$$

So $\{T^\ell\varphi_i, i = 1, \cdots, r, \ell \in \mathbb{Z}\}$ is an orthonormal basis for V_0'. Hence Φ is a r-scaling function. We have proved that $< M, \Phi, \Psi >$ satisfies the conditions (i),(iii) and (Vi) of Definition 1.1. It is not difficult to show that this triple satisfies the conditions (ii),(iV) and (V).

For the converse, let $< M, \Phi, \Psi >$ be an arbitrary given r-MSW-triple in H. Define a mapping V from the orthonormal basis $\{T^\ell\varphi_{0i}, D^nT^\ell\psi_{0i} :$

$i = 1, \cdots, r, n \geq 0, \ell \in \mathbb{Z}\}$ onto the orthonormal basis $\{T^\ell \varphi_i, D^n T^\ell \psi_i, i = 1, \cdots, r,\ n \geq 0, \ell \in \mathbb{Z}\}$ by

$$VT^\ell \Phi_0 = T^\ell \Phi, \ell \in \mathbb{Z}, \text{i.e.,}$$
$$VT^\ell \varphi_{0i} = T^\ell \varphi_i,\ i = 1, \cdots, r,\ \ell \in \mathbb{Z},$$
$$VD^n T^\ell \Psi_0 = D^n T^\ell \Psi,\ n \geq 0, \ell \in \mathbb{Z}, \text{ i.e.,}$$
$$VD^n T^\ell \psi_{0i} = D^n T^\ell \psi_i,\ i = 1, \cdots, r,\ n \geq 0, \ell \in \mathbb{Z}.$$

The mapping V extends to a unitary operator, still denote it by V. Since Ψ_0 and Ψ are r-orthonormal multiwavelets, by Lemma 2.1, there is unique unitary operator $V' \in \mathcal{U}_{\Psi_0}(D, T)$, such that $V'\Psi_0 = \Psi$ and $V' D^n T^\ell \Psi_0 = D^n T^\ell V' \Psi_0 = D^n T^\ell \Psi$. For $n \geq 0$ and $\ell \in \mathbb{Z}$, we have

$$\begin{aligned} VD^n T^\ell \Psi_0 &= D^n T^\ell \Psi \\ &= V' D^n T^\ell \Psi_0. \end{aligned}$$

So $V = V'$ on $V_0^\perp$. Hence $VP_{\Phi_0}^\perp = V' P_{\Phi_0}^\perp$, we have $V = VP_{\Phi_0} + VP_{\Phi_0}^\perp = VP_{\Phi_0} + V' P_{\Phi_0}^\perp$, that is $V \in B(H) P_{\Phi_0} + \mathcal{U}_{\Psi_0}(D, T) P_{\Phi_0}^\perp$.

For $\ell \in \mathbb{Z}$,

$$\begin{aligned} (VT)(T^\ell \Phi_0) &= VT^{\ell+1}\Phi_0 = T^{\ell+1}\Phi \\ &= T^{\ell+1} V\Phi_0 = T^{\ell+1}(VP_{\Phi_0} + VP_{\Phi_0}^\perp)\Phi_0 \\ &= TT^\ell \Phi = T(VT^\ell \Phi_0) \\ &= T^{\ell+1}(V\Phi_0) = (TV)(T^\ell \Phi_0). \end{aligned}$$

For $n \geq 0$, $\ell \in \mathbb{Z}$, since $TD = DT^2$,

$$\begin{aligned} (VT)(D^n T^\ell \Psi_0) &= VD^n T^{2^n + \ell} \Psi_0 \\ &= D^n T^{2^n+\ell} \Psi = D^n T^{2^n+\ell} V\Psi_0 \\ &= TD^n T^\ell V \Psi_0 \\ &= (TV)(D^n T^\ell \Psi_0). \end{aligned}$$

So $(VT - TV)u = 0$, $\forall u \in \{T^\ell \varphi_{0i}, D^n T^\ell \psi_{0i}, i = 1, \cdots, r, n \geq 0, \ell \in \mathbb{Z}\}$, thus $V \in \{T\}'$.

