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SOLVING HYBRID FUZZY DIFFERENTIAL EQUATIONS USING TAYLOR SERIES METHOD

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Abstract: In this research paper, the numerical method for solving hybrid fuzzy differential equations using Taylor series method under generalized Hukuhara differentiability is presented. The exact and approximate solutions are discussed with examples and this is followed by a complete error analysis and the proposed method provides an efficient and powerful mathematical tool for solving hybrid fuzzy differential equations.

AMS Subject Classification: 34A07, 34A38
Key Words: fde, hybrid fde, taylor series method, error analysis

1. Introduction

Fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models and was introduced by Lotfi A. Zadeh in 1965. In particular, fuzzy differential equations are natural way to model dynamic system subject to uncertainties and these topics have been developed in theoretical and numerical point of view in recent years. The term Fuzzy Differential Equation (FDE) was introduced in 1978 by Kandel and Byatt. There have been many suggestions for fuzzy derivative to study fuzzy differential equation. The fuzzy differential equations and fuzzy initial value problem were regularly treated by O.Kaleva and S.Seikkala. Hybrid sys-

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tems are devoted to modeling, designing and validating of interactive systems of computer programs and continuous systems. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are named as hybrid fuzzy differential systems (HFDEs). Some applications of numerical methods in FDE and Hybrid FDE are investigated by various authors and from them we observed that the Taylor series method is a simple and efficient tool to solve numerically related ODEs, especially in engineering and physical problems. Taylor series method has a convergence in comparison to other one step methods.

In order to obtain numerical solutions for fuzzy equations under Hukuhara differentiability, it is not necessary to rewrite the numerical methods for the ordinary differential equations in fuzzy setting. Instead one can use the numerical methods directly for the ordinary differential equations by the use of characterization theorems. In the present paper, we deal with numerical solution of hybrid fuzzy differential equations using Taylor series method and further the numerical solutions are calculated and the results were tabulated and shown graphically.

2. Preliminaries

Definition 2.1. (Triangular Fuzzy Number) A Triangular fuzzy number is a fuzzy set 'u' in E that is characterized by an ordered triple$(u^l, u^c, u^r) \in \mathrm{R}^3$ with $u^l \leq u^c \leq u^r$ such that $[u]^0 = \left[u^l, u^r\right]$ and $[u]^l = [u]^c$. The α-level set of a triangular fuzzy number u is given by

$$[u]^\alpha = [u^c - (1-\alpha)(u^c - u^l), u^c + (1-\alpha)(u^r - u^c)],$$

for any $\alpha \in \mathrm{I} = [0, 1]$.

Definition 2.2. (Fuzzy Process) A fuzzy process X_t is said to have independent increments if $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$ are independent fuzzy variables for any times $t_0 < t_1 \cdots < t_k$. A fuzzy process X_t is said to have stationary increments if, for any given $t > 0$, $X_{s+t} - X_s$ are identically distributed fuzzy variables for all $s > 0$.

Definition 2.3. (Hybrid Process) A hybrid process X_t is said to have independent increments if $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$ are independent hybrid variables for any times $t_0 < t_1 \cdots < t_k$. A hybrid process X_t is said to have stationary increments if, for any given $t > 0$, $X_{s+t} - X_s$ are identically distributed hybrid variables for all $s > 0$.

3. Analysis of Hybrid Fuzzy Differential Equations

3.1. Hybrid Differential Equations

Suppose B_t is a standard Brownian motion, C_t is a standard C process, and f, g_1, g_2 are some given functions. Then $dX_t = f(t, X_t)dt + g_1 f(t, X_t)dB_t + g_2 f(t, X_t)dC_t$ is called a hybrid differential equation. A solution is a hybrid process that satisfies identically in t.

3.2. Fuzzy Differential Equations

Consider the first-order fuzzy differential equation $y' = f(t, y)$, where y is a fuzzy function of $t, f(t, y)$, is a fuzzy function of crisp variable t and fuzzy variable y, and y' is Hukuhara fuzzy derivatives of y. If an initial value $y(t_0) = y_0 \in \Re_F$ is given, a fuzzy Cauchy Problem of first order will be obtained as follows:

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad t_0 \leq t \leq T \\ y(t_0) &= y_0 \end{aligned} \tag{1}$$

Replace (1) by equivalent system

$$\begin{aligned} \underline{y}'(t) &= \underline{f}(t, \underline{y}, \overline{y}) \qquad \underline{y}(t_0) = \underline{y_0} \\ \overline{y}'(t) &= \overline{f}(t, y) = G(t, \underline{y}, \overline{y}), \qquad \overline{y}(t_0) = \overline{y_0} \end{aligned} \tag{2}$$

