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GRÜSS TYPE INEQUALITIES INVOLVING THE GENERALIZED GAUSS HYPERGEOMETRIC FUNCTIONS

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Abstract: In this paper, we establish certain generalized Grüss type inequality for generalized fractional integral inequalities involving the generalized Gauss hypergeometric function. Moreover, we also consider their relevances for other related known results.

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1. Introduction and Preliminares

In 1935, D. Grüss, proved the following integral inequality which gives an estimation of a product in terms of the product of integrals [6]:

$$\left|\frac{1}{(b-a)}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)}\int_a^b f(x)dx\,\frac{1}{(b-a)}\int_a^b g(x)dx\right| \leq \frac{1}{4}\left(L-l\right)\left(M-m\right); \tag{1.1}$$

provided that f and g are two functions which are defined and integrable on

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$[a,b]$ and satisfying the condition

$$l \leq f(x) \leq L,\ m \leq g(x) \leq M \tag{1.2}$$

for all $x \in [a,b]$.

The costant $\frac{1}{4}$ is the best possible.

Definition 1. ([26, 27]) Let $h(x)$ be an increasing and positive monotone function on $[0,\infty)$, also derivative $h'(x)$ is continuous on $[0,\infty)$ and $h(0) = 0$. The space $X_h^p(0,\infty)\,(1 \leq p < \infty)$ of those real-valued Lebesque measurable functions f on $[0,\infty)$ for which

$$||f||_{X_h^p} = \left(\int_0^\infty |f(t)|^p h'(t)\,dt\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p \leq \infty \tag{1.3}$$

and for the case $p = \infty$

$$||f||_{X_h^\infty} = ess \sup_{1\leq t<\infty} \left[f(t)h'(t)\right]. \tag{1.4}$$

In particular, when $h(x) = x\ \ (1 \leq p < \infty)$ the space $X_h^p(0,\infty)$ coincides with the $L_p[0,\infty)-$space and also if we take $h(x) = \dfrac{x^{k+1}}{k+1}\ (1 \leq p < \infty,\ k \geq 0)$ the space $X_h^p(0,\infty)$ coincides with the $L_{p,k}[0,\infty)-$space.

Definition 2. ([28]) Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h'(x)$ on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ for which

$$\left(J_{a^+,h}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x [h(x)-h(t)]^{\alpha-1} h'(t)\,f(t)\,dt,\ x \geq a, \tag{1.5}$$

and

$$\left(J_{b^-,h}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b [h(t)-h(x)]^{\alpha-1} h'(t)\,f(t)\,dt,\ x \leq b. \tag{1.6}$$

Definition 3. Let $f \in X_h^1$. For $\alpha > 0,\ \delta > -1,\ \beta,\eta \in \mathbb{R}$ and $h(x)$ be an increasing and positive monotone function on $(0,x]$, having a continuous derivative $h^{'}(x)$ on $(0,x)$. Then the generalized fractional integral $I_{h(t)}^{\alpha,\beta,\eta,\delta}$ of order α for real-valued continuous function $f(t)$, is defined by

$$\begin{aligned} I_{h(t)}^{\alpha,\beta,\eta,\delta}\{f(x)\} &= \frac{h(x)^{-\alpha-\beta-2\delta}}{\Gamma(\alpha)}\int_0^x h(t)^{\delta}(h(x)-h(t))^{\alpha-1} \\ &\times \left({}_2F_1(\alpha+\beta+\delta,-\eta;\alpha;1-\frac{h(t)}{h(x)})\right)h^{'}(t)f(t)dt, \end{aligned} \tag{1.7}$$

where the function ${}_2F_1(.)$ appearing as a kernel for the operator (1.7) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;h(t)) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{[h(t)]^n}{n!}, \tag{1.8}$$

where,

$${}_2F_1(0,b;c;h(t)) = {}_2F_1(a,b;c;0) = 1.$$

The Pochhammer symbol $(a)_n$ is defined by $(n \in \mathbb{N})$,

$$(a)_n = a(a+1)\cdots(a+n-1); \quad (a)_0 = 1. \tag{1.9}$$

Where $\mathbb{N}$ denotes the set of positive integers. The above integral (1.7) has the following commutative property:

$$I_{h(t)}^{\alpha,\beta,\eta,\delta} I_{h(t)}^{a,b,c,d} f(x) = I_{h(t)}^{a,b,c,d} I_{h(t)}^{\alpha,\beta,\eta,\delta} f(x). \tag{1.10}$$

In the sequel, we use the following well-known result to establish our main results in the present paper:

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \left(\Re(c-a-b)>0;\ c\in\Xi/\mathbb{Z}_0^-\right), \tag{1.11}$$

where and Ξ and $/\mathbb{Z}_0^-$ denotes the sets of complex numbers and nonpositive integers, respectively.

