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GENERALIZED HYERS-ULAM STABILITY FOR A MIXED ADDITIVE-QUADRATIC-CUBIC-QUARTIC (AQCQ) FUNCTIONAL EQUATION IN QUASI-BANACH SPACES

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Abstract: In this paper we establish the general solution and investigate the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation

$$\begin{aligned} &f(3x+2y+z)+f(3x+2y-z)+f(3x-2y+z)+f(3x-2y-z)\\ &=48[f(x+y)+f(x-y)]+24[f(-x+y)+f(-x-y)]\\ &\quad+12[f(x+z)+f(x-z)]+6[f(-x+z)+f(-x-z)]\\ &\quad+4[\tilde{f}(y+z)+\tilde{f}(y-z)]+20f(2x)+4f(-2x)-160f(x)\\ &\quad-80f(-x)+2\tilde{f}(2y)-80\tilde{f}(y)-24\tilde{f}(z) \end{aligned} \tag{1}$$

in quasi-Banach spaces where $\tilde{f}(x)=f(x)+f(-x)$.

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1. Introduction and Preliminaries

Ulam [37] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. A very famous question concerning the stability of homomorphisms: *Let* $(\mathcal{G}_1, *)$ *be a group and let* $(\mathcal{G}_2, \odot, d)$ *be a metric group with metric* $d(.,.)$. *Given* $\epsilon > 0$, *does there exist a* $\delta(\epsilon) > 0$ *such that if a mapping* $h : \mathcal{G}_1 \to \mathcal{G}_2$ *satisfies the inequality*

$$d(h(x * y), h(x) \odot h(y)) < \delta$$

for all $x, y \in \mathcal{G}_1$, *then there is a homomorphism* $H : \mathcal{G}_1 \to \mathcal{G}_2$ *with* $d(h(x), H(x)) < \epsilon$ *for all* $x \in \mathcal{G}_1$? Hyers [16] solved this problem for Banach spaces. Aoki [2] solved this problem for additive mappings. Th.M. Rassias [27] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded.* A generalization of all the above stability results was obtained by Găvruţa [15] by replacing the unbounded Cauchy difference by a general control function $\psi(x, y)$. The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2}$$

is called a *quadratic functional equation* and every solution of the quadratic functional equation is said to be a *quadratic mapping.* A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [35] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [10] proved the generalized Hyers-Ulam stability of the quadratic functional equation. K. Jun and H. Kim [18] considered the following functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{3}$$

which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping.* In [21], Lee et al. considered the following functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y) \tag{4}$$

which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping.* Balamurugan et al. ([3], [4], [5],[6]) introduced the following two mixed type of additive-cubic and quadratic-quartic functional equations

$$f(3x + 2y + z) + f(3x + 2y - z) + f(3x - 2y + z) + f(3x - 2y - z)$$

$$= 24[f(x+y)+f(x-y)] + 6[f(x+z)+f(x-z)] + 16f(2x) - 80f(x) \quad (5)$$

$$\begin{aligned} &f(3x+2y+z)+f(3x+2y-z)+f(3x-2y+z)+f(3x-2y-z) \\ &= 72[f(x+y)+f(x-y)] + 18[f(x+z)+f(x-z)] + 8f(y+z) \\ &\quad + 8f(y-z) + 24f(2x) + 4f(2y) - 240f(x) - 160f(y) - 48f(z) \quad (6) \end{aligned}$$

and they investigated the generalized Hyers-Ulam stability for the above two functional equations. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1], [8], [11], [12], [13], [14], [17], [19], [20], [22], [23], [25], [26], [28], [29], [30], [31], [32], [33], [36], [38], [39], [40]). It is easy to see that the mapping $f(x) = ax^4 + bx^3 + cx^2 + dx$ is a solution of the functional equation (1). The main purpose of this paper is to establish the general solution of Eq. (1) and investigate the generalized Hyers-Ulam stability for Eq. (1). We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1. (See [7],[34]). Let $\mathcal{X}$ be a real linear space. A quasi-norm on $\mathcal{X}$ is a real-valued function on $\mathcal{X}$ satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in \mathcal{X}$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in \mathcal{X}$.

(iii) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in \mathcal{X}$.

The pair $(\mathcal{X}, \|.\|)$ is called *quasi-normed* space if $\|.\|$ is a quasi-norm on $\mathcal{X}$. The smallest possible K is called the *modulus of concavity* of $\|.\|$. A *quasi-Banach* space is a complete quasi-normed space. A quasi-norm $\|.\|$ is called a p-norm $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in \mathcal{X}$. In this case, a quasi-Banach space is called a p-Banach space. By the Aoki-Rolewicz theorem [34] (see also [7]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

2. General Solutions of (1)

Throughout this section, $\mathcal{X}$ and $\mathcal{Y}$ will be real vector spaces.

Lemma 2. *[3] If a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation* (5) *for all $x, y, z \in \mathcal{X}$, then the mapping $g : \mathcal{X} \to \mathcal{Y}$ defined by $g(x) = f(2x) - 8f(x)$ for all $x \in \mathcal{X}$ is additive.*

Lemma 3. *[3] If a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation* (5) *for all $x, y, z \in \mathcal{X}$, then the mapping $h : \mathcal{X} \to \mathcal{Y}$ defined by $h(x) = f(2x) - 2f(x)$ for all $x \in \mathcal{X}$ is cubic.*

Lemma 4. *[5] If a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation* (6) *for all $x, y, z \in \mathcal{X}$, then the mapping $g : \mathcal{X} \to \mathcal{Y}$ defined by $g(x) = f(2x) - 16f(x)$ for all $x \in \mathcal{X}$ is quadratic.*

Lemma 5. *[5] If a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation* (6) *for all $x, y, z \in \mathcal{X}$, then the mapping $h : \mathcal{X} \to \mathcal{Y}$ defined by $h(x) = f(2x) - 4f(x)$ for all $x \in \mathcal{X}$ is quartic.*

Theorem 6. *If an odd mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation* (1) *for all $x, y, z \in \mathcal{X}$, then the mapping $f : \mathcal{X} \to \mathcal{Y}$ is a cubic-additive mapping.*

Proof. If f satisfies (1), then it satisfies (5) for all $x, y, z \in \mathcal{X}$ by the oddness of f. Hence, by Lemma 2, the mapping f_a defined by $f_a = f(2x) - 8f(x)$ is additive and by Lemma 9, f_c defined by $f_c = f(2x) - 2f(x)$ is cubic. So, $f(x) = \frac{1}{6} f_c(x) - \frac{1}{6} f_a(x)$ and therefore the Theorem follows. □

Theorem 7. *If an even mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation* (1) *for all $x, y, z \in \mathcal{X}$, then the mapping $f : \mathcal{X} \to \mathcal{Y}$ is a quartic-quadratic mapping.*

Proof. If f satisfies (1), then it satisfies (6) for all $x, y, z \in \mathcal{X}$ by the evenness of f. Hence, by Lemma 3 the mapping f_b defined by $f_b(x) = f(2x) - 16f(x)$ is quadratic and by Lemma 5, f_d defined by $f_d(x) = f(2x) - 4f(x)$ is quartic. So, $f(x) = \frac{1}{12} f_d(x) - \frac{1}{12} f_b(x)$ and therefore the Theorem follows. □