The parametric form of (2) is given by

$$\underline{y}'(t;\alpha) = F\left(t, \underline{y}(t;\alpha), \overline{y}(t;\alpha)\right) \qquad \underline{y}(t_0;\alpha) = \underline{y_0^\alpha} \tag{3}$$

$$\overline{y}'(t;\alpha) = G(t, \underline{y}(t;\alpha), \overline{y}(t;\alpha)), \qquad \overline{y}(t_0;\alpha) = \overline{y_0^\alpha} \tag{4}$$

for $0 \leq \alpha \leq 1$. In some cases the system given by (3) and (4) can be solved but analytical solutions may not be found, and a numerical approach must be considered.

3.3. Hybrid Fuzzy Differential Equations

Consider the hybrid fuzzy differential equation

$$\begin{aligned} y'(t) &= f(t, y(t), \lambda_k(y_k)), \qquad t \in [t_k, t_{k+1}] \quad k = 0, 1, 2, ... \\ y(t_0) &= y_0 \end{aligned} \tag{5}$$

where $t_{k}{}_{k=0}^{\infty}$ is strictly increasing and unbounded, y_k denotes $y(t_k)$, $f : [t_0, \infty) \times \Re_F \times \Re_F \to \Re_F$ is continuous, and each $\lambda_k : \Re_F \to \Re_F$ is a continuous function. A solution y to (5) will be function $y : [t_0, \infty) \to \Re_F$ satisfying (5). For $k = 0, 1, 2, \ldots$, let $f_k : [t_k, t_{k+1}) \times \Re_F \to \Re_F$, where, $f(t, y_k(t)) = f(t, y(t), \lambda_k(y_k))$. The hybrid fuzzy differential equation in (5) can be written in expanded form as

$$y'(t) = \begin{cases} y_0'(t) = f(t, y_0(t), \lambda_0(y_0)) = f_0(t, y_0(t)), & y_0(t_0) = y_0 \quad t_0 \leq t \leq t_1 \\ y_1'(t) = f(t, y_1(t), \lambda_1(y_1)) = f_1(t, y_1(t)), & y_1(t_1) = y_1 \quad t_1 \leq t \leq t_2 \\ \quad . & \quad . \qquad\qquad . \\ \quad . & \quad . \qquad\qquad . \\ y_k'(t) = f(t, y_k(t), \lambda_k(y_k)) = f_k(t, y_k(t)), & y_k(t_k) = y_k \quad t_k \leq t \leq t_{k+1} \\ \quad . & \quad . \qquad\qquad . \\ \quad . & \quad . \qquad\qquad . \end{cases} \tag{6}$$

and a solution of (5) can be expressed as

$$y(t) = \begin{cases} y_0(t) = y_0 & t_0 \leq t \leq t_1 \\ y_1(t) = y_1 & t_1 \leq t \leq t_2 \\ \quad . & \quad . \\ \quad . & \quad . \\ y_k(t) = y_k & t_k \leq t \leq t_{k+1} \\ \quad . & \quad . \\ \quad . & \quad . \end{cases} \tag{7}$$

We note that the solution y of (5) is continuous and piecewise differentiable over $[t_0, \infty)$ and differentiable on each interval (t_k, t_{k+1}) for any fixed $y_k \in \Re_F$ and $k = 0, 1, 2, \cdots$.

3.4. Theorem

Suppose for that each $f_k : [t_k, t_{k+1}) \times \Re_F \times \Re_F$ is such that

$$[f_k(t, y)]^\alpha = \left[\underline{f}_k^\alpha(t, \underline{y}^\alpha, \overline{y}^\alpha), \overline{f}_k^\alpha(t, \underline{y}^\alpha, \overline{y}^\alpha)\right] \tag{8}$$

If for each $k = 0, 1, 2,$ there exists $L_k > 0$ such that

$$\left|\underline{f}_k^\alpha(t, x_1, y_1) - \underline{f}_k^\alpha(t, x_2, y_2)\right| \leq L_k \max\{|t_2 - t_1|, |x_2 - x_1|, |y_2 - y_1|\} \tag{9}$$

$$\left|\overline{f}_k^{\alpha}(t,x_1,y_1)-\overline{f}_k^{\alpha}(t,x_2,y_2)\right| \le L_k \max\{|t_2-t_1|,|x_2-x_1|,|y_2-y_1|\} \quad (10)$$

