International Journal of Pure and Applied Mathematics
Volume 106 No. 5 2016, 11-20
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v106i5.2

ijpam.eu

RETRIAL QUEUEING SYSTEM WITH RETENTION OF RENEGING CUSTOMERS

S. Pavai Madheswari[1], P. Suganthi[2], S.A. Josephine[3]
[1]Department of Computer Science and Engineering
[2]Department of Mathematics
[1,2]R.M.K. Engineering College
Kavaraipettai, 601 206, Tamil Nadu, INDIA
[3]Department of Mathematics
Loyala-ICAM College of Engineering and Technology
Nungambakkam, Chennai-34, INDIA

Abstract: Managing customer impatience plays a vital role in improving the efficiency of queueing systems. Reneging of impatient customers leads to loss of potential customers, which results in the loss of business. A reneged customer can be retained in many cases by employing various convincing strategies to continue in the queue until completion of service. In this paper we present an analysis of a single server retrial queue with retention of reneging customers from the orbit. The generating function technique has been used to derive the steady state probabilities of the system. Performance measures have been derived and system efficiency discussed by numerical results and graphical illustrations to demonstrate how the various parameters influence the behavior of the system. Some of the existing results have been deduced.

AMS Subject Classification: 60K25, 60K30, 90B22
Key Words: retrial queues, reneging, retention, Kummer confluent function

1. Introduction

Study of queueing systems helps to managing waiting lines and the construc-

Received: January 23, 2016
Published: February 7, 2016

tion of an optimal system to balance customer waiting time with the idle time of the server. Impatient customers are great importance as a lost customer is loss of business. On arrival if a customer finds the expected waiting time in the system to be more than his available time, he refuses to join the system is said to be balked. The reluctance of a customer to continue in queue after joining and waiting is termed as reneging. Researchers have studied widely the effect of balking and reneging on queueing systems. To increase the revenue of a firm, it is essential to bring back customers who leave the system due to impatience. Companies have worked out strategies to retain reneging customers. Researchers have been done on the retention of reneging customers in Markovian queueing models. The combined effects of balking and reneging in an M/M/c queue have been investigated by R.O. Al-Seedy, S.A. El-Shehawy [3]. E.R. Obert [6] has analyzed single server queues with reneging. Ancker and Gafarian [7] have studied M/M/1/N queuing system with balking and reneging. Krishna Kumar and Pavai Madheswari [9] have analyzed the M/G/1 retrial queues with feedback and starting failures. Amina Angelika Bouchentouf, Mokhtar Kadi, Abbes Rabhi [5] analysed the analytical solution of the M/M/2/N queue with balking, reneging and feedback. Medhi Pallabi, Choudhury Amit [10] presented explicit results for the M/M/c/c queueing system assuming that customers are reneging type and few re-designed performance measures also presented. Mahdy S. El-Paoumy, Hossam A. Nabwey [11] has analyzed the Markovian model with balking, reneging and heterogeneous servers with finite capacity. Dequan Yue, Yan Zhang, Wuyi Yue [4] analyzed an M/M/1/N queueing system with balking, reneging and server vacations. The detailed overview of the related references with retrial queue can be found in the book of Falin and Templeton [12], [13] and the survey papers, Artalejo[1],[2]. Retrial queueing with retention of reneging customers contains real-world situations more closely. Such models commonly occur in areas of data transfer via telephone networks, computer and communications and manufacturing systems. This motivates us to analyze the retrial queueing system with retention of reneging customer in the orbit.

Rest of this paper is organized as follows. In Section 2, the mathematical model is described. The steady state distribution of the server and the performance measures are derived in Section 3. In Section 4, we show that our results are consistent with those already known in literature when ν=1, ξ=0. Graphical illustrations and numerical results of the influence of the arrival rate, reneging rate with probability p and retrial rate parameters on the stability condition are investigated in Section 5. In Section 6, conclusion is given.