Theorem 8. *A mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation* (1) *if and only if there exist a unique symmetric multi-additive mapping $\mathcal{D} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$, a unique mapping $\mathcal{C} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$, a unique symmetric bi-additive mapping $\mathcal{B} : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ and a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ such that $f(x) = \mathcal{D}(x, x, x, x) + \mathcal{C}(x, x, x) + \mathcal{B}(x, x) + \mathcal{A}(x)$ for all $x \in \mathcal{X}$, where* *the mapping $\mathcal{C}$ is symmetric for each fixed one variable and is additive for fixed two variables.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ satisfies (1). We decompose f into the even part and odd part by putting $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ and $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in \mathcal{X}$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in \mathcal{X}$. It is easy to show that the mappings f_e and f_o satisfy (1). By Theorems 6 and 7, f_e and f_o are quartic-quadratic and cubic-additive respectively. Thus there exist a unique symmetric multi-additive mapping $\mathcal{D} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ and a unique symmetric bi-additive mapping $\mathcal{B} : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ such that $f_e(x) = \mathcal{D}(x, x, x, x) + \mathcal{B}(x, x)$ for all $x \in \mathcal{X}$ and also there exist a unique mapping $\mathcal{C} : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ and a unique additive mapping $\mathcal{A} : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ such that $f_o(x) = \mathcal{C}(x, x, x, x) + \mathcal{A}(x, x)$ for all $x \in \mathcal{X}$, where the mapping $\mathcal{C}$ is symmetric for each fixed one variable and is additive for fixed two variables. Hence, we get $f(x) = \mathcal{D}(x, x, x, x) + \mathcal{C}(x, x, x) + \mathcal{B}(x, x) + \mathcal{A}(x)$ for all $x \in \mathcal{X}$. The proof of the converse is trivial. $\square$

3. Stability of (1): Odd Case

Throughout this section, assume that $\mathcal{X}$ is a quasi-normed space with quasi-norm $\|.\|_{\mathcal{X}}$ and that $\mathcal{Y}$ is a $p-$Banach space with $p-$norm $\|.\|_{\mathcal{Y}}$. Let K be the modulus of concavity of $\|.\|_{\mathcal{Y}}$. In this section, using an idea of Găvruţa [15] we prove the stability of functional equation (1). For convenience we use the following abbreviation for a given mapping $f : \mathcal{X} \to \mathcal{Y}$:

$$\begin{aligned} Df(x,y,z) =& f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) \\ &+ f(3x-2y-z) - 48f(x+y) - 48f(x-y) \\ &- 24[f(-x+y) + f(-x-y)] - 12[f(x+z) + f(x-z)] \\ &- 6[f(-x+z) + f(-x-z)] - 4[\tilde{f}(y+z) + \tilde{f}(y-z)] \\ &- 20f(2x) - 4f(-2x) + 160f(x) \\ &+ 80f(-x) - 2\tilde{f}(2y) + 80\tilde{f}(y) - 24\tilde{f}(z) \end{aligned}$$

for all $x, y, z \in \mathcal{X}$ where $\tilde{f}(x) = f(x) + f(-x)$. we will use the following lemma in this section.

Lemma 9. *[24] Let $0 < p \leq 1$ and let $x_1, x_2, ..., x_n$ be non-negative real numbers. Then*

$$\left(\sum_{i=1}^{n} x_i \right)^p \leq \left(\sum_{i=1}^{n} x_i^p \right)$$

Theorem 10. *Let $j \in \{-1,1\}$ and $\psi_a, M_a : \mathcal{X}^3 \to [0,\infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi_a\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{4^{nj}} = 0 \quad \forall x,y,z \in \mathcal{X} \tag{7}$$

$$\text{and} \quad M_a(x,y,z) = \sum_{i=0}^{\infty} \frac{\psi_a^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{4^{pij}} < \infty \quad \forall x,y,z \in \mathcal{X}. \tag{8}$$

Suppose that an odd mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x,y,z)\|_{\mathcal{Y}} \le \psi_a(x,y,z) \tag{9}$$

for all $x,y,z \in \mathcal{X}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(2x) - 8f(x) - \mathcal{A}(x)\|_{\mathcal{Y}} \le \frac{K}{4}\left[\tilde{\psi_a}(x)\right]^{\frac{1}{p}} \quad \forall x \in \mathcal{X}, \tag{10}$$

where

$$\begin{aligned}\tilde{\psi_a}(x) =& M_a(x,2x,x) + K^p M_a(x,x,x) + \left(\frac{7}{2}\right)^p K^{2p} M_a(x,0,x)\\ &+ 6^p K^{2p} M_a(x,0,0) \quad \forall x \in \mathcal{X}.\end{aligned}$$

Proof. Assume that $j = 1$. Replacing (x,y,z) by $(x,2x,x)$, (x,x,x), $(x,0,x)$ and $(x,0,0)$ in (9), respectively and using oddness of f, we get the following inequalities

$$\|f(8x) + f(6x) - 24f(3x) - 23f(2x) + 104f(x)\|_{\mathcal{Y}} \le \psi_a(x,2x,x) \tag{11}$$

for all $x \in \mathcal{X}$.

$$\|f(6x) + f(4x) - 45f(2x) + 80f(x)\|_{\mathcal{Y}} \le \psi_a(x,x,x) \tag{12}$$

for all $x \in \mathcal{X}$.

$$\|2f(4x) - 20f(2x) + 32f(x)\|_{\mathcal{Y}} \le \psi_a(x,0,x) \tag{13}$$

for all $x \in \mathcal{X}$.

$$\|4f(3x) - 16f(2x) + 20f(x)\|_{\mathcal{Y}} \le \psi_a(x,0,0) \tag{14}$$

for all $x \in \mathcal{X}$. Let $g_o, \xi_a : \mathcal{X} \to \mathcal{Y}$ be mappings defined by $g_o(x) = f(2x) - 8f(x)$ for all $x \in \mathcal{X}$ and

$$\xi_a(x) = K[\psi_a(x,2x,x) + K\psi_a(x,x,x) + \left(\frac{7}{2}\right) K^2\psi_a(x,0,x)$$

$$+ 6K^2\psi_a(x,0,0)] \tag{15}$$

for all $x \in \mathcal{X}$. It follows from (11) – (15) that

$$\|f(8x) - 8f(4x) - 4f(2x) + 32f(x)\|_{\mathcal{Y}} \leq \xi_a(x) \tag{16}$$

for all $x \in \mathcal{X}$. Therefore (16) means

$$\|g_o(4x) - 4g_o(x)\|_{\mathcal{Y}} \leq \xi_a(x) \tag{17}$$

for all $x \in \mathcal{X}$. By Lemma 9, (7) and (8) we infer that

$$\sum_{i=0}^{\infty} \frac{\xi_a^p\left(4^i x\right)}{4^{pi}} < \infty, \qquad \lim_{n\to\infty} \frac{\xi_a\left(4^n x\right)}{4^n} = 0 \tag{18}$$

for all $x \in \mathcal{X}$. Replacing x by $4^n x$ in (17) and dividing both sides of (17) by 4^{n+1}, we get

$$\left\|\frac{1}{4^{n+1}} g_o(4^{n+1}x) - \frac{1}{4^n} g_o(4^n x)\right\|_{\mathcal{Y}} \leq \frac{1}{4^{n+1}} \xi_a(4^n x) \tag{19}$$