For all $\alpha \in [0,1]$ then (5) and the hybrid system of ODEs

$$\left(\underline{y}_k^{\alpha}(t)\right)' = \underline{f}_k^{\alpha}(t,\underline{y}_k^{\alpha}(t),\overline{y}_k^{\alpha}(t)) \quad (11)$$

$$(\overline{y}_k^{\alpha}(t))' = \overline{f}_k^{\alpha}(t,\underline{y}_k^{\alpha}(t),\overline{y}_k^{\alpha}(t)) \quad (12)$$

$$\underline{y}_k^{\alpha}(t_k) = \underline{y}_{k-1}^{\alpha}(t_k), \quad if \;\; k>0, \quad \underline{y}_0^{\alpha}(t_0)=\underline{y}_0^{\alpha} \quad (13)$$

$$\overline{y}_k^{\alpha}(t_k) = \overline{y}_{k-1}^{\alpha}(t_k), \quad if \;\; k>0, \quad \overline{y}_0^{\alpha}(t_0)=\overline{y}_0^{\alpha} \quad (14)$$

are equivalent.

4. Methodology and Example

In this section, we present the algorithm of the Taylor series method and illustrate the numerical technique for solving the hybrid fuzzy differential equations.

4.1. Taylor Series Method

Suppose a first order differential equation with initial condition is given by

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0)=y_0 \quad (15)$$

Now, to approximate $y(x_1)$ as a solution of (15) for $x_1 = x_0 + h$. We use Taylor Series expansion

$$y(x_0) = y_0 + hy'(x_1) + \frac{h^2}{2}y''(x_1) + \cdots + \frac{h^k}{k!}y^k(x_0) \quad (16)$$

Since, $\frac{dy}{dx} = f(x,y)$, we denote $y'(x_0) = f(x_0, y(x_0)) = f(x_0,y_0), y(x_0)$ by y_0 and now further differentiating $\frac{dy}{dx} = f(x,y)$, the values of $y'(x_0), y''(x_0), y^k(x_0)$ can be computed. On substituting the values in Taylor Series leaving error term

$$y(x_1) = y(x_0) + hy'(x_1) + \frac{h^2}{2!}y''(x_1) + \cdots + \frac{h^k}{k!}y^k(x_0) \quad (17)$$

Therefore $y(x)$ is approximated by y_1 and

$$y(x_1) = y_0 + hy_0' + \cdots + \frac{h^k}{k!}y_0^k \quad (18)$$

is obtained. This shows that using the values of y at x_0 only, the value of $y_1 = y(x_0+h)$ i.e., at next step is computed.

4.2. Numerical Example

Consider the hybrid fuzzy initial value problem

$$\begin{aligned} y'(t) &= y(t) + m(t)\lambda_k(y(t_k)), \quad t \in [t_k, t_{k+1}],\ t_k = k, \\ y(0) &= [0.75 + 0.25\alpha, 1.125 - 0.125\alpha] \quad k = 0, 1, 2, \ldots \end{aligned} \tag{19}$$

where

$$m(t) = \begin{cases} 2(t(mod1)) & t(mod1 \leq 0.5, \\ 2(1 - t(mod1)) & t(mod1 \leq 0.5, \end{cases} \tag{20}$$

We will solve the hybrid fuzzy equations corresponding to (16) by the Taylor series method to obtain numerical solution to (16).

Case 1. When $k = 0$, the solution of (19) in the interval $[0, 1]$: When $k = 0$, the hybrid fuzzy initial value problem (19) becomes

$$\begin{cases} y'(t) = y(t), t \in [0, 1] \\ y(t) = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha] \end{cases} \tag{21}$$

The fuzzy initial value problem (21) is equivalent to the following system of fuzzy initial value problem

$$\begin{cases} y^{l'}(t) = y^r(t), \\ y^{c'}(t) = y^c(t), \\ y^{r'}(t) = y^r(t), \qquad t \in [0, 1] \\ y^l(0) = 0.75 + 0.25\alpha, \quad y^c(0) = 1, y^r(0) = 1.125 - 0.125\alpha \end{cases}$$

In order to solve the fuzzy system (21) by using the Taylor series method, we consider the following system of fuzzy problem

$$\begin{aligned} y'(t) &= y(t), \qquad t \in [0, 1] \\ y(0) &= y^l(0) = 0.75 + 0.25\alpha, \quad y^r(0) = 1.125 - 0.125\alpha \end{aligned} \tag{22}$$