2. Model Description

We consider an M/M/1 retrial queuing system with retention of reneging impatient customer from the orbit. A primary customer who enters the queueing system, arrives according to a Poisson process with rate $\lambda > 0$ and receives service immediately if the server is idle, otherwise joins a pool of waiting customers called orbit. A secondary customer in the orbit repeatedly requests service with an exponentially distributed retrial time with rate $\nu > 0$. Service time follows exponential distribution with mean μ. Customers in the orbit are said to renege at a rate exponentially distributed with probability for reneging from the orbit p and probability for retention of a reneging customer to the orbit being q so that, p+q=1. If the server is not available for the customer before the impatience timer expires, the customer abandons the orbit i.e., the system never to return. To derive a system of differential equation for the probabilities $p_{in}(t)$, $i = 0, 1, n = 0, 1, 2, 3, \ldots,$ $p_{0n}(t)$=Pr[Server is idle and there are n customers in the orbit at time t], $p_{1n}(t)$=Pr[Server is busy and there are n customers in the orbit at time t]. The pair (i, n) represents possible states of the retrial queue with $i = 0$ or 1 corresponding to the server being idle or busy respectively, and the n being the number of customers in the orbit.

3. Steady State Probabilities for the System

In this section, the steady-state probabilities are obtained by the Markov process methods. Let $p_{in}(t)$ denotes the probability of having n number of customers in the orbit at time t, when the server is in state i. The differential-difference equations of the model are

$$p_{0n}^{'}(t) = -(\lambda + n\nu)\ p_{0n}(t) + \mu\ p_{1n}(t), \quad n \geq 0 \tag{1}$$

$$p_{10}^{'}(t) = -(\lambda + \mu)\ p_{10}(t) + \lambda\ p_{00}(t) + \nu\ p_{01}(t) + p\xi\ p_{11}(t), \quad n = 1 \tag{2}$$

$$\begin{aligned} p_{1n}^{'}(t) = &-(\lambda + \mu + np\xi)\ p_{1n}(t) + \lambda\ p_{1n-1}(t) + \lambda\ p_{0n}(t) \\ &+(n+1)\nu\ p_{0n+1}(t) + (n+1)p\xi\ p_{1n+1}(t), \quad n \geq 1. \end{aligned} \tag{3}$$

The colsed form of solution is obtained. when λ, μ, ν and ξ are constants, $\lambda < \mu$ when t becomes infinite. In the steady state, $p_{in}(t)$ is independent of time t. Therefore we obtained the following set of steady- state equations

$$(\lambda + n\nu)\ p_{0n} = \mu\ p_{1n}, \quad n \geq 0 \tag{4}$$

$$(\lambda + \mu)\ p_{10} = \lambda\ p_{00} + \nu\ p_{01} + p\xi\ p_{11}, \quad n = 1 \tag{5}$$

$$(\lambda{+}\mu{+}np\xi)\ p_{1n} = \lambda\ p_{1n-1}{+}\lambda\ p_{0n}{+}(n{+}1)\nu\ p_{0n+1}{+}(n{+}1)p\xi\ p_{1n+1}, \quad n \geq 1 \tag{6}$$

To solve the system of the equations, we follow the approach given in Falin and Templeton (1997) [8], Solving for p_{1n} in (4) and substituting the result into (5) gives

$$p_{1n} = \tfrac{[\lambda+n\nu]}{\mu} p_{0n}$$

substituting above in (6) this gives,

$$(\lambda + \mu + np\xi)\tfrac{[\lambda+n\nu]}{\mu} p_{0n} = \lambda\tfrac{[\lambda+(n-1)\nu]}{\mu} p_{0n-1} + \lambda\ p_{0n}$$

$$+(n+1)\nu\ p_{0n+1} + (n+1)p\xi\tfrac{[\lambda+(n+1)\nu]}{\mu} p_{0n+1}, \quad n \geq 1.$$