for all $x \in \mathcal{X}$ and all non-negative integers n. Since $\mathcal{Y}$ is a $p-$Banach space, we have

$$\begin{aligned} \left\|\frac{1}{4^{n+1}} g_o(4^{n+1}x) - \frac{1}{4^m} g_o(4^m x)\right\|_{\mathcal{Y}}^p &\leq \sum_{i=m}^{n} \left\|\frac{1}{4^{i+1}} g_o(4^{i+1}x) - \frac{1}{4^i} g_o(4^i x)\right\|_{\mathcal{Y}}^p \\ &\leq \frac{1}{4^p} \sum_{i=m}^{n} \frac{1}{4^{pi}} \xi_a^p(4^i x) \end{aligned} \tag{20}$$

for all $x \in \mathcal{X}$ and all non-negative integers n and m with $n \geq m$. Therefore we conclude from (18) and (20) that the sequence $\left\{\frac{1}{4^n} g_o(4^n x)\right\}$ is a Cauchy sequence in $\mathcal{Y}$ for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\left\{\frac{1}{4^n} g_o(4^n x)\right\}$ converges in $\mathcal{Y}$ for all $x \in \mathcal{X}$. So one can define the mapping $\mathcal{A} : X \to \mathcal{Y}$ by

$$\mathcal{A}(x) = \lim_{n\to\infty} \frac{g_o\left(4^n x\right)}{4^n} \tag{21}$$

for all $x \in \mathcal{X}$. Letting $m = 0$ and passing the limit $n \to \infty$ in (20) and applying Lemma 9, we get (10). Now, we show that $\mathcal{A}$ is an additive mapping. It follows from (18),(19) and (21) that

$$\|\mathcal{A}(4x) - 4\mathcal{A}(x)\|_{\mathcal{Y}} = \lim_{n\to\infty} \left\|\frac{1}{4^n} g_o(4^{n+1}x) - \frac{1}{4^{n-1}} g_o(4^n x)\right\|_{\mathcal{Y}}$$

$$=4\left\|\frac{1}{4^{n+1}}g_o(4^{n+1}x)-\frac{1}{4^n}g_o(4^nx)\right\|_{\mathcal{Y}}$$
$$=\lim_{n\to\infty}\frac{\xi_a\left(4^nx\right)}{4^n}=0$$

for all $x\in\mathcal{X}$. So

$$\mathcal{A}(4x)=4\mathcal{A}(x)\quad\forall x\in\mathcal{X}. \tag{22}$$

On the other hand it follows from (7), (9) and (21) that

$$\begin{aligned}
&\left\|D\mathcal{A}(x,y,z)\right\|_{\mathcal{Y}}\\
&=\lim_{n\to\infty}\frac{1}{4^n}\left\|Dg_o(4^nx,4^ny,4^nz)\right\|_{\mathcal{Y}}\\
&=\lim_{n\to\infty}\frac{1}{4^n}\left\|Df(4^{n+1}x,4^{n+1}y,4^{n+1}z)-8Df(4^nx,4^ny,4^nz)\right\|_{\mathcal{Y}}\\
&\le\lim_{n\to\infty}\frac{K}{4^n}\left\|Df(4^n(2x),4^n(2y),4^n(2z))\|_{\mathcal{Y}}+8\|Df(4^nx,4^ny,4^nz)\right\|_{\mathcal{Y}}\\
&\le\lim_{n\to\infty}\frac{K}{4^n}\left[\psi_a\left(4^n(2x),4^n(2y),4^n(2z)\right)+8\psi_a(4^nx,4^ny,4^nz)\right]=0
\end{aligned}$$

for all $x,y,z\in\mathcal{X}$. Hence the mapping $\mathcal{A}$ satisfies (1). So by Theorem 6, the mapping $x\mapsto\mathcal{A}(4x)-64\mathcal{A}(x)$ is cubic-additive. Therefore (22) implies that the mapping $\mathcal{A}$ is additive. To prove the uniqueness of $\mathcal{A}$, let $\mathcal{S}:\mathcal{X}\to\mathcal{Y}$ be another additive mapping satisfying (10). It follows from (7) and (8) that

$$\lim_{n\to\infty}\frac{1}{4^{np}}M_a(4^nx,4^ny,4^nz)=\lim_{n\to\infty}\sum_{i=n}^{\infty}\frac{1}{4^{ip}}\psi_a^p(4^nx,4^ny,4^nz)=0$$

for all $x,y,z\in\mathcal{X}$. Hence $\lim\limits_{n\to\infty}\frac{1}{4^{np}}\tilde{\psi}_a(4^nx)=0$ for all $x\in\mathcal{X}$. So it follows from (10) and (21) that

$$\begin{aligned}
\left\|\mathcal{A}(x)-\mathcal{S}(x)\right\|_{\mathcal{Y}}^p&=\lim_{n\to\infty}\frac{1}{4^{np}}\left\|g_o(4^nx)-\mathcal{S}(4^nx)\right\|_{\mathcal{Y}}^p\\
&\le\frac{K^p}{4^p}\lim_{n\to\infty}\tilde{\psi}_a(4^nx)=0
\end{aligned}$$

for all $x\in\mathcal{X}$. So $\mathcal{A}=\mathcal{S}$. Hence the theorem holds for $j=1$. For $j=-1$, we can prove the theorem by a similar technique. □

Corollary 11. *Let ν,r,s and t be non-negative real numbers such that r,s and t are all $\neq 1$. Suppose that an odd mapping $f:\mathcal{X}\to\mathcal{Y}$ with $f(0)=0$*

satisfies the inequality

$$\|Df(x,y,z)\|_{\mathcal{Y}} \leq \begin{cases} \nu, & \\ \nu \|x\|_{\mathcal{X}}^{r}, & r>0, s=0, t=0; \\ \nu \|y\|_{\mathcal{X}}^{s}, & r=0, s>0, t=0; \\ \nu \|z\|_{\mathcal{X}}^{t}, & r=0, s=0, t>0; \\ \nu \left\{ \|x\|_{\mathcal{X}}^{r} + \|y\|_{\mathcal{X}}^{s} + \|z\|_{\mathcal{X}}^{t} \right\}, & r>0, s>0, t>0; \end{cases} \tag{23}$$

for all $x, y, z \in \mathcal{X}$. *Then there exists a unique additive mapping* $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(2x) - 8f(x) - \mathcal{A}(x)\|_{\mathcal{Y}} \leq \begin{cases} \alpha_a, & \\ \beta_a(x), & r>0, s=0, t=0; \\ \gamma_a(x), & r=0, s>0, t=0; \\ \delta_a(x), & r=0, s=0, t>0; \\ \zeta_a(x), & r>0, s>0, t>0; \end{cases} \tag{24}$$

for all $x \in \mathcal{X}$, *where*

$$\alpha_a = K\nu \left\{ \frac{1 + K^p + \left(\frac{7}{2}\right)^p K^{2p} + 6^p K^{2p}}{|4^p - 1|} \right\}^{\frac{1}{p}},$$

$$\beta_a(x) = K\nu \left(\frac{4^r}{4}\right) \left\{ \frac{1 + K^p + \left(\frac{7}{2}\right)^p K^{2p} + 6^p K^{2p}}{|4^p - 4^{pr}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^{r},$$

$$\gamma_a(x) = K\nu \left(\frac{4^s}{4}\right) \left\{ \frac{2^{ps} + K^p}{|4^p - 4^{ps}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^{s},$$

$$\delta_a(x) = K\nu \left(\frac{4^t}{4}\right) \left\{ \frac{1 + K^p + \left(\frac{7}{2}\right)^p K^{2p}}{|4^p - 4^{pt}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^{t}$$

$$\text{and} \quad \zeta_a(x) = \{\beta_a^p(x) + \gamma_a^p(x) + \delta_a^p(x)\}^{\frac{1}{p}} \quad \text{for all } x \in \mathcal{X}.$$