The approximate solution of (22) is given by

$$y(t; \alpha) = ((0.75 + 0.25\alpha)e^t, (1.125 - 0.125)e^t) \tag{23}$$

The approximate solution of (22) at $t = 0.1$ can be written as

$$y(0.1; \alpha) = ((0.75 + 0.25\alpha)e^{0.1}, (1.125 - 0.125)e^{0.1}) \tag{24}$$

Case 2. When $k = 1$, the solution of (19) in the interval $[1, 1.15]$: When $k = 1$, the above fuzzy problem can be written as

$$\begin{aligned} y'(t) &= y(t) + m(t)\lambda_1(y(t)), t \in [1, 1.15], \\ y(t) &= [0.75 + 0.25\alpha, 1.125 - 0.125\alpha] \end{aligned} \tag{25}$$

This is equivalent to the system of fuzzy differential equations

$$\begin{cases} y^{l'}(t) = y^r(t) + m(t)y^l(t), \\ y^{c'}(t) = y^c(t) + m(t)y^c(t), \\ y^{r'}(t) = y^r(t) + m(t)y^r(t), \qquad t \in [1, 1.15] \\ \quad y^l(0) = 0.75 + 0.25\alpha \qquad y^c(0) = 1, y^r(0) = 1.125 - 0.125\alpha \end{cases} \tag{26}$$

Similar to the previous Case I, in order to solve the fuzzy system (25) we consider the following system of fuzzy equations

$$\begin{aligned} y'(t) &= y(t) + m(t)\lambda_1(y(t)), t \in [1, 1.15], \\ y(t) &= [0.75 + 0.25\alpha,\ 1.125 - 0.125\alpha] \end{aligned} \tag{27}$$

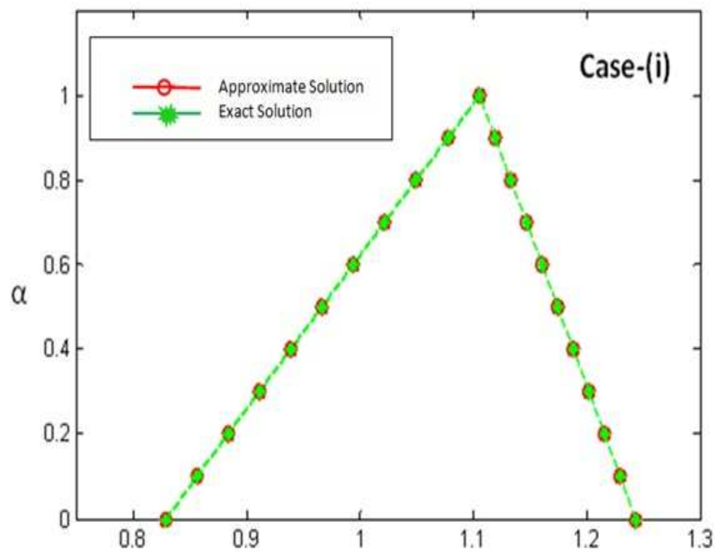
The approximate solution of (27) is

$$\begin{aligned} y(t;\alpha) &= y(1)(3e^{t-1} - 2t) \\ y(1.1;\alpha) &= [(0.75 + 0.25\alpha,\ 1.125 - 0.125\alpha)(3e^{(1.1)-1} - 2(1.1))] \end{aligned} \tag{28}$$

CASE: I. Approximate Solution, Exact Solution & Error Analysis in the interval [0, 1]

α	Approximate solution		Exact Solution		Error Analysis	
	$\underline{y}$	$\overline{y}$	$\underline{y}$	$\overline{y}$	$\underline{y}$	$\overline{y}$
0	0.828878125	1.243317188	0.828878189	1.243317283	$0.640000e^{-6}$	$0.950000e^{-6}$
0.1	0.856507396	1.229502553	0.856507468	1.229502646	$0.660000e^{-6}$	$0.930000e^{-6}$
0.2	0.884136669	1.215687917	0.884136734	1.215688012	$0.670000e^{-6}$	$0.940000e^{-6}$
0.3	0.911765938	1.201873282	0.911766017	1.201873373	$0.690000e^{-6}$	$0.910000e^{-6}$
0.4	0.939339520	1.188058646	0.939395280	1.188058737	$0.710000e^{-6}$	$0.900000e^{-6}$
0.5	0.967024487	1.174244020	0.967024559	1.174244106	$0.730000e^{-6}$	$0.890000e^{-6}$
0.6	0.994653753	1.160429376	0.994653826	1.160429464	$0.760000e^{-6}$	$0.890000e^{-6}$
0.7	1.022283024	1.146614747	1.022283090	1.146614829	$0.780000e^{-6}$	$0.880000e^{-6}$
0.8	1.049912292	1.132800115	1.049912381	1.132800195	$0.800000e^{-6}$	$0.860000e^{-6}$
0.9	1.077541563	1.118985469	1.077541645	1.118985558	$0.820000e^{-6}$	$0.840000e^{-6}$
1	1.105170834	1.105170834	1.105170918	1.105170918	$0.840000e^{-6}$	$0.810000e^{-6}$