We define a fractional integral operator $K_{h(t)}^{\alpha,\beta,\eta,\delta}$ associated with the Gauss hypergeometric function as follows.

Definition 4. Let $f \in X_h^1$. For $\alpha > max\{0,-(\delta+\eta+1)\}, \beta-1<\eta<0, \beta<1$ and $\delta>-1$ we define a fractional integral $K_{h(t)}^{\alpha,\beta,\eta,\delta} f$ as follows:

$$\left(K_{h(t)}^{\alpha,\beta,\eta,\delta} f\right)(x) = \frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)}h(x)^{\beta+\delta}\left(I_{h(t)}^{\alpha,\beta,\eta,\delta} f\right)(x), \tag{1.12}$$

where $I_{h(t)}^{\alpha,\beta,\eta,\delta}$ is the Gauss hypergeometric fractional integral of order α and is defined in the following.

Definition 5. Two functions f and g are said to be synchronous functions on $[0,\infty)$ if

$$A(x,y) = (f(x) - f(y))\,(g(x) - g(y)) \geq 0; \tag{1.13}$$

for any $x, y \in [0,\infty)$.

Lemma 6. *For $\mu > max\{0, -(\eta-\beta)\} - 1, \alpha > max\{0, -(\delta+\eta+1)\}$; and $\eta - \beta > -1, \beta < 1,\ \delta + \mu > -1,\ h\,(x)$ be an increasing and positive monotone function on $(0,x]$, having a continuous derivative $h^{'}\,(x)$ on $(0,x)$, $h(0) = 0$, we have*

$$K_{h(t)}^{\alpha,\beta,\eta,\delta}\left(h^{\mu}\,(x)\right) = \frac{\Gamma\,(1-\beta)\,\Gamma\,(\alpha+\delta+\eta+1)\,\Gamma\,(\delta+\mu+1)\,\Gamma\,(\mu-\beta+\eta+1)}{\Gamma\,(\eta-\beta+1)\,\Gamma\,(\delta+1)\,\Gamma\,(\mu-\beta+1)\,\Gamma\,(\mu+\delta+\alpha+\eta+1)} h^{\mu}\,(x) \tag{1.14}$$

and

$$K_{h(t)}^{\alpha,\beta,\eta,\delta}\,(C) = C, \tag{1.15}$$

where C is constant.

Proof. Using the result (1.7), (1.12) reduces to

$$K_{h(t)}^{\alpha,\beta,\eta,\delta}\left(h^{\mu}\,(x)\right) = \frac{\Gamma\,(1-\beta)\,\Gamma\,(\alpha+\delta+\eta+1)\,[h\,(x)]^{\beta+\delta}}{\Gamma\,(\eta-\beta+1)\,\Gamma\,(\delta+1)} \frac{[h\,(x)]^{-\alpha-\beta-2\delta}}{\Gamma\,(\alpha)}$$
$$\times \int_0^x [h\,(t)]^{\delta+\mu}\,(h\,(x) - h\,(t))^{\alpha-1} \left({}_2F_1(\alpha+\beta+\delta, -\eta; \alpha; 1 - \frac{h\,(t)}{h\,(x)})\right) h^{'}\,(t)\,dt. \tag{1.16}$$