Corollary 12. *Let* $\nu \geq 0$ *and* r, s *and* t *which are all* > 0 *be real numbers such that* $\lambda = r + s + t \neq 1$. *Suppose that an odd mapping* $f : \mathcal{X} \to \mathcal{Y}$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|Df(x,y,z)\|_{\mathcal{Y}} \leq \begin{cases} \nu \ \|x\|_{\mathcal{X}}^{r} \|y\|_{\mathcal{X}}^{s} \|z\|_{\mathcal{X}}^{t} \\ \nu \left(\|x\|_{\mathcal{X}}^{r} \|y\|_{\mathcal{X}}^{s} \|z\|_{\mathcal{X}}^{t} + \|x\|_{\mathcal{X}}^{\lambda} + \|y\|_{\mathcal{X}}^{\lambda} + \|z\|_{\mathcal{X}}^{\lambda} \right) \end{cases} \tag{25}$$

for all $x, y, z \in \mathcal{X}$. *Then there exists a unique additive mapping* $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(2x) - 8f(x) - \mathcal{A}(x)\|_{\mathcal{Y}} \leq \begin{cases} \rho_a(x), \\ \tau_a(x) \end{cases} \tag{26}$$

for all $x \in \mathcal{X}$*, where*

$$\rho_a(x) = K\nu\left(\frac{4^\lambda}{4}\right)\left\{\frac{2^{ps}+K^p}{|4^p-4^{p\lambda}|}\right\}^{\frac{1}{p}}\|x\|_{\mathcal{X}}^{\lambda} \quad \text{and}$$

$$\tau_a(x) = K\nu\left(\frac{4^\lambda}{4}\right)\left\{\frac{2+2^{ps}+2^{p\lambda}+4K^p+2\left(\frac{7}{2}\right)^p K^{2p}+6^pK^{2p}}{|4^p-4^{p\lambda}|}\right\}^{\frac{1}{p}}\|x\|_{\mathcal{X}}^{\lambda}$$

for all $x \in \mathcal{X}$

Similarly, we can obtain the following. We will omit the proof.

Theorem 13. *Let* $j \in \{-1, 1\}$ *and* $\psi_c, M_c : \mathcal{X}^3 \to [0, \infty)$ *be mappings such that*

$$\lim_{n\to\infty}\frac{\psi_c\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{64^{nj}} = 0 \quad \forall x, y, z \in \mathcal{X} \tag{27}$$

$$\text{and} \quad M_c(x,y,z) = \sum_{i=0}^{\infty}\frac{\psi_c^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{64^{pij}} < \infty \quad \forall x, y, z \in \mathcal{X}. \tag{28}$$

Suppose that an odd mapping $f : \mathcal{X} \to \mathcal{Y}$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|Df(x,y,z)\|_{\mathcal{Y}} \le \psi_c(x,y,z) \tag{29}$$

for all $x, y, z \in \mathcal{X}$*. Then there exists a unique cubic mapping* $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(2x) - 2f(x) - \mathcal{C}(x)\|_{\mathcal{Y}} \le \frac{K}{64}[\tilde{\psi}_c(x)]^{\frac{1}{p}} \tag{30}$$

for all $x \in \mathcal{X}$*, where*

$$\tilde{\psi}_c(x) = M_c(x,2x,x) + K^pM_c(x,x,x) + \left(\frac{1}{2}\right)^p K^{2p}M_c(x,0,x) + 6^pK^{2p}M_c(x,0,0) \quad \forall x \in \mathcal{X}$$

Corollary 14. *Let* ν, r, s *and* t *be non-negative real numbers such that* r, s *and* t *are all* $\neq 3$*. Suppose that an odd mapping* $f : \mathcal{X} \to \mathcal{Y}$ *with* $f(0) = 0$ *satisfies the inequality* (23) *for all* $x, y, z \in \mathcal{X}$*. Then there exists a unique cubic mapping* $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(2x) - 2f(x) - \mathcal{C}(x)\|_{\mathcal{Y}} \le \begin{cases} \alpha_c, & \\ \beta_c(x), & r > 0, s = 0, t = 0; \\ \gamma_c(x), & r = 0, s > 0, t = 0; \\ \delta_c(x), & r = 0, s = 0, t > 0; \\ \zeta_c(x), & r > 0, s > 0, t > 0; \end{cases} \tag{31}$$

for all $x \in \mathcal{X}$, *where*

$$\alpha_c = K\nu \left\{ \frac{1 + K^p + \left(\frac{1}{2}\right)^p K^{2p} + 6^p K^{2p}}{|64^p - 1|} \right\}^{\frac{1}{p}},$$

$$\beta_c(x) = K\nu \left(\frac{4^r}{64}\right) \left\{ \frac{1 + K^p + \left(\frac{1}{2}\right)^p K^{2p}}{|64^p - 4^{pr}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^r,$$

$$\gamma_c(x) = K\nu \left(\frac{4^s}{64}\right) \left\{ \frac{2^{ps} + K^p}{|64^p - 4^{ps}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^s,$$

$$\delta_c(x) = K\nu \left(\frac{4^t}{64}\right) \left\{ \frac{1 + K^p + \left(\frac{1}{2}\right)^p K^{2p}}{|64^p - 4^{pt}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^t$$

$$\text{and} \quad \zeta_c(x) = \{\beta_c^p(x) + \gamma_c^p(x) + \delta_c^p(x)\}^{\frac{1}{p}} \quad \text{for all } x \in \mathcal{X}.$$

Corollary 15. *Let* $\nu \geq 0$ *and* r, s *and* t *which are all* > 0 *be real numbers such that* $\lambda = r + s + t \neq 3$. *Suppose that an odd mapping* $f : \mathcal{X} \to \mathcal{Y}$ *with* $f(0) = 0$ *satisfies the inequality* (25) *for all* $x, y, z \in \mathcal{X}$. *Then there exists a unique cubic mapping* $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(2x) - 2f(x) - \mathcal{C}(x)\|_{\mathcal{Y}} \leq \begin{cases} \rho_c(x), \\ \tau_c(x) \end{cases} \tag{32}$$

for all $x \in \mathcal{X}$, *where*

$$\rho_c(x) = K\nu \left(\frac{4^\lambda}{64}\right) \left\{ \frac{2^{ps} + K^p}{|64^p - 4^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^\lambda \quad \text{and}$$

$$\tau_c(x) = K\nu \left(\frac{4^\lambda}{64}\right) \left\{ \frac{2 + 2^{ps} + 2^{p\lambda} + 4K^p + 2\left(\frac{1}{2}\right)^p K^{2p} + 6^p K^{2p}}{|64^p - 4^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^\lambda$$

for all $x \in \mathcal{X}$.