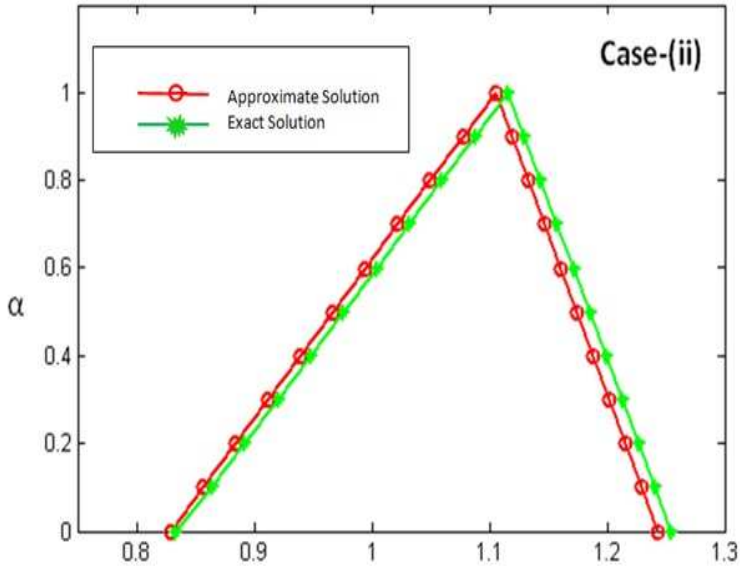
Graphical Representation of Exact and Approximate Solution in the interval $[0, 1]$



CASE: II. Approximate Solution, Exact Solution & Error Analysis in the interval [1, 1.5]

α	Approximate solution		Exact Solution		Error Analysis	
	$\underline{y}$	$\overline{y}$	$\underline{y}$	$\overline{y}$	$\underline{y}$	$\overline{y}$
0	0.828878125	1.243317188	0.833663456	1.254951848	0.4785331e^{-2}	0.11634292e^{-1}
0.1	0.856507396	1.229502553	0.864522387	1.241007939	0.8014988e^{-2}	0.11505386e^{-1}
0.2	0.884136669	1.215687917	0.892410212	1.227064029	0.8273536e^{-2}	0.11376112e^{-1}
0.3	0.911765938	1.201873282	0.920298022	1.213120124	0.8532084e^{-2}	0.11246838e^{-1}
0.4	0.939339520	1.188058646	0.948185841	1.199176221	0.8790632e^{-2}	0.11117565e^{-1}
0.5	0.967024487	1.174244020	0.976073664	1.185232310	0.9049180e^{-2}	0.10988290e^{-1}
0.6	0.994653753	1.160429376	1.0039614797	1.171288392	0.9307729e^{-2}	0.10859016e^{-1}
0.7	1.022283024	1.146614747	1.031849297	1.157344485	0.9566276e^{-2}	0.10729742e^{-1}
0.8	1.049912292	1.132800115	1.059737116	1.143400573	0.9824824e^{-2}	0.10600468e^{-1}
0.9	1.077541563	1.118985469	1.087624938	1.129456664	0.1008337e^{-2}	0.10471194e^{-1}
1	1.105170834	1.105170834	1.115512754	1.115512754	0.1034192e^{-2}	0.10341920e^{-1}

Graphical Representation of Exact and Approximate Solution in the interval $[1, 1.15]$



5. Conclusion

In this paper, a simple yet convincing method was established by considering the hybrid fuzzy differential equations and solved by applying Taylor series under generalized Hukuhara differentiability. We proved that hybrid fuzzy differential equation initial value problem can be numerically solved using Taylor series explicit numerically stable method for ODEs by first converting the original system to a hybrid system of ordinary differential equations. The result shows that our proposed is a simple and reliable method because this method is useful for finding a concrete approximation of the exact solution of hybrid fuzzy differential equations and the same can be used for solving higher order equations.

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