Using (1.8), (1.16) reduces to the following form:

$$\begin{aligned}
&K_{h(t)}^{\alpha,\beta,\eta,\delta}\left(h^{\mu}\,(x)\right) \\
&= \frac{\Gamma\,(1-\beta)\,\Gamma\,(\alpha+\delta+\eta+1)\,[h\,(x)]^{-\alpha-\delta}}{\Gamma\,(\eta-\beta+1)\,\Gamma\,(\delta+1)\,\Gamma\,(\alpha)} \\
&\times \sum_{n=0}^{\infty} \frac{(a+\beta+\delta)_n\,(-\eta)_n}{(\alpha)_n\,n!}\,[h\,(x)]^{-n} \int_0^x [h\,(t)]^{\delta+\mu}\,(h\,(x)-h\,(t))^{\alpha+n-1}\,h^{'}\,(t)\,dt \\
&= \frac{\Gamma\,(1-\beta)\,\Gamma\,(\alpha+\delta+\eta+1)\,[h\,(x)]^{-\alpha-\delta}}{\Gamma\,(\eta-\beta+1)\,\Gamma\,(\delta+1)\,\Gamma\,(\alpha)} \sum_{n=0}^{\infty} \frac{(a+\beta+\delta)_n\,(-\eta)_n}{(\alpha)_n\,n!}\,[h\,(x)]^{-n} \\
&\times [h\,(x)]^{\delta+\mu+\alpha+n+1-1}\,B(\delta+\mu+1, \alpha+n) \\
&= \frac{\Gamma\,(1-\beta)\,\Gamma\,(\alpha+\delta+\eta+1)\,\Gamma\,(\delta+\mu+1)}{\Gamma\,(\eta-\beta+1)\,\Gamma\,(\delta+1)\,\Gamma\,(\alpha+\delta+\mu+1)} \sum_{n=0}^{\infty} \frac{(a+\beta+\delta)_n\,(-\eta)_n}{(\alpha+\delta+\mu+1)_n\,n!} h^{\mu}\,(x) \\
&= \frac{\Gamma\,(1-\beta)\,\Gamma\,(\alpha+\delta+\eta+1)\,\Gamma\,(\delta+\mu+1)}{\Gamma\,(\eta-\beta+1)\,\Gamma\,(\delta+1)\,\Gamma\,(\alpha+\delta+\mu+1)}
\end{aligned} \tag{1.17}$$

$$\times\left[{}_2F_1(a+\beta+\delta,-\eta;\alpha+\delta+\mu+1;1)\right]h^{\mu}(x)$$
$$=\frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)\Gamma(\delta+\mu+1)}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)\Gamma(\alpha+\delta+\mu+1)}$$
$$\times\frac{\Gamma(\alpha+\delta+\mu+1)\Gamma(\alpha+\delta+\mu+1-(a+\beta+\delta)-(-\eta))}{\Gamma(\alpha+\delta+\mu+1-(a+\beta+\delta))\Gamma(\alpha+\delta+\mu+1-(-\eta))}h^{\mu}(x)$$
$$=\frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)\Gamma(\delta+\mu+1)}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)\Gamma(\alpha+\delta+\mu+1)}$$
$$\times\frac{\Gamma(\alpha+\delta+\mu+1)\Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta)\Gamma(\alpha+\delta+\mu+1+\eta)}h^{\mu}(x)$$
$$=\frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)\Gamma(\delta+\mu+1)\Gamma(\mu-\beta+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)\Gamma(\mu-\beta+1)\Gamma(\mu+\delta+\alpha+\eta+1)}h^{\mu}(x)$$

Proof is done.

To prove (1.15), we again use the result (1.7), and (1.12) reduces to

$$\begin{aligned}K_{h(t)}^{\alpha,\beta,\eta,\delta}(C) &= \frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)[h(x)]^{\beta+\delta}}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)} \\ &\times\frac{[h(x)]^{-\alpha-\beta-2\delta}}{\Gamma(\alpha)}\int_0^x [h(t)]^{\delta}(h(x)-h(t))^{\alpha-1} \\ &\times\left({}_2F_1(\alpha+\beta+\delta,-\eta;\alpha;1-\frac{h(t)}{h(x)})\right)h'(t)\,Cdt.\end{aligned} \tag{1.18}$$