We need to prove the uniqueness. Assume that there are two unitary operators U and V in $\mathcal{U}(M_0, \Phi_0, \Psi_0)$ such that

$$U < M_0, \Phi_0, \Psi_0 >= V < M_0, \Phi_0, \Psi_0 > .$$

By $UT = TU$ and Lemma 2.2 for operators in $\mathcal{U} < M_0, \Phi_0, \Psi_0 >$, $Uu = Vu$ for all u in the orthonormal basis $\{T^\ell \Phi_0,\ D^n T^\ell \Psi_0,\ n \geq 0,\ \ell \in \mathbb{Z}\}$, hence $U = V$. $\square$

3. The Operational Parameterization of Parseval r-MSW-Triple

In this section, we observe the parametrization of r-FMRA.

Let $< M_0, \Phi_0, \Psi_0 >$ be an orthonormal r-MSW-triple. Let (M, Φ) be a r-FMRA, and Ψ be the r-frame multiwavelet related to M, we call $\langle M, \Phi, \Psi\rangle$ a r-FMSW-triple and denote $\mathcal{FMSW}$ the set of all r-FMSW-triples. We write $M_0 = \{V_n\}$, $M = \{V_n'\}$, $n \in \mathbb{Z}$.

We say that an operator A maps a r-MSW-triple $< M_0, \Phi_0, \Psi_0 >$ into a r-FMSW-triple $< M, \Phi, \Psi >$ if

$$\begin{aligned}
&\Phi = A(\Phi_0) : \varphi_i = A\varphi_{0i},\ i = 1, \cdots, r,\\
&\Psi = A\Psi_0 : \psi_i = A\psi_{0i}, i = 1, \cdots, r,\\
&V_0' = AV_0,\ V_n' = D^n V_0',\ n \in \mathbb{Z},\\
&M = \{V_n',\ n \in \mathbb{Z}\}.
\end{aligned}$$

We write $< M, \Phi, \Psi >= A < M_0, \Phi_0, \Psi_0 >$.

For any operator A, $A < M_0, \Phi_0, \Psi_0 >$ is not necessary be a r-FMSW-triple. For a r-MSW-triple $< M_0, \Phi_0, \Psi_0 >$, define

$$\begin{aligned}
&C(M_0, \Phi_0, \Psi_0) = \{T\}' \bigcap \{B(H)P_{\Phi_0} + \mathcal{U}^C_{\Psi_0}(D, T)P^{\perp}_{\Phi_0}\}\\
&S(M_0, \Phi_0, \Psi_0) = \text{co-isometry operators in}\, C(M_0, \Phi_0, \Psi_0).
\end{aligned}$$

Where $\mathcal{U}^C_{\Psi_0}(D, T)$ denotes the co-isometry operators in $C_{\Psi_0}(D, T)$.

Lemma 3.1. ([8]) Let Ψ be an orthonormal multiwavelet with multiplicity r for H. Then $\eta = \{\eta_1, \cdots, \eta_r\}$ is a Parseval frame multiwavelet for H if and only if there is a (unique) co-isometry $A \in C_\Psi(D, T)$ such that $A\Psi = \eta$.

Lemma 3.2. Let $A \in S(M_0, \Phi_0, \Psi_0)$. Then A has the following properties:

(i) $AD^nT^\ell\Psi_0 = D^nT^\ell A\Psi_0$, $\forall n \geq 0, \ell \in \mathbb{Z}$.

(ii) $\{AT^\ell\Phi_0, AD^nT^\ell\Psi_0 : n \geq 0, \ell \in \mathbb{Z}\} = \{AT^\ell\varphi_{0j}, AD^nT^\ell\psi_{0j}, j = 1, \cdots, r, n \geq 0, \ell \in \mathbb{Z}\}$ is a Parseval frame for $L^2(\mathbb{R})$.