After algebraic simplification above system of equations is rearranged in the form of

$$(n+1)[\mu\nu + p\xi[\lambda + (n+1)\nu]]\ p_{0n+1} - \lambda(\lambda + n\nu)\ p_{0n} = n[\mu\nu$$

$$+p\xi(\lambda + \nu)]\ p_{0n+1} - \lambda(\lambda + (n-1)\nu)\ p_{0n-1}. \tag{7}$$

This can be rewritten as

$$x_{n+1}\ p_{0n+1} - y_n\ p_{0n} = x_n\ p_{0n} - y_{n-1}\ p_{0n-1} \tag{8}$$

where $x_n = n[\mu\nu + p\xi(\lambda + \nu)]$ and $y_n = \lambda(\lambda + n\nu)$, Let us assume in the (8) R.H.S is equal to C

$$x_n\ p_{0n} - y_{n-1}\ p_{0n-1} = C. \tag{9}$$

Therefore equation (8) becomes

$$x_{n+1}\ p_{0n+1} - y_n\ p_{0n} = C. \tag{10}$$

The constant C can be determined from (4)and (5) as folows: Solve p_{10} and p_{11} in (4) and substitute in (5). This gives

$$[\mu\nu + p\xi(\lambda + \nu)]\ p_{01} - \lambda^2\ p_{00} = 0$$

which can be writern as

$$x_1\ p_{01} - y_0\ p_{00} = 0 \tag{11}$$

substituting $n = 1$ (9) this gives

$$x_1\ p_{01} - y_0\ p_{00} = C \tag{12}$$

comparing equations (11) and (12), $C = 0$. Which is true for all $n \geq 1$

$$p_{0n+1} = (\frac{y_n}{x_{n+1}})\ p_{0n}$$

for some $n \geq 0$, $p_{0n} = \prod_{i=0}^{n-1} \frac{y_i}{x_{i+1}}\ p_{00}$

$$= \ p_{00}\frac{\lambda^n}{n!(p\xi)^n}\prod_{i=0}^{n-1}\frac{\frac{\lambda}{\nu}+i}{[\frac{\mu}{p\xi}+(\frac{\lambda}{\nu}+1+i)]}$$

$$p_{0n} = \frac{\lambda^n}{n!(p\xi)^n}\frac{(a)_n}{(b)_n}\ p_{00} \tag{13}$$

where $a = \frac{\lambda}{\nu}, b = \frac{\mu}{p\xi} + a + 1$, $(a)_n \equiv a(a+1)(a+2)........(a+n-1)$ is the rising factorial function. To obtain probability of n customers when the system is busy

$$p_{1n} = \frac{[\lambda + n\nu]}{\mu}\ p_{0n}$$

$$= \frac{\nu}{\mu}a\frac{\lambda^n}{n!(p\xi)^n}\frac{(a+1)(a+2)(a+3)........(a+n-1)(a+n)}{(b)_n}\ p_{00}$$

$$= \frac{\nu}{\mu}\frac{\lambda^n}{n!(p\xi)^n}\frac{(a+1)_n}{(b)_n}\ p_{00}$$

$$p_{1n} = \frac{\nu}{\mu}\frac{\lambda^n}{n!(p\xi)^n}\frac{(a+1)_n}{(b)_n}\ p_{00} \tag{14}$$

we defined the probability generating functions for the probabilities p_{0n} and p_{1n} as

$$P_0(z) = \sum_{n=0}^{\infty} p_{0n}z^n \quad \text{and} \quad P_1(z) = \sum_{n=0}^{\infty} p_{1n}z^n \tag{15}$$

substituting (13) and (14) in (15), we get

$$P_0(z) = \ p_{00} \sum_{n=0}^{\infty} \tfrac{\lambda^n}{n!(p\xi)^n} \tfrac{(a)_n}{(b)_n} z^n = \ p_{00} \sum_{n=0}^{\infty} \tfrac{(a)_n}{n!(b)_n} \tfrac{\lambda z}{(p\xi)^n}$$

$$P_1(z) = \ p_{00} \sum_{n=0}^{\infty} \tfrac{\lambda \lambda^n}{\mu n!(p\xi)^n} \tfrac{(a)_{n+1}}{(b)_n} z^n = \ p_{00} \tfrac{\lambda}{\mu} \sum_{n=0}^{\infty} \tfrac{(a+1)_n}{n!(b)_n} \tfrac{\lambda z}{(p\xi)^n}.$$