Using (1.8), (1.18) gets the following form:

$$K_{h(t)}^{\alpha,\beta,\eta,\delta}(C)$$
$$=C\frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)[h(x)]^{-\alpha-\delta}}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)}\frac{1}{\Gamma(\alpha)}$$
$$\times\sum_{n=0}^{\infty}\frac{(a+\beta+\delta)_n(-\eta)_n}{(\alpha)_n n!}[h(x)]^{-n}\int_0^x [h(t)]^{\delta}(h(x)-h(t))^{\alpha+n-1}h'(t)\,dt$$
$$=C\frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)[h(x)]^{-\alpha-\delta}}{\Gamma(\eta-\beta+1)\Gamma(\delta+1)}\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(a+\beta+\delta)_n(-\eta)_n}{(\alpha)_n n!}[h(x)]^{-n}$$
$$\times[h(x)]^{\delta+\alpha+n+1-1}B(\delta+1,\alpha+n)$$
$$=C\frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\alpha+\delta+1)}\sum_{n=0}^{\infty}\frac{(a+\beta+\delta)_n(-\eta)_n}{(\alpha+\delta+1)_n n!}$$
$$=C\frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\alpha+\delta+1)}\left[{}_2F_1(\alpha+\beta+\delta,-\eta;\alpha+\delta+1;1)\right]$$
$$=C\frac{\Gamma(1-\beta)\Gamma(\alpha+\delta+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\alpha+\delta+1)}\frac{\Gamma(\alpha+\delta+1)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(\alpha+\delta+1+\eta)}$$
$$=C \tag{1.19}$$

This completes the proof of the Lemma 6. $\square$

Lemma 7. *Let $g \in X_h^1$ and $m, M \in \mathbb{R}$ with $m \leq g(x) \leq M$. Then we have*

$$K_{h(t)}^{\alpha,\beta,\eta,\delta} g^2(x) - \left(K_x^{\alpha,\beta,\eta,\delta} tg(x)\right)^2 \\ = \left(M - K_{h(t)}^{\alpha,\beta,\eta,\delta} g(x)\right)\left(K_{h(t)}^{\alpha,\beta,\eta,\delta} g(x) - m\right) - K_{h(t)}^{\alpha,\beta,\eta,\delta}(M - g(x))(g(x) - m), \tag{1.20}$$

for all $x \in [0, \infty)$; $\alpha > 0$, $\delta > -1$, and $\beta, \eta \in \mathbb{R}$ with $\alpha + \beta + \delta \geq 0$ and $\eta \leq 0$.

Proof. Let $g \in X_h^1$ and $m, M \in \mathbb{R}$; $m \leq g(x) \leq M$, for all $x \in [0, \infty)$. Then, for any $u, v \in [0, \infty)$, we have

$$(M - g(u))(g(v) - m) + (M - g(v))(g(u) - m) - (M - g(u))(g(u) - m) \\ -(M - g(v))(g(v) - m) = g^2(u) + g^2(v) - 2g(u)g(v). \tag{1.21}$$

If $g \in X_h^1$, then g is integrable on $[0, x]$, $x > 0$.

Thus multiplying (1.21) by

$$\frac{[h(u)]^\delta (h(x) - h(u))^{\alpha-1}}{\Gamma(\alpha)} h'(u) \left({}_2F_1(\alpha + \delta + \beta, -\eta; \alpha; 1 - \frac{h(u)}{h(x)})\right);$$

using $u \in (0, x)$; $x > 0$, and integrating with respect to u from 0 to x, and then applying Definition 4 and Lemma 6, we obtain

$$\left(M - K_{h(t)}^{\alpha,\beta,\eta,\delta} g(x)\right)(g(v) - m) + (M - g(v))\left(K_{h(t)}^{\alpha,\beta,\eta,\delta} g(x) - m\right) \\ -K_{h(t)}^{\alpha,\beta,\eta,\delta}(M - g(x))(g(x) - m) - (M - g(v))(g(v) - m) \\ = K_{h(t)}^{\alpha,\beta,\eta,\delta} g^2(x) + g^2(v) - 2K_{h(t)}^{\alpha,\beta,\eta,\delta} g(x)g(v) \tag{1.22}$$

Again multiplying (1.22) by

$$\frac{[h(v)]^\delta (h(x) - h(v))^{\alpha-1}}{\Gamma(\alpha)} h'(v) \left({}_2F_1(\alpha + \delta + \beta, -\eta; \alpha; 1 - \frac{h(v)}{h(x)})\right),$$

$v \in (0, x)$; $x > 0$, then integrating with respect to v from 0 to x, we obtain the required result (1.20). This completes the proof of Lemma 7. $\square$

2. Main Results

Theorem 8. *Let f and g be two functions defined and integrable on $[a,b]$ with f , $g \in X_h^1$ and satisfying the condition (1.1) on $[0,\infty)$. Thus we have*

$$\left|K_{h(t)}^{\alpha,\beta,\eta,\delta} fg(x) - K_{h(t)}^{\alpha,\beta,\eta,\delta} f(x) K_{h(t)}^{\alpha,\beta,\eta,\delta} g(x)\right| \leq \tfrac{1}{4}(L-l)(M-m), \tag{2.1}$$

for all $x \in [0,\infty)$; $\alpha > 0$, $\delta > -1$, and β, $\eta \in \mathbb{R}$ with $\alpha+\beta+\delta \geq 0$ and $\eta \leq 0$.