Proof. (i) Since $A \in \{T\}' \bigcap \{B(H)P_{\Phi_0} + \mathcal{U}^C_{\Psi_0}(D, T)P^{\perp}_{\Phi_0}\}$, $A = KP_{\Phi_0} + GP^{\perp}_{\Phi_0}$, for some $K \in B(H)$, $G \in \mathcal{U}^C_{\Psi_0}(D, T)$. When $n \geq 0, W_n \perp V_0$, where $W_n = \overline{span}\{D^nT^\ell\psi_{0i}, i = 1, \cdots, r, \ell \in \mathbb{Z}\}$. So $P_{\Phi_0}D^nT^\ell\Psi_0 = 0$, $P^{\perp}_{\Phi_0}(D^nT^\ell\Psi_0) = D^nT^\ell\Psi_0$, when $n \geq 0$. Thus for $n \geq 0$,

$$\begin{aligned}
AD^nT^\ell\Psi_0 &= (KP_{\Phi_0} + GP^{\perp}_{\Phi_0})D^nT^\ell\Psi_0 = GD^nT^\ell\Psi_0\\
&= D^nT^\ell G\Psi_0 = D^nT^\ell(KP_{\Phi_0}\Psi_0 + GP^{\perp}_{\Phi_0}\Psi_0) = D^nT^\ell A\Psi_0.
\end{aligned}$$

(ii) It is obvious by Proposition 1.9 (i) in [9]. □

Theorem 3.3. For any r-FMRA-triple $< M, \Phi, \Psi >$, there exists unique $A \in S(M_0, \Phi_0, \Psi_0)$ with $A < M_0, \Phi_0, \Psi_0 >=< M, \Phi, \Psi >$.

Proof. Let $< M, \Phi, \Psi >$ be any arbitrary FMSW-triple. Define a mapping A from $\{T^\ell \varphi_{0j}, D^n T^\ell \psi_{0j} : j = 1, \cdots, r, n \geq 0, n, \ell \in \mathbb{Z}\}$ onto the complete Parseval frame $\{T^\ell \varphi_j, D^n T^\ell \psi_j : j = 1, \cdots, r, n \geq 0, \ell \in \mathbb{Z}\}$ by

$$AT^\ell \varphi_{0j} = T^\ell \varphi_j, \ell \in \mathbb{Z}, j = 1, \cdots, r,$$
$$AD^n T^\ell \psi_{0j} = D^n T^\ell \psi_j, n \geq 0, \ell \in \mathbb{Z}.$$

Linearly extending A to H denoted it by A as follows,

$$Ax = \sum_{i=1}^{r} \sum_{n \geq 0, \ell \in \mathbb{Z}} < x, D^n T^\ell \psi_{0j} > D^n T^\ell \psi_j + \sum_{j=1}^{r} \sum_{\ell \in \mathbb{Z}} < x, T^\ell \varphi_{0j} > T^\ell \varphi_j,$$

$\forall x \in H$. It is obvious that A is bounded. By Proposition 1.9 in [9], A is co-isometry.

Since Ψ_0 is an orthonormal multiwavelet, Ψ is a Parseval frame multiwavelet. By Lemma 3.1, there exists unique co-isometry $G \in C_{\Psi_0}(D, T)$ such that $G\Psi_0 = \Psi$, and $GD^n T^\ell \Psi_0 = D^n T^\ell G\Psi_0 = D^n T^\ell \Psi, n, \ell \in \mathbb{Z}$. For any $n \geq 0$ and $\ell \in \mathbb{Z}$, we have

$$AD^n T^\ell \Psi_0 = D^n T^\ell \Psi = GD^n T^\ell \Psi_0.$$

So $A = G$ on $V_0^\perp$, hence $AP_{\Phi_0}^\perp = GP_{\Phi_0}^\perp$. So we have $A = AP_{\Phi_0} + GP_{\Phi_0}^\perp$, that is $A \in B(H)P_{\Phi_0} + U_{\Psi_0}^C(D, T)P_{\Phi_0}^\perp$.