Theorem 16. *Let* $j \in \{-1, 1\}$ *and* $\psi, M_a, M_c : \mathcal{X}^3 \to [0, \infty)$ *be mappings such that*

$$\lim_{n\to\infty} \frac{\psi\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{4^{nj}} = 0 = \lim_{n\to\infty} \frac{\psi\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{64^{nj}}, \tag{33}$$

$$M_a(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{4^{pij}} < \infty \quad \text{and}$$

$$M_c(x,y,z) = \sum_{i=0}^{\infty} \frac{\psi^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{64^{pij}} < \infty \tag{34}$$

for all $x, y, z \in \mathcal{X}$. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x,y,z)\|_{\mathcal{Y}} \le \psi(x,y,z) \tag{35}$$

for all $x, y, z \in \mathcal{X}$. Then there exist a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ and a unique cubic mapping $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - \mathcal{A}(x) - \mathcal{C}(x)\|_{\mathcal{Y}} \le \frac{K^2}{384}\left\{16[\tilde{\psi}_a(x)]^{\frac{1}{\mathsf{p}}} + [\tilde{\psi}_c(x)]^{\frac{1}{\mathsf{p}}}\right\} \quad \forall x \in \mathcal{X}, \tag{36}$$

where $\tilde{\psi}_a(x)$ and $\tilde{\psi}_c(x)$ for all $x \in \mathcal{X}$ are defined as in Theorems 10 and 13 respectively.

Proof. Let $j = 1$. By Theorems 10 and 13, there exist an additive mapping $\mathcal{A}_0 : \mathcal{X} \to \mathcal{Y}$ and a cubic mapping $\mathcal{C}_0 : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(2x) - 8f(x) - \mathcal{A}_0(x)\|_{\mathcal{Y}} \le \frac{K}{4}[\tilde{\psi}_a(x)]^{\frac{1}{\mathsf{p}}} \qquad \text{and}$$

$$\|f(2x) - 2f(x) - \mathcal{C}_0(x)\|_{\mathcal{Y}} \le \frac{K}{64}[\tilde{\psi}_c(x)]^{\frac{1}{\mathsf{p}}}, \qquad \forall x \in \mathcal{X}.$$

Therefore it follows from the last two inequalities that

$$\left\|f(x) + \frac{1}{6}\mathcal{A}_0(x) - \frac{1}{6}\mathcal{C}_0(x)\right\|_{\mathcal{Y}} \le \frac{K^2}{384}\left\{[16\tilde{\psi}_a(x)]^{\frac{1}{\mathsf{p}}} + [\tilde{\psi}_c(x)]^{\frac{1}{\mathsf{p}}}\right\}$$

for all $x \in \mathcal{X}$. So we obtain (36) by letting $\mathcal{A}(x) = -\frac{1}{6}\mathcal{A}_0(x)$ and $\mathcal{C}(x) = \frac{1}{6}\mathcal{C}_0(x), \forall x \in \mathcal{X}$. To prove the uniqueness of $\mathcal{A}$ and $\mathcal{C}$, let $\mathcal{A}_1, \mathcal{C}_1 : \mathcal{X} \to \mathcal{Y}$ be another additive and cubic mappings satisfying (36). Let $\mathcal{A}' = \mathcal{A} - \mathcal{A}_1$ and $\mathcal{C}' = \mathcal{C} - \mathcal{C}_1$. So

$$\begin{aligned}
&\left\|\mathcal{A}'(x) + \mathcal{C}'(x)\right\|_{\mathcal{Y}} \\
&\quad \le K[\|f(x) - \mathcal{A}(x) - \mathcal{C}(x)\|_{\mathcal{Y}} + \|f(x) - \mathcal{A}_1(x) - \mathcal{C}_1(x)\|_{\mathcal{Y}}] \\
&\quad \le \frac{K^3}{192}\left\{[16\tilde{\psi}_a(x)]^{\frac{1}{\mathsf{p}}} + [\tilde{\psi}_c(x)]^{\frac{1}{\mathsf{p}}}\right\}, \quad \forall x \in \mathcal{X}.
\end{aligned} \tag{37}$$

Since $\lim_{n\to\infty} \frac{1}{64^{np}}\tilde{\psi}_c(x) = \frac{1}{4^{np}}\tilde{\psi}_a(x) = 0$, for all $x \in \mathcal{X}$, then (37) implies that $\lim_{n\to\infty} \frac{1}{64^n}\left\|\mathcal{A}'(4^n x) + \mathcal{C}'(4^n x)\right\|_{\mathcal{Y}} = 0$, for all $x \in \mathcal{X}$. Therefore $\mathcal{C}' = 0$. So, it follows from (37) that $\|\mathcal{A}'(x)\|_{\mathcal{Y}} \le \frac{17K^3}{192}[\tilde{\psi}_a(x)]^{\frac{1}{\mathsf{p}}}$ for all $x \in \mathcal{X}$. Therefore $\mathcal{A}' = 0$. For $j = -1$, we can prove a similar stability result. □

Corollary 17. *Let ν, r, s and t be non-negative real numbers such that r, s and t are all $\neq 1$ and 3. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality* (23) *for all $x, y, z \in \mathcal{X}$. Then there exist a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ and a unique cubic mapping $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - \mathcal{A}(x) - \mathcal{C}(x)\|_{\mathcal{Y}} \leq \frac{K}{6}\begin{cases} \alpha_a + \alpha_c, & \\ \beta_a(x) + \beta_c(x), & r > 0, s = 0, t = 0; \\ \gamma_a(x) + \gamma_c(x), & r = 0, s > 0, t = 0; \\ \delta_a(x) + \delta_c(x), & r = 0, s = 0, t > 0; \\ \zeta_a(x) + \zeta_c(x), & r > 0, s > 0, t > 0; \end{cases}$$

for all $x \in \mathcal{X}$, where α_a, α_c, $\beta_a(x)$, $\beta_c(x)$, $\gamma_a(x)$, $\gamma_c(x)$,$\delta_a(x)$, $\delta_c(x)$,$\zeta_a(x)$ and $\zeta_c(x)$ are defined as in Corollaries 11 and 14

Corollary 18. *Let $\nu \geq 0$ and r, s and t which are all > 0 be real numbers such that $\lambda = r + s + t \neq 1$ and 3. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality* (25) *for all $x, y, z \in \mathcal{X}$. Then there exist a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ and a unique cubic mapping $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - \mathcal{A}(x) - \mathcal{C}(x)\|_{\mathcal{Y}} \leq \frac{K}{6}\begin{cases} \rho_a(x) + \rho_c(x), \\ \tau_a(x) + \tau_c(x) \end{cases} \tag{38}$$

for all $x \in \mathcal{X}$, where $\rho_a(x), \rho_c(x), \tau_a(x), \tau_c(x)$ are defined as in Corollaries 12 and 15

4. Stability of (1): Even Case

Theorem 19. *Let $j \in \{-1, 1\}$ and $\psi_b, M_b : \mathcal{X}^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi_b\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{16^{nj}} = 0, \quad \forall x, y, z \in \mathcal{X}, \quad \text{and} \tag{39}$$

$$M_b(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi_b^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{16^{pij}} < \infty, \quad \forall x, y, z \in \mathcal{X}. \tag{40}$$

Suppose that an even mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x, y, z)\|_{\mathcal{Y}} \leq \psi_b(x, y, z), \quad \forall x, y, z \in \mathcal{X}. \tag{41}$$