Proof. Let us define a function

$$A(u,v) = (f(u)-f(v))\,(g(u)-g(v)) \quad (u,v \in [0,\infty))\,. \tag{2.2}$$

First multiplying (2.2) by

$$\frac{(h(u)h(v))^{\delta}\,(h(x)-h(u))^{\alpha-1}\,(h(x)-h(v))^{\alpha-1}}{(\Gamma(\alpha))^2} h'(u)h'(v)$$
$$\times({}_2F_1(\alpha+\delta+\beta,-\eta;\alpha;1-\frac{h(u)}{h(x)})({}_2F_1(\alpha+\delta+\beta,-\eta;\alpha;1-\frac{h(v)}{h(x)})$$

and then integrating twice with respect to u and v from 0 to x, we obtain the following result with the aid of (1.7), (1.12), and property (1.8):

$$\begin{aligned}
&\frac{1}{(\Gamma(\alpha))^2}\int_0^x\int_0^x (h(u)h(v))^{\delta}\,(h(x)-h(u))^{\alpha-1}\,(h(x)-h(v))^{\alpha-1} \\
&\times({}_2F_1(\alpha+\delta+\beta,-\eta;\alpha;1-\frac{h(u)}{h(x)}) \\
&\times({}_2F_1(\alpha+\delta+\beta,-\eta;\alpha;1-\frac{h(v)}{h(x)})A(u,v)h'(u)h'(v)dudv \\
&= 2K_{h(t)}^{\alpha,\beta,\eta,\delta} fg(x) - 2K_{h(t)}^{\alpha,\beta,\eta,\delta} f(x) K_{h(t)}^{\alpha,\beta,\eta,\delta} g(x).
\end{aligned} \tag{2.3}$$

Making use of the well-known Cauchy-Schwarz inequality for a linear operator, we find that

$$\begin{aligned}
&\left(K_{h(t)}^{\alpha,\beta,\eta,\delta} fg(x) - K_{h(t)}^{\alpha,\beta,\eta,\delta} f(x) K_{h(t)}^{\alpha,\beta,\eta,\delta} g(x)\right)^2 \\
&\leq (K_{h(t)}^{\alpha,\beta,\eta,\delta} f^2(x) - \left(K_{h(t)}^{\alpha,\beta,\eta,\delta} f(x)\right)^2)(K_{h(t)}^{\alpha,\beta,\eta,\delta} g^2(x) - \left(K_{h(t)}^{\alpha,\beta,\eta,\delta} g(x)\right)^2).
\end{aligned} \tag{2.4}$$

Since

$$(L-f(x))\,(f(x)-l) \geq 0 \quad \text{and} \quad (M-f(x))\,(f(x)-m) \geq 0,$$

we therefore have

$$K_{h(t)}^{\alpha,\beta,\eta,\delta}\left(L-f(x)\right)\left(f(x)-l\right)\geq 0 \quad \text{and} \quad K_{h(t)}^{\alpha,\beta,\eta,\delta}\left(M-f(x)\right)\left(f(x)-m\right)\geq 0. \tag{2.5}$$

Thus by using Lemma 7, we have

$$K_{h(t)}^{\alpha,\beta,\eta,\delta}f^2(x)-\left(K_{h(t)}^{\alpha,\beta,\eta,\delta}f(x)\right)^2\leq\left(L-K_{h(t)}^{\alpha,\beta,\eta,\delta}f(x)\right)(K_{h(t)}^{\alpha,\beta,\eta,\delta}f(x)-l) \tag{2.6}$$

and

$$K_{h(t)}^{\alpha,\beta,\eta,\delta}g^2(x)-\left(K_{h(t)}^{\alpha,\beta,\eta,\delta}g(x)\right)^2\leq\left(M-K_{h(t)}^{\alpha,\beta,\eta,\delta}g(x)\right)(K_{h(t)}^{\alpha,\beta,\eta,\delta}g(x)-m). \tag{2.7}$$