This can be writtern as

$$P_0(z) = \Phi(a, b; \frac{\lambda z}{(p\xi)^n})\ p_{00}, \qquad P_1(z) = \rho\Phi(a+1, b; \frac{\lambda z}{(p\xi)^n})\ p_{00} \tag{16}$$

where Φ is the Kummer confluent function $\Phi(a.b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{a^n}{n!}$, we can find p_{00} with normalizing condition $P_0(1) + P_1(1) = 1$, implying

$$p_{00} = \frac{1}{\Phi(a, b; \frac{\lambda}{p\xi}) + \rho\Phi(a+1, b; \frac{\lambda}{p\xi})}. \tag{17}$$

The various performance measures can be derived from the probability generating functions. The fraction of the time the server is busy is

$$p_1 \equiv \sum_{n=0}^{\infty} p_{1n} = \ P_1(1) = \frac{\rho\Phi(a+1, b; \frac{\lambda}{p\xi})}{\Phi(a, b; \frac{\lambda}{p\xi}) + \rho\Phi(a+1, b; \frac{\lambda}{p\xi})} = \frac{\rho^{\square}}{1+\rho^{\square}}$$

$$p_1 = \frac{\rho^{\square}}{1+\rho^{\square}}, \quad \text{where } \rho^{\square} = \frac{\rho\Phi(a+1, b; \frac{\lambda}{p\xi})}{\Phi(a, b; \frac{\lambda}{p\xi})}. \tag{18}$$

We now obtain the derivation for the mean number of customers in orbit L_0, using probability generating function $L_0 = p_0'(1) + p_1'(1)$,
Differentating $P_0(z)$ and $P_1(z)$ we get

$$P_0'(z) = \ p_{00} \tfrac{\lambda}{p\xi} \tfrac{a}{b} \sum_{n=0}^{\infty} \tfrac{(a+1)_n}{n!(b+1)_n} (\tfrac{\lambda z}{p\xi})^n$$

$$P_0'(z) = \ p_{00} \frac{\lambda}{p\xi} \frac{a}{b} \Phi(a+1, b+1; \frac{\lambda z}{p\xi}) \tag{19}$$

$$P_1'(z) = \rho\ p_{00} \tfrac{\lambda}{p\xi} \tfrac{(a+1)}{b} \sum_{n=0}^{\infty} \tfrac{(a+2)_n}{n!(b+1)_n} (\tfrac{\lambda z}{p\xi})^n$$

$$P_1'(z) = \rho\ p_{00} \frac{\lambda}{p\xi} \frac{(a+1)}{b} \Phi(a+2, b+1; \frac{\lambda z}{p\xi}) \tag{20}$$

substituting $z = 1$ in (19) and (20), we get

$$P_0'(1) = \ p_{00} \frac{\lambda}{p\xi} \frac{a}{b} \Phi(a+1, b+1; \frac{\lambda}{p\xi}) \tag{21}$$

$$P_1^{'}(1) = \rho\ p_{00}\frac{\lambda}{p\xi}\frac{(a+1)}{b}\Phi(a+2,b+1;\frac{\lambda}{p\xi}) \tag{22}$$

combining these two we get

$$L_0 = \ p_{00}\frac{\lambda}{pb\xi}\{a\Phi(a+1,b+1;\frac{\lambda}{p\xi}) + \rho(a+1)\Phi(a+2,b+1;\frac{\lambda}{p\xi}). \tag{23}$$

The mean number of customer in the system is $L = L_0 + p_1$.
W_0 and W can be determined from L_0 and L using Littles law $W_0 = \frac{L_0}{\lambda}$ and $W = \frac{L}{\lambda}$.