We compute, for $\ell \in \mathbb{Z}$:

$$\begin{aligned} (AT)(T^\ell \Phi_0) &= AT^{\ell+1}\Phi_0 = T^{\ell+1}\Phi \\ &= TT^\ell \Phi = TA(T^\ell \Phi_0). \end{aligned}$$

Since $TD = DT^2$, for $n \geq 0, \ell \in \mathbb{Z}$, we have

$$\begin{aligned} (AT)(D^n T^\ell \Psi_0) &= AD^n T^{2^n+\ell}\Psi_0 = D^n T^{2^n+\ell}\Psi \\ &= D^n T^{2^n+\ell} A\Psi_0 = TD^n T^\ell A\Psi_0 \\ &= TA(D^n T^\ell \Psi_0). \end{aligned}$$

So $(AT - TA)u = 0, \forall u \in span\{T^\ell \Phi_0, D^n T^\ell \Psi_0 : n \geq 0, n, \ell \in \mathbb{Z}\}$. Thus $A \in \{T\}'$ and $A \in S(M_0, \Phi_0, \Psi_0)$.

It is clear that $< M, \Phi, \Psi >= A < M_0, \Phi_0, \Psi_0 >$.

Finally we prove the uniqueness. Assume that there are two co-isometries A and B in $S(M_0.\Phi_0, \Psi_0)$ such that $A < M_0, \Phi_0, \Psi_0 >= B < M_0, \Phi_0, \Psi_0 >$. For

$$x = \sum_{j=1}^{r} \sum_{n \geq 0, \ell \in \mathbb{Z}} < x, D^n T^\ell \psi_{0j} > D^n T^\ell \psi_{0j} + \sum_{j=1}^{r} \sum_{\ell \in \mathbb{Z}} < x, T^\ell \varphi_{0j} > T^\ell \varphi_{0j} \in H,$$

by Lemma 3.2, we have

$$\begin{aligned} Ax &= \sum_{j=1}^{r} \sum_{n \geq 0, \ell \in \mathbb{Z}} < x, D^n T^\ell \psi_{0j} > D^n T^\ell \psi_j \\ &+ \sum_{j=1}^{r} \sum_{\ell \in \mathbb{Z}} < x, T^\ell \varphi_{0j} > T^\ell \varphi_j = Bx, \end{aligned}$$

Thus $A = B$. □

Remark 1. From Theorem 3.3, we know that the mapping $T : \mathcal{FMSW} \to S(M_0, \Phi_0, \Psi_0)$ is injective, but we do not know whether it is surjective. The key is that we do not know whether $\bigcap_{j \in \mathbb{Z}} D^j V_0 = \{0\}$ for a shift invariant subspace $V_0 = \overline{span}\{D^n T^\ell \psi_i,\, i = 1, \cdots, r,\, n < 0, n, \ell \in \mathbb{Z}\}$, where $\Psi = (\psi_1, \cdots, \psi_r)$ is a Parseval r-frame multiwavelet, see [24].

4. The Operational Parameterization of Riesz r-MSW-Triples

In this section, we discuss the parameterization of Riesz r-MSW-triples.

Assume $< M_0, \Phi_0, \Psi_0 >$ be an orthonormal r-MSW-triple. Let (M, Φ) be a $r-$RMRA, and Ψ be the Riesz multiwavelet related to M. We call $< M, \Phi, \Psi >$ a r-RMSW-triple and denote $\mathcal{RMSW}$ the set of all $r-$RMSW-triples. We write $M_0 = \{V_n\}$, $M = \{V'_n\}, n \in \mathbb{Z}$.

Similily, we say an operator A maps a $r-$MSW-triple into a r-RMSW-triple $< M, \Phi, \Psi >$ if

$$\begin{aligned} &\Phi = A(\Phi_0) : \varphi_i = A\varphi_{0i},\, i = 1, \cdots, r, \\ &\Psi = A\Psi_0 : \psi_i = A\psi_{0i}, i = 1, \cdots, r, \\ &V'_0 = AV_0,\, V'_n = D^n V'_0,\, n \in \mathbb{Z}, \\ &M = \{V'_n,\, n \in \mathbb{Z}\}. \end{aligned}$$

We write $< M, \Phi, \Psi >= A < M_0, \Phi_0, \Psi_0 >$.