Using the inequalities (2.6) and (2.7), (2.4) reduces to the following form:

$$\begin{aligned}&\left(K_{h(t)}^{\alpha,\beta,\eta,\delta}fg(x)-K_{h(t)}^{\alpha,\beta,\eta,\delta}f(x)K_{h(t)}^{\alpha,\beta,\eta,\delta}g(x)\right)^2\\&\leq(L-K_{h(t)}^{\alpha,\beta,\eta,\delta}f(x))(K_{h(t)}^{\alpha,\beta,\eta,\delta}f(x)-l)(M-K_{h(t)}^{\alpha,\beta,\eta,\delta}g(x))\\&\quad\times(K_{h(t)}^{\alpha,\beta,\eta,\delta}g(x)-m).\end{aligned} \tag{2.8}$$

Applying the well-known inequality $4ab\leq(a+b)^2$; and using $a,b\in\mathbb{R}$ in the right-hand side of the inequality (2.8),

$$\begin{aligned}&\left(K_{h(t)}^{\alpha,\beta,\eta,\delta}fg(x)-K_{h(t)}^{\alpha,\beta,\eta,\delta}f(x)K_{h(t)}^{\alpha,\beta,\eta,\delta}g(x)\right)^2\\&\leq\frac{\left[(L-K_{h(t)}^{\alpha,\beta,\eta,\delta}f(x)+K_{h(t)}^{\alpha,\beta,\eta,\delta}f(x)-l\right]^2}{4}\frac{\left[M-K_{h(t)}^{\alpha,\beta,\eta,\delta}g(x)+K_{h(t)}^{\alpha,\beta,\eta,\delta}g(x)-m\right]^2}{4}\end{aligned}$$

and simplifying it,

$$\begin{aligned}&\left(K_{h(t)}^{\alpha,\beta,\eta,\delta}fg(x)-K_{h(t)}^{\alpha,\beta,\eta,\delta}f(x)K_{h(t)}^{\alpha,\beta,\eta,\delta}g(x)\right)\\&\leq\frac{1}{4}\left(L-l\right)\left(M-m\right)\end{aligned}$$

we obtain the required result (2.1). This completes the proof of Theorem 8. □

Theorem 9. *Let f and g be two synchronous functions on $[0,\infty)$. Then the following inequality holds:*

$$K_{h(t)}^{\alpha,\beta,\eta,\delta}fg(x)\geq K_{h(t)}^{\alpha,\beta,\eta,\delta}f(x)K_{h(t)}^{\alpha,\beta,\eta,\delta}g(x), \tag{2.9}$$

for all $x\in[0,\infty)$; $\alpha>0,\delta>-1$,and $\beta,\eta\in R$ with $\alpha+\beta+\delta\geq 0$ and $\eta\leq 0$.

Proof. For the synchronous function f and g, the inequality (1.13) holds for all $u, v \in [0, \infty)$. This implies that

$$f(u)g(u) - f(v)g(v) \geq f(u)g(v) + f(v)g(u). \tag{2.10}$$

Following the procedure of the Lemma 7 for applying the fractional integral $K_{h(t)}^{\alpha,\beta,\eta,\delta}$, after a little simplification, we arrive at the required result (2.9). This completes the proof of Theorem 9. □

3. Concluding Remarks

We consider some consequences of the results derived in the previous section.

$$\begin{aligned} I_{0,h(t)}^{\alpha,\beta,\eta}\left\{f(t)\right\} &= I_{h(t)}^{\alpha,\beta,\eta,0}\left\{f(t)\right\} \\ &= \frac{[h(x)]^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (h(t) - h(\tau))^{\alpha-1} \\ &\times \left({}_2F_1(\alpha+\beta, -\eta; \alpha; 1 - \frac{h(\tau)}{h(t)})\right) h^{'}(\tau) f(\tau) d\tau, \end{aligned} \tag{3.1}$$