Then there exists a unique quadratic mapping $\mathcal{B}:\mathcal{X}\to\mathcal{Y}$ such that

$$\|f(2x)-16f(x)-\mathcal{B}(x)\|_{\mathcal{Y}} \le \frac{K}{16}\left[\tilde{\psi_b}(x)\right]^{\frac{1}{p}} \tag{42}$$

for all $x\in\mathcal{X}$, where

$$\begin{aligned}\tilde{\psi_b}(x) =& M_b(x,2x,x)+K^pM_b(x,x,x)+\left(\frac{11}{2}\right)^p K^{2p}M_b(x,0,x)\\ &+20^pK^{2p}M_b(x,0,0) \quad \forall x\in\mathcal{X}.\end{aligned}$$

Proof. Assume that $j=1$. Replacing (x,y,z) by $(x,2x,x)$, (x,x,x), $(x,0,x)$ and $(x,0,0)$ in (36), respectively and using evenness of f, we get the following inequalities

$$\begin{aligned}&\|f(8x)+f(6x)-4f(4x)-80f(3x)+119f(2x)+202f(x)\|_{\mathcal{Y}}\\ &\qquad\le \psi_b\left(x,2x,x\right), \forall x\in\mathcal{X}.\end{aligned} \tag{43}$$

$$\begin{aligned}&\|f(6x)+f(4x)-125f(2x)+448f(x)\|_{\mathcal{Y}}\\ &\qquad\le \psi_b\left(x,x,x\right), \forall x\in\mathcal{X}.\end{aligned} \tag{44}$$

$$\|2f(4x)-40f(2x)+128f(x)\|_{\mathcal{Y}} \le \psi_b\left(x,0,x\right), \quad \forall x\in\mathcal{X}. \tag{45}$$

$$\|4f(3x)-24f(2x)+60f(x)\|_{\mathcal{Y}} \le \psi_b\left(x,0,0\right), \quad \forall x\in\mathcal{X}. \tag{46}$$

Let $g_e,\xi_b:\mathcal{X}\to\mathcal{Y}$ be mappings defined by $g_e(x)=f(2x)-16f(x)$ for all $x\in\mathcal{X}$ and

$$\begin{aligned}\xi_b(x) =& K[\psi_b(x,2x,x)+K\psi_b(x,x,x)+\left(\frac{11}{2}\right)K^2\psi_b(x,0,x)\\ &+20K^2\psi_b(x,0,0)]\end{aligned} \tag{47}$$

for all $x\in\mathcal{X}$. It follows from (43) – (47) that

$$\|f(8x)-16f(4x)-16f(2x)+256f(x)\|_{\mathcal{Y}} \le \xi_b(x), \forall x\in\mathcal{X}. \tag{48}$$

Therefore (48) means

$$\|g_e(4x)-16g_e(x)\|_{\mathcal{Y}} \le \xi_b(x), \quad \forall x\in\mathcal{X}. \tag{49}$$

The rest of the proof is similar to the proof of Theorem 13 □

Corollary 20. *Let ν, r, s and t be non-negative real numbers such that r, s and t are all $\neq 2$. Suppose that an even mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality (23) for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(2x) - 16f(x) - \mathcal{B}(x)\|_{\mathcal{Y}} \leq \begin{cases} \alpha_b, & \\ \beta_b(x), & r > 0, s = 0, t = 0; \\ \gamma_b(x), & r = 0, s > 0, t = 0; \\ \delta_b(x), & r = 0, s = 0, t > 0; \\ \zeta_b(x), & r > 0, s > 0, t > 0; \end{cases} \tag{50}$$

for all $x \in \mathcal{X}$, where

$$\alpha_b = K\nu \left\{ \frac{1 + K^p + \left(\frac{11}{2}\right)^p K^{2p} + 20^p K^{2p}}{|16^p - 1|} \right\}^{\frac{1}{p}},$$

$$\beta_b(x) = K\nu \left(\frac{4^r}{16}\right) \left\{ \frac{1 + K^p + \left(\frac{11}{2}\right)^p K^{2p} + 20^p K^{2p}}{|16^p - 4^{pr}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^r,$$

$$\gamma_b(x) = K\nu \left(\frac{4^s}{16}\right) \left\{ \frac{2^{ps} + K^p}{|16^p - 4^{ps}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^s,$$

$$\delta_b(x) = K\nu \left(\frac{4^t}{16}\right) \left\{ \frac{1 + K^p + \left(\frac{11}{2}\right)^p K^{2p}}{|16^p - 4^{pt}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^t \quad \text{and}$$

$$\zeta_b(x) = \{\beta_b^p(x) + \gamma_b^p(x) + \delta_b^p(x)\}^{\frac{1}{p}} \quad \text{for all } x \in \mathcal{X}.$$

Corollary 21. *Let $\nu \geq 0$ and r, s and t which are all > 0 be real numbers such that $\lambda = r + s + t \neq 2$. Suppose that an even mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality (25) for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(2x) - 16f(x) - \mathcal{B}(x)\|_{\mathcal{Y}} \leq \begin{cases} \rho_b(x), \\ \tau_b(x) \end{cases} \tag{51}$$

for all $x \in \mathcal{X}$, where

$$\rho_b(x) = K\nu \left(\frac{4^\lambda}{16}\right) \left\{ \frac{2^{ps} + K^p}{|16^p - 4^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^\lambda \quad \text{and}$$

$$\tau_b(x) = K\nu \left(\frac{4^\lambda}{16}\right) \left\{ \frac{2 + 2^{ps} + 2^{p\lambda} + 4K^p + 2\left(\frac{11}{2}\right)^p K^{2p} + 20^p K^{2p}}{|16^p - 4^{p\lambda}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^\lambda,$$

for all $x \in \mathcal{X}$.

Similarly, we can obtain the following. We will omit the proof.

Theorem 22. *Let $j \in \{-1, 1\}$ and $\psi_d, M_d : \mathcal{X}^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{\psi_d\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{256^{nj}} = 0, \quad \forall x, y, z \in \mathcal{X}, \quad \text{and} \tag{52}$$

$$M_d(x, y, z) = \sum_{i=0}^{\infty} \frac{\psi_d^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{256^{pij}} < \infty, \quad \forall x, y, z \in \mathcal{X}. \tag{53}$$

Suppose that an even mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x, y, z)\|_{\mathcal{Y}} \leq \psi_d\left(x, y, z\right), \quad \forall x, y, z \in \mathcal{X}. \tag{54}$$

Then there exists a unique quartic mapping $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(2x) - 4f(x) - \mathcal{D}(x)\|_{\mathcal{Y}} \leq \frac{K}{256}[\tilde{\psi_d}(x)]^{\frac{1}{p}} \tag{55}$$

for all $x \in \mathcal{X}$, where

$$\begin{aligned}\tilde{\psi_d}(x) =& M_d(x, 2x, x) + K^p M_d(x, x, x) + \left(\frac{1}{2}\right)^p K^{2p} M_d(x, 0, x) \\ &+ 20^p K^{2p} M_d(x, 0, 0) \qquad \forall x \in \mathcal{X}.\end{aligned}$$