For any operator A, $A < M_0, \Phi_0, \Psi_0 >$ is not necessary be a $r-$ RMSW-triple. For a $r-$MSW-triple $< M_0, \Phi_0, \Psi_0 >$, define

$$C(M_0, \Phi_0, \Psi_0) = \{T\}' \bigcap \{B(H)P_{\Phi_0} + \mathcal{U}^V_{\Psi_0}(D, T)P^{\perp}_{\Phi_0}\}$$
$$V(M_0, \Phi_0, \Psi_0) = \text{invertible operators in}\, C(M_0, \Phi_0, \Psi_0).$$

Where $\mathcal{U}^V_{\Psi_0}(D, T)$ denotes the invertible operators in $C_{\Psi_0}(D, T)$.

Lemma 4.1. Suppose that $\Psi = (\psi_1, \psi_2, \cdots, \psi_r)$ is an orthonormal multiwavelet for H. Then a r-tuple $\eta = (\eta_1, \cdots, \eta_r)$ is a Riesz multiwavelet for H if and only if there is unique invertible operator $A \in C_\Psi(D, T)$ such that $A\Psi = \eta$.

Proof. Following the argument in [23], It is obvious. □

Lemma 4.2. Let $A \in V(M_0, \Phi_0, \Psi_0)$. Then A has the following properties,

(i) $AD^nT^\ell\Psi_0 = D^nT^\ell A\Psi_0, \forall n \geq 0, \ell \in \mathbb{Z}$.

(ii) $\{AT^\ell\Phi_0, AD^nT^\ell\Psi_0 : n \geq 0, \ell \in \mathbb{Z}\} = \{AT^\ell\phi_{0,j},\ AD^nT^\ell\psi_{0,j}, j = 1, 2, \cdots, r, n \geq 0, \ell \in \mathbb{Z}\}$ is a Riesz basis for $L^2(\mathbb{R})$.

Proof. (i) Following the argument of Lemma 3.2.

(ii) Because A is bouned and invertible, and $\{T^\ell\Phi_0, D^nT^\ell\Psi_0 : n \geq 0, \ell \in \mathbb{Z}\}$ is an orthonormal basis. □

Theorem 4.3. For any $r-$RMSW-triple $< M, \Phi, \Psi >$, there exists unique $A \in V(M_0, \Phi_0, \Psi_0)$ with $A < M_0, \Phi_0, \Psi_0 >=< M, \Phi, \Psi >$.

Proof. Let $< M, \Phi, \Psi >$ be any arbitrary RMSW-triple. Define a mapping A from $\{T^\ell\varphi_{0j}, D^nT^\ell\psi_{0j} : j = 1, \cdots, r, n \geq 0, n, \ell \in \mathbb{Z}\}$ onto the Riesz basis $\{T^\ell\varphi_j, D^nT^\ell\psi_j : j = 1, \cdots, r, n \geq 0, \ell \in \mathbb{Z}\}$ by

$$AT^\ell\varphi_{0j} = T^\ell\varphi_j, \ell \in \mathbb{Z}, j = 1, \cdots, r,$$
$$AD^nT^\ell\psi_{0j} = D^nT^\ell\psi_j, n \geq 0, \ell \in \mathbb{Z}.$$

Linearly extending A to H denoted it by A. By the definition of Riesz basis, it is obvious that A is bounded and invertible.

Since Ψ_0 is an orthonormal multiwavelet,Ψ is a Riesz multiwavelet, by Lemma 4.1, there exists unique invertible $G \in C_{\Psi_0}(D, T)$ such that $G\Psi_0 = \Psi$. By Lemma 4.2, the rest argument is the same as the proof of Theorem 3.3. □

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