for $\delta = 0$,

$$\begin{aligned} I^{\alpha,\eta}\left\{f(t)\right\} &= I_{h(t)}^{\alpha,0,\eta,0}\left\{f(t)\right\} \\ &= \frac{[h(x)]^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^t (h(t) - h(\tau))^{\alpha-1} \left[h(\tau)\right]^{\eta} h^{'}(\tau) f(\tau) d\tau, \end{aligned} \tag{3.2}$$

for $\delta = 0$ and $\beta = 0$, and

$$\begin{aligned} R^{\alpha}\left\{f(t)\right\} &= I_{h(t)}^{\alpha,-\alpha,\eta,0}\left\{f(t)\right\} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (h(t) - h(\tau))^{\alpha-1} h^{'}(\tau) f(\tau) d\tau \qquad (\alpha > 0), \end{aligned} \tag{3.3}$$

for $\delta = 0$ and $\beta = -\alpha$.

Here if we choose $h(x) = x$, following we, the operator (1.7) would reduce immediately to the extensively investigated Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, given by the following relationships (see also [17], [19], [20]):

$$\begin{aligned} I_{0,t}^{\alpha,\beta,\eta}\left\{f(t)\right\} &= I_{t}^{\alpha,\beta,\eta,0}\left\{f(t)\right\} \\ &= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int\limits_{0}^{t}(t-\tau)^{\alpha-1} \\ &\times\left({}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t})\right)f(\tau)d\tau, \end{aligned} \tag{3.4}$$

for $\delta=0$,

$$\begin{aligned} I^{\alpha,\eta}\left\{f(t)\right\} &= I_{t}^{\alpha,0,\eta,0}\left\{f(t)\right\} \\ &= \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int\limits_{0}^{t}(t-\tau)^{\alpha-1}\tau^{\eta}f(\tau)d\tau, \end{aligned} \tag{3.5}$$

for $\delta=0$ and $\beta=0$, and

$$\begin{aligned} R^{\alpha}\left\{f(t)\right\} &= I_{t}^{\alpha,-\alpha,\eta,0}\left\{f(t)\right\} \\ &= \frac{1}{\Gamma(\alpha)}\int\limits_{0}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau \qquad (\alpha>0), \end{aligned} \tag{3.6}$$

for $\delta=0$ and $\beta=-\alpha$.

Remark 10. We obtain the special cases of the operator $K_{h(t)}^{\alpha,\beta,\eta,\delta}$ as follows by setting $\delta=0$, $\delta=\beta=0$, $\delta=0$, $\beta=-\alpha$ and $h(x)=x$ in Definition 3. Immediately Definition 3. would reduce to the Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, given as follows:

$$\left(K_{t}^{\alpha,\beta,\eta}f\right)(x)=\frac{\Gamma(1-\beta)\Gamma(\alpha+\eta+1)}{\Gamma(\eta-\beta+1)}x^{\beta}\left(I_{0,t}^{\alpha,\beta,\eta}f\right)(x), \tag{3.7}$$

$$\left(K_{t}^{\alpha,\eta}f\right)(x)=\frac{\Gamma(\alpha+\eta+1)}{\Gamma(\eta+1)}x^{\beta}\left(I^{\alpha,\eta}f\right)(x) \tag{3.8}$$

and

$$\left(K_{t}^{\alpha}f\right)(x)=\frac{\Gamma(\alpha+1)}{x^{\alpha}}\left(R^{\alpha}f\right)(x), \tag{3.9}$$

where $\left(I_{0,t}^{\alpha,\beta,\eta}\right)$, $\left(I^{\alpha,\eta}\right)$ and $\left(R^{\alpha}\right)$ are given by (3.1), (3.2), and (3.3), respectively.

We conclude our present investigation by remarking further that the results obtained here are useful in deriving various fractional integral inequalities involving such relatively more familiar fractional integral operators. For example, if we consider $\delta = 0$, $h(x) = x$ and make use of (3.1), Theorems 8. and 9. provide, respectively, the known fractional integral inequalities due to Kalla and Rao [12].

Similarly, if we choose $\delta = 0$, $\beta = 0$ and $h(x) = x$ in Theorems 8. and 9 and making use of the relation (3.2), Theorems 8. and 9. provide, respectively, the known fractional integral inequalities owing to Kalla and Rao [12].

Finally, if we choose $\delta = 0$, $h(x) = x$ and $\beta = -\alpha$ in Theorems 8. and 9. yields the known result owing to Dahmani et al. [5].

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