4. Particular Case

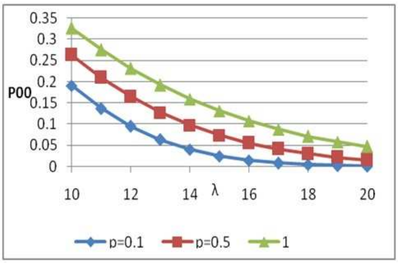
Case 1. Reneging rate $\xi = 0$ our model becomes M/M/1 retrial queueing studied by Donald Gross and John F. Shortle [8].

Case 2. Reneging rate $\xi = 0, \nu = 0$ our model becomes M/M/1l queueing studied by Donald Gross and John F. Shortle [8].

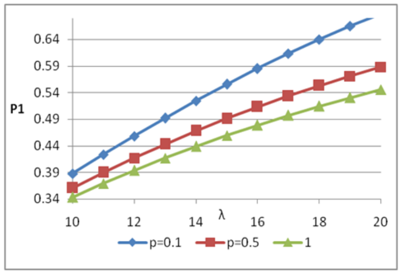
Case 3. Reneging rate $\xi = \gamma, p = 1$ our model becomes M/M/1 retrial queueing with impatience studied by Donald Gross and John F. Shortle [8].

5. Numerical Illustrations

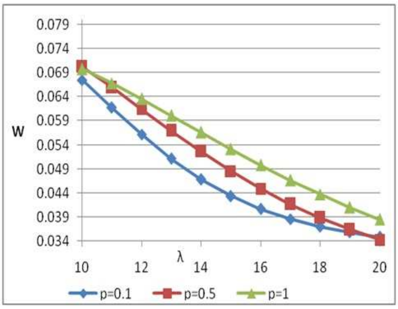
In this section, we present some numerical examples to describe the various system parameters influenced by the arrival rate λ, retrial rate ν and reneging rate ξ with probability p. Choosing arbitrary values for the parameters $\lambda = 10, 11, 12, \ldots, 20, \mu = 25, \xi = 4, \nu = 4, p = 0.1, 0.5, 1.0$ and various values of the parameters p_{00}, p_1 and W such that the stability condition is satisfied, two dimensional graphs are drawn in (a)-(c). The idle probability p_{00} decreasing for the various arrival rate λ as shown in graph (a). In graph (b), shows that the fraction of the time the system is busy, probability P_1 increases for increasing arrival rate λ. In graph (c), shows the mean waiting time of the system L decreasing because of the influence of the reneging rate ξ =4 with various probabilities p=0.1, 0.5 and 1 for increasing arrival rate λ.



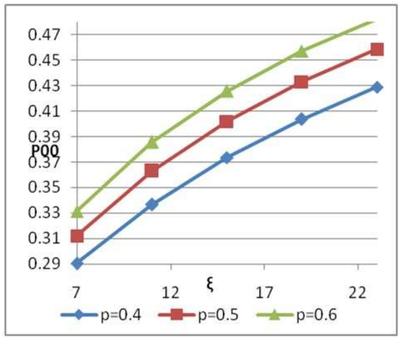
(a) P_{00} versus λ for $p = 0.1, 0.5, 1.0$