Corollary 23. *Let ν, r, s and t be non-negative real numbers such that r, s and t are all $\neq 4$. Suppose that an even mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality (23) for all $x, y, z \in \mathcal{X}$. Then there exists a unique quartic mapping $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(2x) - 4f(x) - \mathcal{D}(x)\|_{\mathcal{Y}} \leq \begin{cases} \alpha_d, & \\ \beta_d(x), & r > 0, s = 0, t = 0; \\ \gamma_d(x), & r = 0, s > 0, t = 0; \\ \delta_d(x), & r = 0, s = 0, t > 0; \\ \zeta_d(x), & r > 0, s > 0, t > 0; \end{cases} \tag{56}$$

for all $x \in \mathcal{X}$, where

$$\alpha_d = K\nu \left\{ \frac{1 + K^p + \left(\frac{1}{2}\right)^p K^{2p} + 20^p K^p}{|256^p - 1|} \right\}^{\frac{1}{p}},$$

$$\beta_d(x) = K\nu \left(\frac{4^r}{256}\right) \left\{ \frac{1 + K^p + \left(\frac{1}{2}\right)^p K^{2p} + 20^p K^{2p}}{|256^p - 4^{pr}|} \right\}^{\frac{1}{p}} \|x\|_{\mathcal{X}}^r,$$

$$\gamma_d(x) = K\nu\left(\frac{4^s}{256}\right)\left\{\frac{2^{ps}+K^p}{|256^p-4^{ps}|}\right\}^{\frac{1}{p}}\|x\|_{\mathcal{X}}^s,$$

$$\delta_d(x) = K\nu\left(\frac{4^t}{256}\right)\left\{\frac{1+K^p+\left(\frac{1}{2}\right)^p K^{2p}}{|256^p-4^{pt}|}\right\}^{\frac{1}{p}}\|x\|_{\mathcal{X}}^t \qquad \text{and}$$

$$\zeta_d(x) = \{\beta_d^p(x)+\gamma_d^p(x)+\delta_d^p(x)\}^{\frac{1}{p}} \quad \text{for all } x \in \mathcal{X}.$$

Corollary 24. *Let $\nu \geq 0$ and r, s and t which are all > 0 be real numbers such that $\lambda = r+s+t \neq 4$. Suppose that an even mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality (25) for all $x, y, z \in \mathcal{X}$. Then there exists a unique quartic mapping $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(2x)-4f(x)-\mathcal{D}(x)\|_{\mathcal{Y}} \leq \begin{cases} \rho_d(x), \\ \tau_d(x) \end{cases} \tag{57}$$

for all $x \in \mathcal{X}$, where

$$\rho_d(x) = K\nu\left(\frac{4^\lambda}{256}\right)\left\{\frac{2^{ps}+K^p}{|256^p-4^{p\lambda}|}\right\}^{\frac{1}{p}}\|x\|_{\mathcal{X}}^\lambda \quad \text{and}$$

$$\tau_d(x) = K\nu\left(\frac{4^\lambda}{256}\right)\left\{\frac{2+2^{ps}+2^{p\lambda}+4K^p+2\left(\frac{1}{2}\right)^p K^{2p}+20^p K^{2p}}{|256^p-4^{p\lambda}|}\right\}^{\frac{1}{p}}\|x\|_{\mathcal{X}}^\lambda$$

for all $x \in \mathcal{X}$.

Theorem 25. *Let $j \in \{-1, 1\}$ and $\psi, M_b, M_d : \mathcal{X}^3 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty}\frac{\psi\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{16^{nj}} = 0 = \lim_{n\to\infty}\frac{\psi\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{256^{nj}}, \tag{58}$$

$$M_b(x,y,z) = \sum_{i=0}^{\infty}\frac{\psi^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{16^{pij}} < \infty \quad \text{and}$$

$$M_d(x,y,z) = \sum_{i=0}^{\infty}\frac{\psi^p\left(4^{ij}x, 4^{ij}y, 4^{ij}z\right)}{256^{pij}} < \infty, \tag{59}$$

for all $x, y, z \in \mathcal{X}$. Suppose that an even mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality (35) for all $x, y, z \in \mathcal{X}$. Then there exist a unique quadratic mapping $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$ and a unique quartic mapping $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x)-\mathcal{B}(x)-\mathcal{D}(x)\|_{\mathcal{Y}} \leq \frac{K^2}{3072}\left\{16[\tilde{\psi}_b(x)]^{\frac{1}{p}}+[\tilde{\psi}_d(x)]^{\frac{1}{p}}\right\} \tag{60}$$

for all $x \in \mathcal{X}$, where $\tilde{\psi_b}(x)$ and $\tilde{\psi_d}(x)$ for all $x \in \mathcal{X}$ are defined as in Theorems 19 and 22 respectively.

Proof. The proof is similar to the proof of the Theorem 16 □

Corollary 26. *Let ν, r, s and t be non-negative real numbers such that r, s and t are all $\neq 2$ and 4. Suppose that an even mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality (23) for all $x, y, z \in \mathcal{X}$. Then there exist a unique quadratic mapping $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$ and a unique quartic mapping $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - \mathcal{B}(x) - \mathcal{D}(x)\|_{\mathcal{Y}} \leq \frac{K}{12}\begin{cases} \alpha_b + \alpha_d, & \\ \beta_b(x) + \beta_d(x), & r > 0, s = 0, t = 0; \\ \gamma_b(x) + \gamma_d(x), & r = 0, s > 0, t = 0; \\ \delta_b(x) + \delta_d(x), & r = 0, s = 0, t > 0; \\ \zeta_b(x) + \zeta_d(x), & r > 0, s > 0, t > 0; \end{cases}$$

for all $x \in \mathcal{X}$, where α_b, α_d, $\beta_b(x)$, $\beta_d(x)$, $\gamma_b(x)$, $\gamma_d(x)$, $\delta_b(x)$, $\delta_d(x)$, $\zeta_b(x)$ and $\zeta_d(x)$ are defined as in Corollaries 20 and 23

Corollary 27. *Let $\nu \geq 0$ and r, s and t which are all > 0 be real numbers such that $\lambda = r + s + t \neq 2$ and 4. Suppose that an even mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality (25) for all $x, y, z \in \mathcal{X}$. Then there exist a unique quadratic mapping $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$ and a unique quartic mapping $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - \mathcal{B}(x) - \mathcal{D}(x)\|_{\mathcal{Y}} \leq \frac{K}{12}\begin{cases} \rho_b(x) + \rho_d(x), \\ \tau_b(x) + \tau_d(x) \end{cases} \tag{61}$$

for all $x \in \mathcal{X}$, where $\rho_b(x), \rho_d(x), \tau_b(x), \tau_d(x)$ are defined as in Corollaries 12 and 15

Theorem 28. *Let $j \in \{-1, 1\}$ and $\psi, M_a, M_b, M_c, M_d : \mathcal{X}^3 \to [0, \infty)$ be mappings which satisfy (33),(34),(58) and (59). Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality (35) for all $x, y, z \in \mathcal{X}$. Then there exist a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$, a unique quadratic mapping $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$, a unique cubic mapping $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ and a unique quartic mapping $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - \mathcal{A}(x) - \mathcal{B}(x) - \mathcal{C}(x) - \mathcal{D}(x)\|_{\mathcal{Y}}$$
$$\leq \frac{K^4}{6144}\Big\{ \left(128[\tilde{\psi_a}(x) + \tilde{\psi_a}(-x)]^{\frac{1}{\mathsf{p}}} + [\tilde{\psi_c}(x) + \tilde{\psi_c}(-x)]^{\frac{1}{\mathsf{p}}}\right)$$

$$+\left(16[\tilde{\psi}_b(x)+\tilde{\psi}_b(-x)]^{\frac{1}{p}}+[\tilde{\psi}_d(x)+\tilde{\psi}_d(-x)]^{\frac{1}{p}}\right)\Bigg\}$$

for all $x \in \mathcal{X}$, where $\tilde{\psi}_a(x)$, $\tilde{\psi}_b(x)$, $\tilde{\psi}_c(x)$ and $\tilde{\psi}_d(x)$ for all $x \in \mathcal{X}$ are defined as in Theorems 10, 19, 13 and 22 respectively.