(b) P_1 versus λ for $p = 0.1, 0.5, 1.0$



(a) W versus λ for $p = 0.1, 0.5, 1.0$

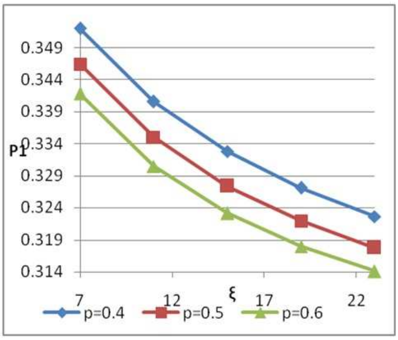


(b) P_{00} versus ξ for $p = 0.4, 0.5, 0.6$

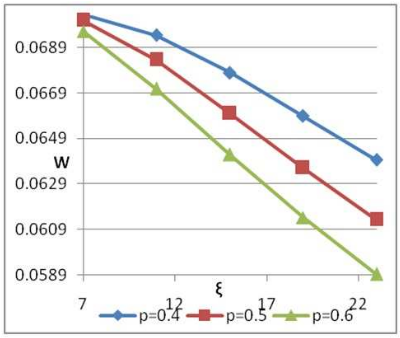
From above analysis we can see the influence of reneging probabilities in the case of arrival rate increase.Now choosing arbitrary values for the parameters $\lambda = 10, \mu = 25, \xi = 7, 8, \ldots, p = 0.4, 0.5$ and 0.6, $\nu = 4$ varying the values of the parameters p_{00}, p_1 and W such that the stability condition is satisfied, two dimensional graphs are drawn in (d)-(f). The idle probability P_{00} increasing for increasing reneging rate ξ is shown in graph (d). In graph (e), shows that the fraction of the time the system is busy probability P_1 increases for increasing the reneging rate ξ. In graph (f), shows the mean waiting time of the system L decreasing because of the influence of the reneging rate ξ =4 with various probabilities p=0.4, 0.5 and 0.6 for increasing arrival rate λ.

6. Conclusion

In this paper, we analyzed a single server retrial queuing system with retention of reneging impatient customers from the orbit. For this model closed form solution is obtained using probability generating function of the system state. Various performance measures such as the probability for fraction of time the



(a) p_1 versus ξ for $p = 0.4, 0.5, 0.6$



(b) W versus ξ for $p = 0.4, 0.5, 0.6$

server is busy and free, mean number of customers in the orbit and in the system, waiting time of the mean number of customers in the orbit and in the system have been derived. Using MATLAB numerical results have been presented to demonstrate how the various parameters of the model influence the behavior of the system.

References

[1] Jesus R. Artalejol, Acccessible bibliography on retrial queues, *Mathematical and computer modeling*, (1991), **30**, 1-6.

[2] Jesus R. Artalejo, Accessible bibliography of research on retrial queues, *Top* **7**, (1990-1999), 187-211.

[3] R.O. Al-Seedy,S.A. El-Shehawy, S.I. Ammar, Transient solution of the M/M/c queue with balking and reneging, *Computers and Mathematics with Applications* 57 (2009), 1280-1285.

[4] Dequan Yue, Yan Zhang, Wuyi Yue, Optimal performance of an M/M/1/N queue system with balking, reneging and server vacation, *International Journal of Pure and Applied Mathematics,* **28**, No. 1 (2006), 101-115.

[5] Amina Angelika Bouchentouf, Mokhtar Kadi,AbbesRabhi, Analysis of two heterogeneous server queueing models with balking, reneging and feedback, *Mathematical Sciences and Applications E-Notes,* **2** (2013), 10-21.

[6] E.R. Obert, Reneging phenomenon of single channel queues, *Mathematics of Operations Research* **4** (1979), 162-178.

[7] Jr.C.J. Ancker,A.V. Gafarian, Some queuing problems with balking and reneging: *I Operations Research,* **11** (1963), 88-100.

[8] Donald Gross, John F. Shortle, James M. Thompson, Carl M. Harris, *Fundamental of queueing theory*, Fourth edition, (2008), John wiley and Sons.

[9] B. Krishna Kumar and S. Pavai Madhewari, A.Vijayakumar, The M/G/1 retrial queues with feedback and starting failures, *Applied Mathematical Modelling*, **26** (2002), 1057-1075.

[10] Medhi Pallabi, Choudhury Amit, Reneging in queues without waiting space, *International Journal of Research in Applied, Natural and Social Science*, (2013), 111-124.

[11] Mahdy S. El-Paoumy, Hossam A. Nabwey,The Poissonian queue with balking function, reneging and two heterogeneous servers, *International Journal of Basics and Applied Sciences*, **6** (2011), 149-152.

[12] G. Falin, J. Templeton, *Retrial Queues* (1997), Chapman and Hal.

[13] G. Falin, A survey on retrial queues, *Queueing systems*, **7** (1990), 127-167.