Proof. Let $j = 1$. Let $f_e(x) = \frac{1}{2}[f(x)+f(-x)]$ for all $x \in \mathcal{X}$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in \mathcal{X}$ and also $\|Df_e(x,y,z)\| \leq \tilde{\psi}(x,y,z)$ for all $x,y,z \in \mathcal{X}$, where

$$\tilde{\psi}(x,y,z) = \frac{K}{2}[\psi(x,y,z)+\psi(-x,-y,-z)]$$

for all $x,y,z \in \mathcal{X}$. So $\tilde{\psi}$ satisfies (58) for all $x,y,z \in \mathcal{X}$. Since $\tilde{\psi}^p(x,y,z) \leq \frac{K^p}{2^p}[\psi^p(x,y,z)+\psi^p(-x,-y,-z)]$, $\tilde{\psi}$ satisfies (59) for all $x,y,z \in \mathcal{X}$. By Theorem 25, there exist a unique quadratic mapping $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$ and a unique quartic mapping $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f_e(x)-\mathcal{B}(x)-\mathcal{D}(x)\|_{\mathcal{Y}} \leq \frac{K^2}{3072}\left\{16[\tilde{\tilde{\psi}}_b(x)]^{\frac{1}{p}}+[\tilde{\tilde{\psi}}_d(x)]^{\frac{1}{p}}\right\} \tag{62}$$

for all $x \in \mathcal{X}$, where $\tilde{\tilde{\psi}}_b(x)$ and $\tilde{\tilde{\psi}}_d(x)$ are getting by replacing ψ by $\tilde{\psi}$ in $\tilde{\psi}_b(x)$ and $\tilde{\psi}_d(x)$ for all $x \in \mathcal{X}$ respectively and also where $\tilde{\psi}_b(x)$ and $\tilde{\psi}_d(x)$ are defined as in Theorems 19 and 22 respectively. It is clear that

$$\tilde{\tilde{\psi}}_b(x) \leq \frac{K^p}{2^p}[\tilde{\psi}_b(x)+\tilde{\psi}_b(-x)] \quad \text{and} \quad \tilde{\tilde{\psi}}_d(x) \leq \frac{K^p}{2^p}[\tilde{\psi}_d(x)+\tilde{\psi}_d(-x)]$$

for all $x \in \mathcal{X}$. Therefore it follows from (62) that

$$\begin{aligned}&\|f_e(x)-\mathcal{B}(x)-\mathcal{D}(x)\|_{\mathcal{Y}}\\&\quad\leq \frac{K^3}{6144}\left\{16[\tilde{\psi}_b(x)+\tilde{\psi}_b(-x)]^{\frac{1}{p}}+[\tilde{\psi}_d(x)+\tilde{\psi}_d(-x)]^{\frac{1}{p}}\right\}\end{aligned} \tag{63}$$

for all $x \in \mathcal{X}$. Let $f_o(x) = \frac{1}{2}[f(x)-f(-x)], \forall x \in \mathcal{X}$. By using the above method and the Theorem 16, it follows that there exist a unique cubic mapping $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ and a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned}&\|f_o(x)-\mathcal{A}(x)-\mathcal{C}(x)\|_{\mathcal{Y}}\\&\quad\leq \frac{K^3}{768}\left\{16[\tilde{\psi}_a(x)+\tilde{\psi}_a(-x)]^{\frac{1}{p}}+[\tilde{\psi}_c(x)+\tilde{\psi}_c(-x)]^{\frac{1}{p}}\right\}\end{aligned} \tag{64}$$

for all $x \in \mathcal{X}$. Hence the Theorem follows from (63) and (64). For $j = -1$, we can prove a similar stability result. □

Corollary 29. *Let ν, r, s and t be non-negative real numbers such that r, s and t are all $\neq 1, 2, 3$ and 4. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality* (23) *for all $x, y, z \in \mathcal{X}$. Then there exist a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$, a unique quadratic mapping $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$, a unique cubic mapping $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ and a unique quartic mapping $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - \mathcal{A}(x) - \mathcal{B}(x) - \mathcal{C}(x) - \mathcal{D}(x)\|_{\mathcal{Y}}$$
$$\leq \frac{K^2}{12} \begin{cases} 2(\alpha_a + \alpha_c) + \alpha_b + \alpha_d, & \\ 2(\beta_a(x) + \beta_c(x)) + \beta_b(x) + \beta_d(x), & r > 0, s = 0, t = 0; \\ 2(\gamma_a(x) + \gamma_c(x)) + \gamma_b(x) + \gamma_d(x), & r = 0, s > 0, t = 0; \\ 2(\delta_a(x) + \delta_c(x)) + \delta_b(x) + \delta_d(x), & r = 0, s = 0, t > 0; \\ 2(\zeta_a(x) + \zeta_c(x)) + \zeta_b(x) + \zeta_d(x), & r > 0, s > 0, t > 0; \end{cases}$$

for all $x \in \mathcal{X}$, where α_a, α_b, α_c, α_d, $\beta_a(x)$, $\beta_b(x)$, $\beta_c(x)$, $\beta_d(x)$, $\gamma_a(x)$, $\gamma_b(x)$, $\gamma_c(x)$, $\gamma_d(x)$, $\delta_a(x)$, $\delta_b(x)$, $\delta_c(x)$, $\delta_d(x)$, $\zeta_a(x)$, $\zeta_b(x)$, $\zeta_c(x)$ and $\zeta_d(x)$ are defined as in Corollaries 11,14,20 and 23

Corollary 30. *Let $\nu \geq 0$ and r, s, t which are all > 0 be real numbers such that $\lambda = r + s + t \neq 1, 2, 3$ and 4. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the inequality* (23) *for all $x, y, z \in \mathcal{X}$. Then there exist a unique additive mapping $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$, a unique quadratic mapping $\mathcal{B} : \mathcal{X} \to \mathcal{Y}$, a unique cubic mapping $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ and a unique quartic mapping $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x) - \mathcal{A}(x) - \mathcal{B}(x) - \mathcal{C}(x) - \mathcal{D}(x)\|_{\mathcal{Y}}$$
$$\leq \frac{K^2}{12} \begin{cases} 2(\rho_a(x) + \rho_c(x)) + \rho_b(x) + \rho_d(x), \\ 2(\tau_a(x) + \tau_c(x)) + \tau_b(x) + \tau_d(x) \end{cases} \tag{65}$$

for all $x \in \mathcal{X}$, where $\rho_a(x)$, $\rho_b(x)$, $\rho_c(x)$, $\rho_d(x)$, $\tau_a(x)$, $\tau_b(x)$, $\tau_c(x)$, and $\tau_d(x)$ are defined as in Corollaries 12, 15, 21 and 24

5. Conclusion

In this paper, we proved the Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (1) in quasi-Banach spaces.

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